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Analysis of Mechanical Systems using Interval Computations applied to Finite Elements Methods.

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This paper addresses the problem of mechanical systems in which parameters are uncertain and bounded. Interval calculation is used to find an envelope of transfer functions for mechanical systems modeled with Finite Elements. Within this context, a new formulation has been developed for Finite Elements problems involving bounded parameters, to avoid the problems of overestimation. An iterative algorithm is introduced, which leads to a conservative solution for linear mechanical problems. A method to ensure the convergence of this algorithm is also proposed. This new algorithm has been tested on simple mechanical systems, and leads to a conservative envelope of the transfer functions.

Nomenclature

x	scalar
$\{x\}$	vector
$[A]$	matrix
$\boldsymbol{x}$	interval
$\underline{\boldsymbol{x}}$	interval lower bound
$\overline{\boldsymbol{x}}$	interval upper bound
$rad(\boldsymbol{x})$	interval radius
$\mathrm{m}(\boldsymbol{x}) = \boldsymbol{x_c} = \frac{\underline{\boldsymbol{x}}+\overline{\boldsymbol{x}}}{2}$	interval center
$\mathrm{w}(\boldsymbol{x}) = \overline{\boldsymbol{x}} - \underline{\boldsymbol{x}}$	interval width
$\boldsymbol{\{x\}}$	interval vector
$\boldsymbol{[A]}$	interval matrix
$[M]$	mass matrix
$[K]$	stiffness matrix
η	hysteretic damping coefficient
E	Young's modulus
DOFF	Degrees Of Freedom
flops	floating point operations

1. INTRODUCTION

The physical parameters used to describe a structure are often uncertain, due to physical and geometrical uncertainties, or modeling inaccuracies. They are for instance Young's modulus, Poisson's ratio, length, volumic mass or thickness of plates. These uncertain parameters are generally identified to random variables, and introduced in a stochastic approach of the problems. Different methods can be used to solve these stochastic problems. A Monte Carlo simulation may for example be carried out. Several other methods exist ([1]), such as the perturbation method, the Neumann expansion series, or a projection on homogeneous chaos. But all of these methods consider stochastic variables for which the density of probability is known (Gaussian variables are mostly used). Furthermore, real variables are bounded, which is not the case for most stochastic variables. The Monte Carlo method is very expensive on a CPU point of view, and the others often encounter convergence problems. Moreover, only the mean value and the moments (often the variance only) are known, and since the density of probability of the solution is not known, these informations are difficult to use. As most of the time, the variables can be bounded, it seems to be judicious to investigate the mechanical problems containing uncertain parameters from the interval arithmetic theory point of view. Thus, interval arithmetic (R.E. Moore [2], G. Alefeld and J. Herzberger [3], Kearfott [4]) will be applied in connection with the Finite Element Methods.
We are interested in solving linear systems of equations, which correspond to the classical mechanical problem of finding the transfer function of a structure:

$$\Big([K](1+i\eta)-\omega^2[M]\Big)\,[H]=[I] \tag{1}$$

where $[K]$ and $[M]$ are the stiffness and mass matrices, η the coefficient of hysteretic damping, ω the excitation frequency, $[H]$ is the dynamic compliance matrix, and $[I]$ the identity matrix.
These problems have already been studied by several researchers (R. Chen and A. C. Ward [5], A. D. Dimarogonas [6], Koyluoglu [7]). They applied numerical methods developed for "reliable computing" based on interval matrices algebra (E. R. Hansen [8], Rump [9], S. Ning and R. B. Kearfott [10]). Elishakoff et al have focused on the bounds of eigenvalues of such dynamic systems ([11–15]). Chen [5] has pointed out the limitations of these formulations, which present a major drawback: the classical formulation does not take into account the way the matrices are built for mechanical problems. In fact, the terms of the matrices are not independent from each other, since they are calculated from the same parameters, for instance Young's modulus or density.
We will first introduce some basic concepts about interval arithmetic, and we will present the problematic of solving linear systems of interval equations. We will then introduce a new formulation of the problem, based on interval parameters which is adapted for the modeling of mechanical systems.
An adaptation of the Rump's algorithm ([9]) will be proposed which takes into account this novel interval formulation. The new algorithm is iterative, therefore the convergence criteria will be evaluated. The algorithm will be tested on a simple case to enable a comparison with the classical formulation. We will finally study frequency response functions for different mechanical systems, and also evaluate the

amount of computations on simple discrete systems, as well as the accuracy of the solutions.

2. RESOLUTION OF INTERVAL LINEAR SYSTEMS

The interval arithmetic has been first introduced by Moore ([2]), who was interested in the error propagation due to truncation of the mantissa in computers. Many publications (in particular the book of Alefeld and Herzberger [3]) give the basic and advanced concepts of this theory.
In this paper, boldface, lower cases, underscores and overscores respectively denote intervals, scalars, lower and upper bounds of intervals.

$$\boldsymbol{x} = [\underline{\boldsymbol{x}}, \overline{\boldsymbol{x}}] \tag{2}$$

The basic interval operations are presented in appendix 7. The interval arithmetic has special properties (in particular the property of sub-distributivity $(x)(\boldsymbol{y}+\boldsymbol{z}) \subseteq \boldsymbol{xy}+\boldsymbol{xz}$), that can lead to problems of overestimation when evaluating functions. We shall then be mindful to that problem in this paper.
One can also define interval vectors and interval matrices. Interval matrices can be expressed as follows:

$$[\boldsymbol{A}] = [A_c] + [-[rad([\boldsymbol{A}])], [rad([\boldsymbol{A}])]] \tag{3}$$

which is a quite convenient form.
The special properties of interval matrices have been investigated for example by Ning & al and Rohn in [10, 16].

2.1. SOLVING LINEAR SYSTEMS

If we are interested in the dynamic behavior of an industrial mechanical structure, one has to consider Finite Element Modeling, which leads to matrices (such as stiffness, mass, or damping matrix). Thus finding frequency response functions corresponds to solving linear systems of equations. If some of the mechanical parameters are uncertain at design stage, they can be modeled using the interval theory. The uncertain parameters can be geometrical ones (length, thickness, clearance...), or physical ones (Young's Modulus...). Then the matrices given by the Finite Element theory are interval matrices, and the problem is generally (static problems, frequency response functions) written as:

$$[A]\{x\} = \{b\} \tag{4}$$

with $[A] \in [\boldsymbol{A}]$ and $\{b\} \in \{\boldsymbol{b}\}$. Although several problems can be distinguished, as done by Chen and Ward in [5] and by Shary in [17], we will focus exclusively in this paper on the solution set of the outer problem which is defined as $\Sigma_{\exists\exists}([\boldsymbol{A}], \{\boldsymbol{b}\})$:

$$\Sigma_{\exists\exists}([\boldsymbol{A}], \{\boldsymbol{b}\}) = \{x \in \mathbb{R}^n | (\exists [A] \in [\boldsymbol{A}]), (\exists \{b\} \in \{\boldsymbol{b}\}) / [A]\{x\} = \{b\}\} \tag{5}$$

where $[\boldsymbol{A}]$ is an interval matrix and $\{\boldsymbol{b}\}$ an interval vector.
In general this set is not an interval vector. It is a non convex polyhedra (see [5] or [17] for examples). The Oettli and Prager theorem [18] gives an expression to get the exact solution set (5):

Theorem 1 (Oettli et Prager Lemma). *Let* $[\boldsymbol{K}] \in \mathbb{IR}^{n\times n}$ *and* $\{\boldsymbol{f}\} \in \mathbb{IR}^n$.

$$\{x\} \in \Sigma_{\exists\exists}([\boldsymbol{K}], \{\boldsymbol{f}\}) \Leftrightarrow |\mathrm{m}([\boldsymbol{K}])\{x\} - \mathrm{m}(\{\boldsymbol{f}\})| \leq rad([\boldsymbol{K}])|\{x\}| + rad(\{\boldsymbol{f}\}) \quad (6)$$

Nevertheless, this expression is quite difficult to use with matrices corresponding to real physical cases in a n-dimensional problem. Most of the time, only the smallest interval vector containing $\Sigma_{\exists\exists}([\boldsymbol{A}], \{\boldsymbol{b}\})$ will be considered, which is defined as $\square\Sigma_{\exists\exists}([\boldsymbol{A}], \{\boldsymbol{b}\})$. In this case, this ensures that the true solution is included in the numerical solution found $\square\Sigma_{\exists\exists}([\boldsymbol{A}], \{\boldsymbol{b}\})$. Within the context of this problematic, equation (4) can be rewritten as:

$$[\boldsymbol{A}]x = \{\boldsymbol{b}\} \quad (7)$$

Several algorithm intend to solve this problem. For example the Gaussian elimination algorithm can be adapted to the resolution of a linear system whose coefficients are interval. Alefeld gives some basic results in [3]. J. Rohn has shown in [19] that this algorithm could lead to an important overestimation of the solution. It even sometimes cannot solve the system because of zero pivot encountered.
Ning and Kearfott have made a review in [10] of existing methods for finding either $\square\Sigma_{\exists\exists}([\boldsymbol{A}], \{\boldsymbol{b}\})$ or an interval vector containing $\square\Sigma_{\exists\exists}([\boldsymbol{A}], \{\boldsymbol{b}\})$. These methods use particular forms of the matrices, that do not exactly correspond to mechanical cases, and are more appropriate for the treatment of numerical uncertainties as they are not well suited for dealing with large uncertainties.
Another useful method is based on a residual iteration, it is called the inclusion method of Rump [9]. It is an iterative method relying on the fixed point theorem, that leads to sharp results quite fast.

3. FORMULATION ADAPTED TO FINITE ELEMENTS METHODS

The existing algorithms used to solve $\Sigma_{\exists\exists}([\boldsymbol{A}], \{\boldsymbol{b}\})$ have been formulated for reliable computing on a numerical point of view. In an interval matrix for instance, each term can vary independently of each other in its interval, which is generally sharp.
If the interval formulation has to be adapted to mechanics, the dependence between the parameters must be taken into account, because many of the terms of the matrices are depending on the same parameters. For example if the Young's modulus varies in $\boldsymbol{E}$, a stiffness matrix could formally be written:

$$\boldsymbol{E}\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \quad (8)$$

which is not the same as

$$\begin{bmatrix} \boldsymbol{E}k_{11} & \boldsymbol{E}k_{12} \\ \boldsymbol{E}k_{21} & \boldsymbol{E}k_{22} \end{bmatrix} \quad (9)$$

that is treated in the classical interval techniques as:

$$\begin{bmatrix} E1k_{11} & E2k_{12} \\ E3k_{21} & E4k_{22} \end{bmatrix} \quad (10)$$

with $E1, E2, E3, E4$ varying in $\boldsymbol{E}$ independently.
When including the parameters in the terms of the matrices and vectors, the width of

$\Sigma_{\exists\exists}([\boldsymbol{A}], \{\boldsymbol{b}\})$ grows substantially (see example in 4.2). If all the matrices $[K] \in [\boldsymbol{K}]$ are considered, it must be noticed that many of them do not physically correspond to stiffness matrices, because stiffness matrices are symmetric positive and definite. For the different interval parameters in the matrix $[\boldsymbol{A}]$ to be put into factor as in equation (8), $[A]$ and $\{b\}$ are developed as follows:

$$[A] = [A_0] + \sum_{n=1}^{N} \boldsymbol{\epsilon_n}[A_n] \quad \{b\} = \{b_0\} + \sum_{p=1}^{P} \boldsymbol{\beta_p}\{b_p\} \tag{11}$$

N and P are the number of interval parameters to be taken into account when building the matrix $[A]$ and the vector $\{b\}$. $\boldsymbol{\epsilon_n}$ and $\boldsymbol{\beta_p}$ are independent centered intervals, generally $[-1, 1]$. $[A_0]$ and $\{b_0\}$ correspond to the matrices and vector built from the mean values of the parameters.
For a mechanical problem, the stiffness matrix will be written with factorized parameters:

$$[K] = [K_0] + \sum_{n=1}^{N} \boldsymbol{\epsilon_n}[K_n] \tag{12}$$

For each value of ϵ_n in $\boldsymbol{\epsilon_n}$, $[K]$ remains symmetric positive and definite, due to the physical character of the parameters.

4. A NEW ALGORITHM OF RESOLUTION

For the particular form of the problem shown in equation (11), where the interval parameters are put into factor in front of the matrices, it is necessary to adapt the algorithms. The new algorithm of resolution proposed here relies on the Rump's technique, that has been presented by Rohn in [21]. His demonstration is reminded in appendix 8. The inclusion method of Rump ([9]) relies on the fixed point theorem, and had to be adapted to avoid the problems of overestimation due to the loss of dependence in interval arithmetic. As the basic method of Rump, our algorithm is iterative, and then subject to convergence criteria that will be analyzed in 4.1.

Let us first consider a system in which only one parameter is an interval, then

$$[\boldsymbol{A}] = [A_0] + \boldsymbol{\alpha}[A_1] \quad \boldsymbol{\alpha} \text{ is centered} \tag{13}$$

is the equation of the system.
The implementation of the algorithm is presented below:

- First, an initialization stage
  $\boldsymbol{\epsilon} = [0.9, 1.1]$ is the so called inflation parameter.
  $[R] = inv(mid[\boldsymbol{A}]) = [A_0]^{-1}$ is an estimation of the inverse of $mid[\boldsymbol{A}]$.
  $\{x_s\} = [R] * \{b\}$ is an estimation of the solution.
  $[B] = [A_0]^{-1}[A_1]$
  $\{\boldsymbol{g}\} = [R] * (\{b\} - [\boldsymbol{A}] * \{x_s\}) = -\boldsymbol{\alpha}[A_0]^{-1}[A_1][A_0]^{-1}\{b\} = -\boldsymbol{\alpha}[B]\{x_s\}$
  $\{\boldsymbol{x_0}\} = \{\boldsymbol{g}\}$ initialization of the solution $\{\boldsymbol{x^*}\}$
  $[\boldsymbol{G}] = [I] - [R] * [\boldsymbol{A}] = -\boldsymbol{\alpha}[B]$ is the iteration matrix in the equation

$$\{\boldsymbol{x^*}\} = [\boldsymbol{G}]\{\boldsymbol{x^*}\} + \{\boldsymbol{g}\} \tag{14}$$

- Second, iterative resolution
  $\{\boldsymbol{y}\} = \boldsymbol{\epsilon} * \{\boldsymbol{x}\}$
  $\{\boldsymbol{x}\} = \{\boldsymbol{g}\} + [\boldsymbol{G}] * \{\boldsymbol{y}\}$
  until $\{\boldsymbol{x}\} \subset \{\boldsymbol{y^0}\}$.
  If the condition $\{\boldsymbol{x}\} \subset \{\boldsymbol{y^0}\}$ is satisfied, then $\{\boldsymbol{x}\}$ is a conservative solution of the equation $[\boldsymbol{A}]\{\boldsymbol{x}\} = \{\boldsymbol{b}\}$.

It must be noticed that all the matrices multiplications and linear system resolutions only concern deterministic matrices (opposed to interval ones). The interval formulation is preserved, and the interval parameters are put into factor in front of deterministic matrices. The control of the intervals is essential to avoid large overestimations of the solutions.
After n iterations, the solutions are given by the equations:

$$\{\boldsymbol{y_n}\} = \{\boldsymbol{y_{n-1}}\} + (-1)^n \boldsymbol{\alpha^n \epsilon^n} [B]^n \{x_s\} \tag{15}$$

$$\{\boldsymbol{x_n}\} = \{\boldsymbol{x_{n-1}}\} + (-1)^{n+1} \boldsymbol{\alpha^{n+1} \epsilon^n} [B]^{n+1} \{x_s\} \tag{16}$$

where the interval parameters have been put in factor in front of the deterministic matrices.
The main difference with the algorithm of Rump is the control of the interval parameters inside the iterative scheme, that avoids dramatic overestimations.
Rohn and Rex have shown in [22, 21] that the algorithm converges if and only if $\rho(|[\boldsymbol{G}]|) < 1$, where $\rho(|[G]|)$ is the spectral radius of the absolute value of $[G]$.
Few iterations are necessary to get a result if the matrix $[\boldsymbol{G}]$ is contracting. If the number of iterations remains small, the overestimation of the solution is not important, and that is why making $[\boldsymbol{G}]$ as much contracting as possible is interesting: it reduces the number of iterations and by the way the overestimation effect.
The method proposed above on a system with one interval parameter can easily be generalized to the problems where

$$[\boldsymbol{A}] = [A_0] + \sum_{i=1}^{N1} \boldsymbol{\alpha_i} [A_i]$$

$$\{\boldsymbol{b}\} = \{b_0\} + \sum_{j=1}^{N2} \boldsymbol{\beta_j} \{b_j\}$$

4.1. CONVERGENCE OF THE METHOD

We have proposed an iterative algorithm for solving the linear systems with interval parameters. This algorithm is based on the fixed point theorem, and the iteration matrix must be contracting. The problem of the convergence of the algorithm is then crucial to get solutions.
We have seen that the equation

$$\{\boldsymbol{x^*}\} = [\boldsymbol{G}]\{\boldsymbol{x^*}\} + \{\boldsymbol{g}\} \tag{17}$$

is convergent if and only if $\rho(|[\boldsymbol{G}]|) < 1$. In the general case, The iteration matrix is given by:

$$[\boldsymbol{G}] = \sum_{i=1}^{N1} \boldsymbol{e_i} [A_0]^{-1} [A_i] \tag{18}$$

and the condition is:

$$\rho(|\sum_{i=1}^{N1} -\boldsymbol{e_i}[A_0]^{-1}[A_i]|) < 1 \tag{19}$$

which is quite difficult to evaluate.
To estimate this value, we use the theorem 2 (see [23]).

Theorem 2 (Perron-Froebenius). *Let* $[A]$ *and* $[B]$ *be two* $n \times n$ *matrices with* $0 \leq |[B]| \leq [A]$.
Then,

$$\rho([B]) \leq \rho([A]) \tag{20}$$

And as it is well known that $|A + B| \leq |A| + |B|$, we can say that if the stronger convergence condition

$$\rho(\sum_i | - \boldsymbol{e_i}||[A_0]^{-1}[A_i]|) < 1 \tag{21}$$

is verified, then equation (19) is also true.

The condition $\rho(|[\boldsymbol{G}]|) < 1$ is not always true, especially for systems with wide interval parameters. We propose a method to avoid this problem and also to improve the contracting level of $[\boldsymbol{G}]$.
For a system with one interval parameter $([A_0] + \boldsymbol{e}[A_1])\{\boldsymbol{x}\} = \{b\}$, the iteration matrix is

$$[\boldsymbol{G}] = -\boldsymbol{e}[A_0]^{-1}[A_1] \tag{22}$$

and the condition of convergence is

$$\rho(|[\boldsymbol{G}]|) = |\boldsymbol{e}|\rho(|[A_0]^{-1}[A_1]|) < 1 \tag{23}$$

$\boldsymbol{e}$ is a centered interval, so that $[A_0]$ is the mean value of $[A_0] + \boldsymbol{e}[A_1]$. $[A_0]$ is depending on the position of the center of $\boldsymbol{e}$.
If $\boldsymbol{e}$ is a relatively wide interval (it means that the terms of $\boldsymbol{e}[A_1]$ are relatively wide with respect to the corresponding terms in $[A_0]$), the condition (23) can be false and the algorithm will be divergent. If this is the case, the strategy proposed is to split the interval into a partition of it, and then work on narrower intervals, on which the condition (23) will be verified. If we consider a partition of the interval $\boldsymbol{e} = \cup \boldsymbol{e_i}$, we have

$$\Sigma_{\exists\exists}([A_0] + \boldsymbol{e}[A_1], \{b\}) = \cup_i \Sigma_{\exists\exists}([A_0] + \boldsymbol{e_i}[A_1], \{b\}) \tag{24}$$

From $\boldsymbol{e}$ to $\boldsymbol{e_i}$, $[A_0]$ becomes $[A_0] + \mathrm{m}(\boldsymbol{e_i})[A_1]$, and $[A_1]$ remains the same. The equation to be solved is:

$$([A_0] + \mathrm{m}(\boldsymbol{e_i})[A_1] + [-rad(e_i), rad(e_i)][A_1])\{\boldsymbol{x_i}\} = \{b\} \tag{25}$$

Let us define d as:

$$d = \underset{e_i \subset \boldsymbol{e}}{Sup}(\rho(|([A_0] + \mathrm{m}(e_i)[A_1])^{-1}[A_1]|)) \tag{26}$$

For all interval $\boldsymbol{e_i}$ such that $\mathrm{w}(\boldsymbol{e_i}) < \frac{1}{d}$,

$$|\boldsymbol{e_i}|\rho(|[A_0]^{-1}[A_1]|) < 1 \tag{27}$$

It is then possible to split the interval $\boldsymbol{e}$ into a partition of it $\cup_i e_i$, where the algorithm is convergent for each interval $\boldsymbol{e_i}$.
For multi interval parameters problems, the same kind of splitting technique can be used, leading to the same result. Moreover, this technique can also be used to accelerate the convergence of the iterative scheme. The smaller the spectral radius of $|[\boldsymbol{G}]|$, the faster the convergence of the algorithm and the smaller the overestimation of the solution.

4.2. TEST OF THE NEW ALGORITHM ON A SIMPLE CASE

A new version of the algorithm of Rump has been developed to handle the case in which the interval parameters are put into factor in front of the matrices. The intervals are then controlled all along the algorithm, to avoid too large an overestimation. Moreover the convergence of the algorithm can be guaranteed, and even improved by splitting the intervals into a partition of them.
We will now test the proposed algorithm on a simple case to emphasize it's efficiency with respect to the basic method.

We have proposed a new interval formulation adapted to mechanical problems. The results found with the modified Rump's algorithm are often much sharper than the ones found with the classical formulation. To show the efficiency of the method for finding the solution of a linear system $[\boldsymbol{A}]\{x\} = \boldsymbol{\{b\}}$, we will consider the very simple example of a clamped free beam:

Figure 1. Clamped free beam

F and M are respectively the shear force and bending momentum applied at the free end of the beam, d and θ correspond to the displacement and slope at the free end of the beam.
The characteristics of the beam are:

$$\text{The Young's modulus } E \in [2.058e11, 2.142e11] \qquad (2.1e11 \pm 2\%) \tag{28}$$

$$\text{The Inertia } I \in [8.82e-8, 9.18e-8] \qquad (9e-8 \pm 2\%) \tag{29}$$

$$\text{The length } l = 1 \tag{30}$$

The shear force and bending momentum are also interval parameters:

$$\boldsymbol{\{f\}} = \begin{Bmatrix} [-10.2, -9.8] \\ [29.4, 30.6] \end{Bmatrix} \tag{31}$$

If we consider the elementary Finite Element matrix of the Euler Bernoulli theory [24], the static matrix equation of the problem is given by:

$$\begin{bmatrix} \frac{2EI}{9l} & \frac{-EI}{3l^2} \\ \frac{-EI}{3l^2} & \frac{2EI}{3l^3} \end{bmatrix} \begin{Bmatrix} d \\ \theta \end{Bmatrix} = \begin{Bmatrix} F \\ M \end{Bmatrix} \tag{32}$$

And from a numerical point of view, the stiffness matrix is an interval matrix:

$$\begin{bmatrix} [4033.68, 4369.68] & [-6554.52, -6050.52] \\ [-6554.52, -6050.52] & [12101.04, 13109.04] \end{bmatrix} \tag{33}$$

The first problem that can be solved is finding the solution set corresponding to the numerical equation:

$$\begin{bmatrix} [4033.68, 4369.68] & [-6554.52, -6050.52] \\ [-6554.52, -6050.52] & [12101.04, 13109.04] \end{bmatrix} \begin{Bmatrix} d \\ \theta \end{Bmatrix} = \begin{Bmatrix} [-10.2, -9.8] \\ [29.4, 30.6] \end{Bmatrix} \tag{34}$$

The Oettli and Prager lemma gives the exact solution set $\Sigma_{\exists\exists}([\boldsymbol{A}], \{\boldsymbol{b}\})$ and $\square\Sigma_{\exists\exists}([\boldsymbol{A}], \{\boldsymbol{b}\})$ (dotted line) shown in Figure 2. All the terms in the matrix are said to be independent.

Let us consider the mechanical problem with factorized interval parameters:

$$\boldsymbol{EI} \begin{bmatrix} 2/9l & -1/3l^2 \\ -1/3l^2 & 2/3l^3 \end{bmatrix} \begin{Bmatrix} d \\ \theta \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{F} \\ \boldsymbol{M} \end{Bmatrix} \tag{35}$$

As this system is quite simple, the solution can be found analytically. The exact mechanical solution set is given in Figure 2. It is called mechanical exact solution set. The hull of this set (which is an interval vector) has also been drawn. The mechanical exact solution set is included in $\Sigma_{\exists\exists}([\boldsymbol{A}], \{\boldsymbol{b}\})$, and is really small in comparison. This shows how important the factorization is for solving mechanical problems.

To test our algorithm, we have computed the result of the modified Rump's algorithm. It is illustrated on Figure 2. As we can see, it is overestimating the exact solution, but it gives a good idea of the size of the solution. Above all it is really smaller than the range computed when considering all the terms in the matrices independent, as in the initial Rump's algorithm.

As it had been noticed in [5], a large overestimation is obtained when including the parameters in the elements of the matrices. For finite element matrices, this overestimation can become critical, and often leads to an insolvable problem. As we have shown above, even on 2×2 matrices, the overestimation can reach 10 times or more. Such an adaptation of this algorithm enables its use for industrial problems involving huge size matrices.

5. APPLICATIONS ON MECHANICAL SYSTEMS

We will now focus on several specific examples to show the efficiency of the new algorithm. Each one is associated to a particular difficulty, for instance the number of parameters, or the development of the matrices into a sum with interval parameters put into factor.

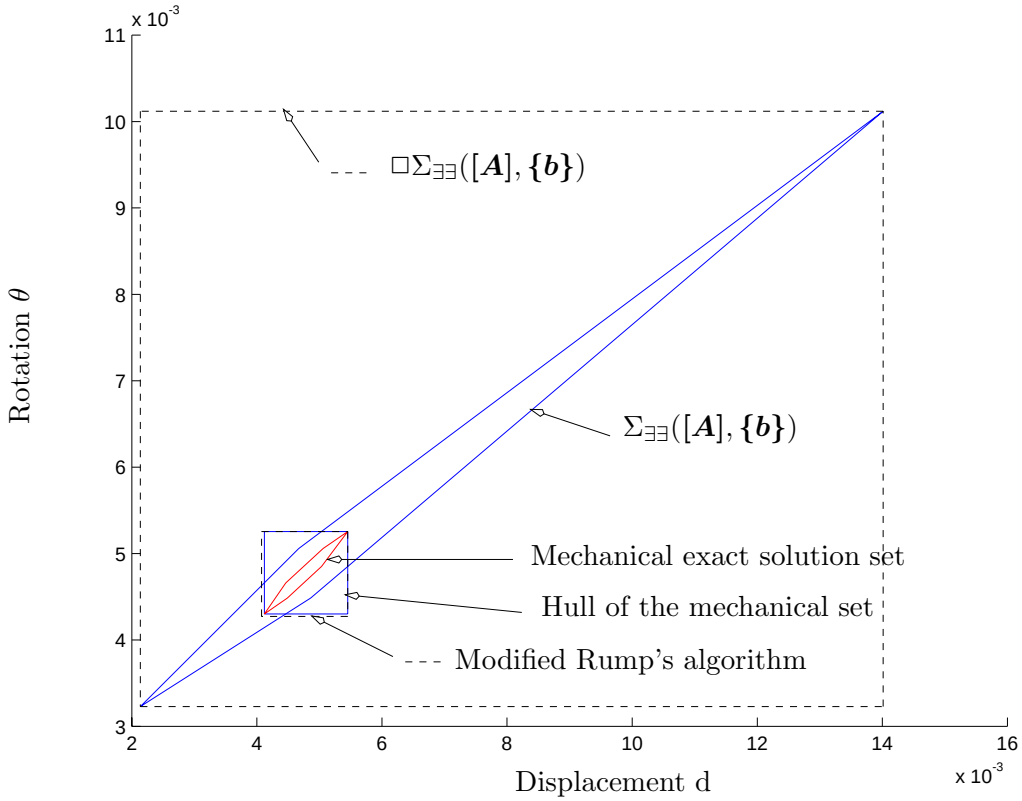

Figure 2. Solution sets for the clamped free beam. EI is uncertain ($\pm 2\%$). Numerical global problem, and reduced mechanical problem, and their respective hulls.

5.1. PROBLEM WITH SEVERAL PARAMETERS

This problem has two Degrees of Freedom, and is presented in Figure 3. The three stiffnesses are uncertain and vary in bounded intervals. We will focus on finding the transfer function envelope of the system.

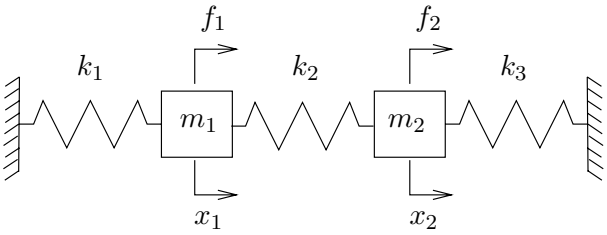

Figure 3. 3 springs system. x_1 and x_2 are the displacements of the masses m_1 and m_2, that are subject to the forces f_1 and f_2 respectively.

Each spring of stiffness k_i is subject to hysteretic damping η_i. Each value of the stiffness is uncertain ($k_i = k_i^0 + [-1, 1]k_i^1$, or $k_i^0 - k_i^1 < k_i < k_i^0 + k_i^1$).
The numerical values of the parameters are:

TABLE 1

Numerical values of the parameters

$k_1^0 = 100\ N.m^{-1}$	$k_2^0 = 10\ N.m^{-1}$	$k_3^0 = 100\ N.m^{-1}$
$k_1^1 = 0.04k_1^0$	$k_2^1 = 0.04k_2^0$	$k_3^1 = 0.04k_3^0$
$\eta_1 = 0.02$	$\eta_2 = 0.02$	$\eta_3 = 0.02$
$m_1 = 1\ kg$	$m_2 = 1\ kg$	

We consider the dynamic problem, and the set of equations for the transfer function is given below:

$$\left(\begin{bmatrix} (1+i\eta_1)k_1^0 + (1+i\eta_2)k_2^0 & -(1+i\eta_2)k_2^0 \\ -(1+i\eta_2)k_2^0 & (1+i\eta_2)k_2^0 + (1+i\eta_3)k_3^0 \end{bmatrix} + \boldsymbol{e_1} \begin{bmatrix} (1+i\eta_1)k_1^1 & 0 \\ 0 & 0 \end{bmatrix} \right.$$
$$\left. + \boldsymbol{e_2} \begin{bmatrix} (1+i\eta_2)k_2^1 & -(1+i\eta_2)k_2^1 \\ -(1+i\eta_2)k_2^1 & (1+i\eta_2)k_2^1 \end{bmatrix} + \boldsymbol{e_3} \begin{bmatrix} 0 & 0 \\ 0 & (1+i\eta_3)k_3^1 \end{bmatrix} - \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \right) \begin{Bmatrix} \boldsymbol{H_1} \\ \boldsymbol{H_2} \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{36}$$

The resolution of this problem will be done on a frequency band including all the modes, represented by 61 points linearly spaced. The use of our algorithm leads to envelope bounds of both real and imaginary parts of the transfer function for each frequency evaluated (see Figure 4). To have a very contracting iterative scheme, the spectral radius is imposed to be less than 0.3, and the inflation parameter is $[0, 2]$. To compare our results with the ones of a Monte Carlo simulation, we have made 10000 stochastic tests on each of the 61 frequency. The Monte Carlo simulation leads to an estimation of the envelope interval which is not conservative. We can then compare the results of both methods for a particular value of the frequency. For $\omega = 9.5$ rad/s, the results are given in table 2. The Monte Carlo simulation gives results that are included in the true bounds, whereas the proposed algorithm can find envelope bounds.

TABLE 2

Real and imaginary parts of the collocated transfer function $H(1, 1)$ for $\omega = 9.5$ rad/s.

	Monte Carlo 5000 tests	Monte Carlo 20000 tests	Proposed algorithm
flops	1380000	5520241	123500
real(H(1,1))	$[0.05020, 0.09722]$	$[0.05007, 0.09755]$	$[0.04829, 0.09964]$
imag(H(1,1))	$[-0.02754, -0.00649]$	$[-0.02773, -0.00644]$	$[-0.02916, -0.00557]$

It must be noticed that the amount of computations for the proposed algorithm is small compared to the amount needed by the Monte Carlo simulation. If the example is computed with a smaller uncertainty ($\pm 2\%$ for instance), the algorithm will be even faster (10500 flops for $\omega = 9.5$ rad/s). For the computation on all the 61 points, a $\pm 2\%$ uncertainty on each spring will use 16, 6 Mflops, and a simulation with $\pm 4\%$ uncertainty 126.5 Mflops. The result is also quite good, the envelope is wrapping the deterministic transfer functions, without overestimating the true envelope too much (see Figures 4 and 5).

5.1.1. *Nonlinear dependence of the parameters*

The decomposition of the finite element problems into a factorized sum as in equation (11) is not obvious. Let us consider a very simple finite element problem.

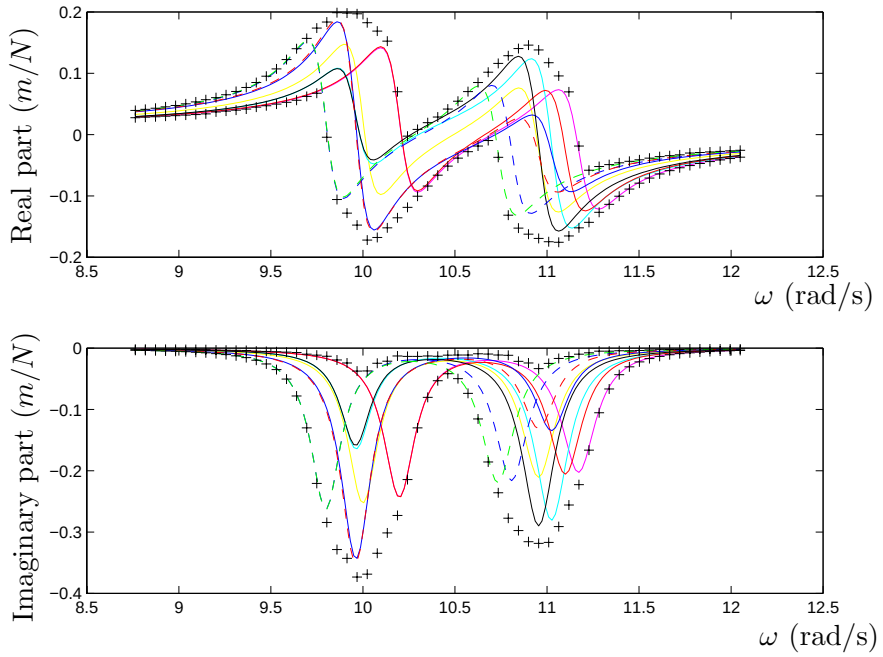

Figure 4. Real and imaginary parts of the collocated transfer function (1,1) of the system shown in Figure 3. Solid lines represent the transfer function for several values of the stiffnesses. Crosses represent the envelope calculated with the modified Rump's algorithm, for ±4% uncertainties.

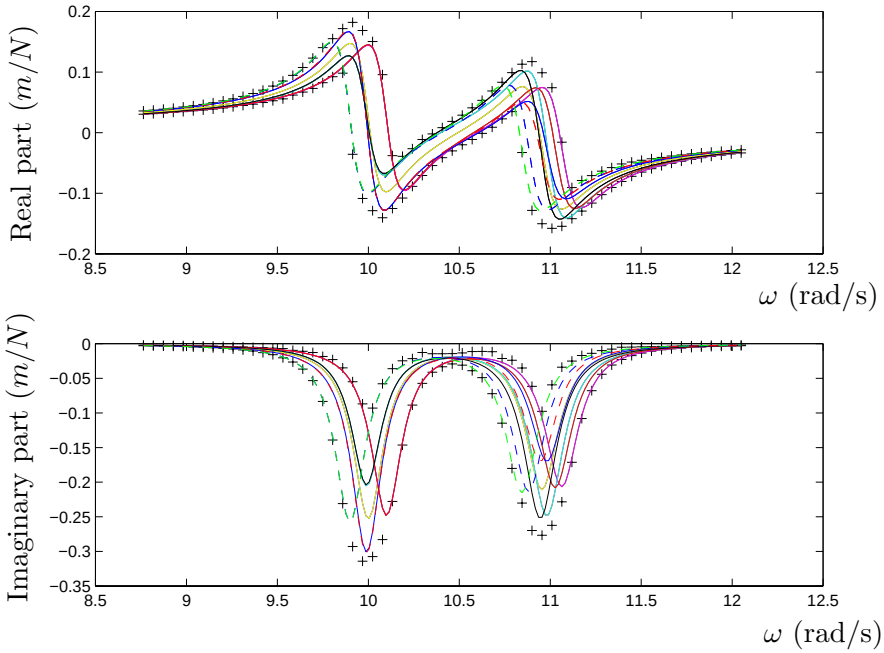

Figure 5. Real and imaginary parts of the collocated transfer function (1,1) of the system shown in Figure 3. Solid lines represent the transfer function for several values of the stiffnesses. Crosses represent the envelope calculated with the modified Rump's algorithm, for ±2% uncertainties.

We have meshed a clamped free plate whose thickness varies in an interval (see Figure 6). If we call t the thickness of the plate, both t and t^3 appear in the elementary matrices of the Love-Kirchhoff theory (the stiffness matrix depends on t^3, and the mass matrix on t).
The dynamic problem is written:

$$\left(\boldsymbol{t^3}[K](1+i\eta) - \omega^2 \boldsymbol{t}[M]\right)\{H\} = \{F\} \tag{37}$$

The intervals $\boldsymbol{t}$ and $\boldsymbol{t^3}$ cannot be considered as independent. Thus we have to use an approximate expression to take this dependence into account :
$\boldsymbol{t}$ can be written $\mathrm{m}(\boldsymbol{t}) + \boldsymbol{[-1,1]}rad(t)$, or $t_0 + \delta t$. Then $\boldsymbol{t^3}$ is

$$(t_0+\delta t)^3 = t_0^3 + 3\delta t.t_0^2 + 3t_0.\delta t^2 + \delta t^3 \tag{38}$$

The matrix equation (37) becomes :

$$\left(\left(t_0^3[K](1+i\eta) - \omega^2 t_0[M]\right) + \delta t\left(3.t_0^2[K](1+i\eta) - \omega^2[M]\right) + \left(3t_0\delta t^2 + \delta t^3\right)[K](1+i\eta)\right)\{H\} = \{F\} \tag{39}$$

Where δt varies in $[-rad(\boldsymbol{t}), rad(\boldsymbol{t})] = [-dt, dt]$.
If δt and $3t_0\delta t^2 + \delta t^3$ are said to be independent (which is false, but for $\delta t << t_0$, $\quad \delta t >> 3t_0\delta t^2 + \delta t^3$), we will get a new equation of form:

$$\left(A_0 + \boldsymbol{\epsilon_1}A_1 + \boldsymbol{\epsilon_2}A_2\right)X = b \tag{40}$$

where $\boldsymbol{\epsilon_1} = [-dt, dt]$ and $\boldsymbol{\epsilon_2} = [0, 3t_0dt^2 + dt^3]$.
The equation (39) as been modified so that the dependence between the preponderant terms is conserved (ie the terms in δt). The other terms are then considered to be independent of δt. Taking these terms into account is anyway essential for the algorithm to lead to conservative results. By treating the new equation with the modified Rump's algorithm, we can get a conservative result of the transfer function of the plate.

The numerical example we have treated is a clamped free plate (dimensions $4m \times 1m$), whose thickness is $t = 5.10^{-2}m \pm 6\%$. The value of the hysteretic damping in the plate is 2%. The plate is meshed with $5 * 3$ elements (see Figure 6).

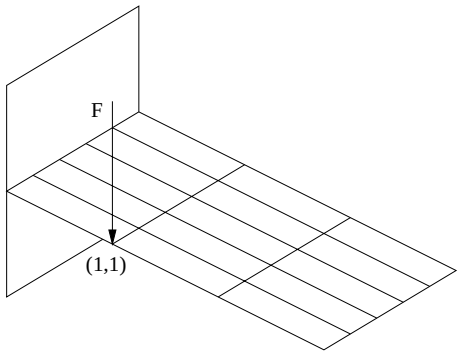

Figure 6. Clamped free plate meshed with 15 elements.

The collocated transfer function calculated in point $(1,1)$ (see Figure 6) is represented in the Figures 7 and 8. The algorithm leads to an envelope of the real and imaginary parts of the transfer function, and the overestimation remains small.

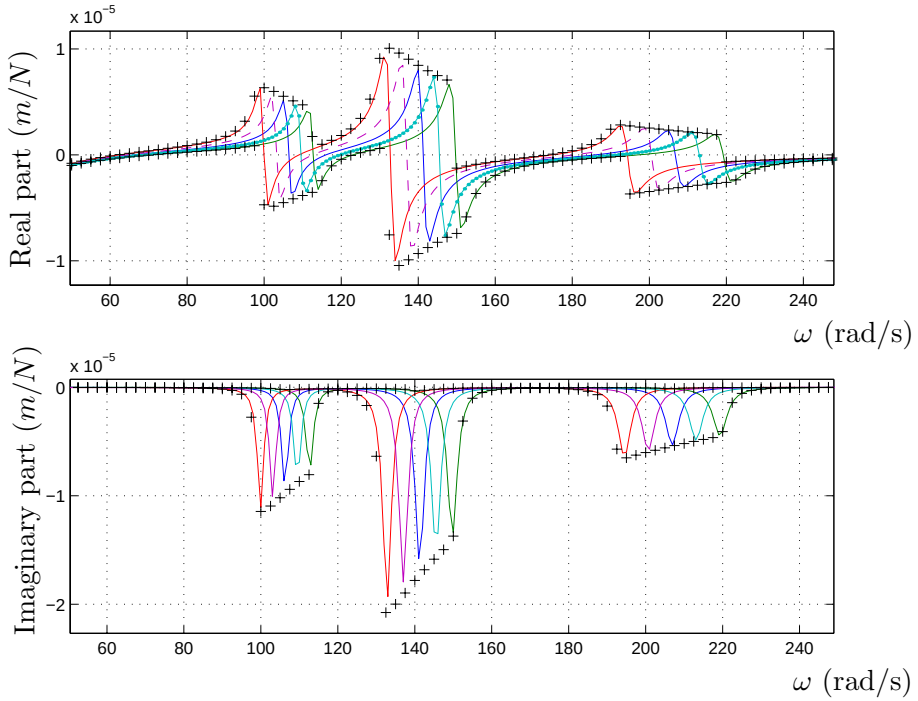

Figure 7. Collocated transfer function (real and imaginary parts) for the plate at node (1,1). Several deterministic transfer functions have been drawn, corresponding to different values of the thickness. The crosses correspond to the robust interval algorithm.

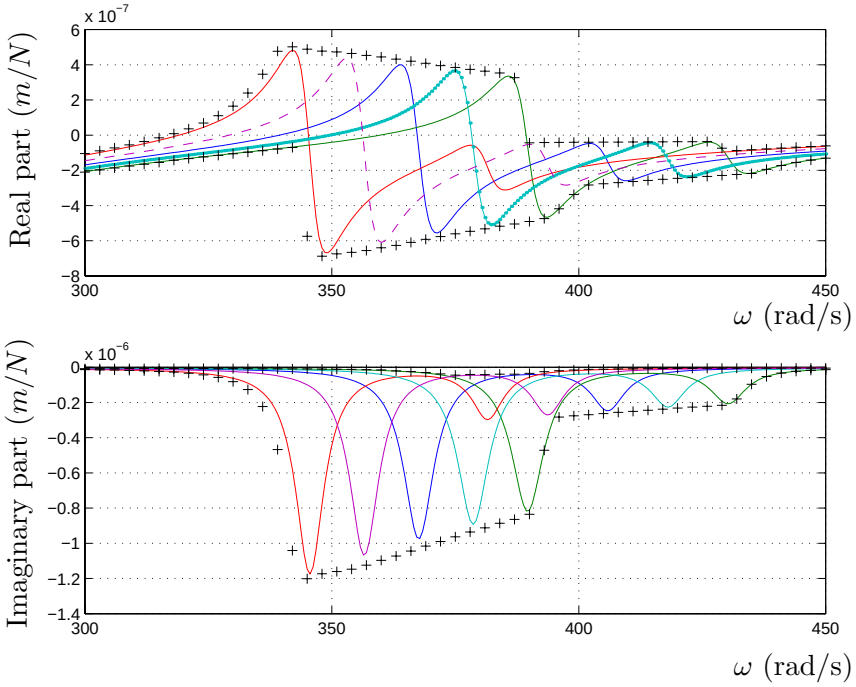

Figure 8. Zoom of the collocated transfer function for the plate.

5.1.2. *System with multiple eigenvalues*

A last example will be treated to show the efficiency of the interval calculus, when taking into account small uncertainties that are inherent to mechanical systems. The new algorithm permits in this case to bring out important effects due to these small fluctuations.

We will consider a three bladed-disk that is modeled with a 7 DOFF system (see Figure 9). The blades are modeled with the Euler-Bernoulli theory, and only hysteretic damping is considered. The values of the parameters are: length $L = 1\ m$, area $S = \pi 10^{-4}\ m^2$, Young's modulus $E_0 = 210\ GPa$, and volumic mass $\rho = 7800\ kg/m^3$. The damping coefficient is $\eta = 2\%$. The Young's modulus of one blade is uncertain ($E = E_0 \pm 10\%$).

As the 3 blades are identical in the crisp tuned model, the eigenfrequencies are found as multiple eigenvalues of a matrix system. If one of the blades is mistuned, then the eigenvalues are no more multiple ones, and new resonances can appear. This is a complete modification of the structure, and that kind of phenomenon is well know in aeronautics (see [25, 26]) and can lead to the appearance of a much stronger dynamics than the one expected for a tuned system. Let us consider the transfer function $H(1, 3)$. When all the 3 blades are identical, the transfer function shows only two resonances. If one of the blades is mistuned (for instance, if it's Young's modulus is not exactly the same than for the other blades) two new resonances appear on the transfer function.

On Figure 10 the modulus of the transfer function $H(1, 3)$ is shown (it's special calculation is explained in appendix 9). Dashed line represent the deterministic case for which all of the three blades are identical, and solid lines the envelope of the transfer function for the system in which one blade has an uncertain Young's modulus ($E = E_0 \pm 10\%$). The envelope shows four resonance zones. This is due to the mistuning phenomenon. This deep modification of the spectrum due to a small perturbation brings out the efficiency of the method, that can predict a priori non expected phenomena.

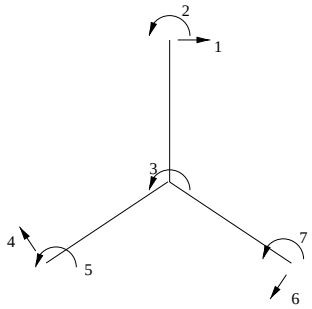

Figure 9. 3 bladed-disk, and the 7 DOFF.

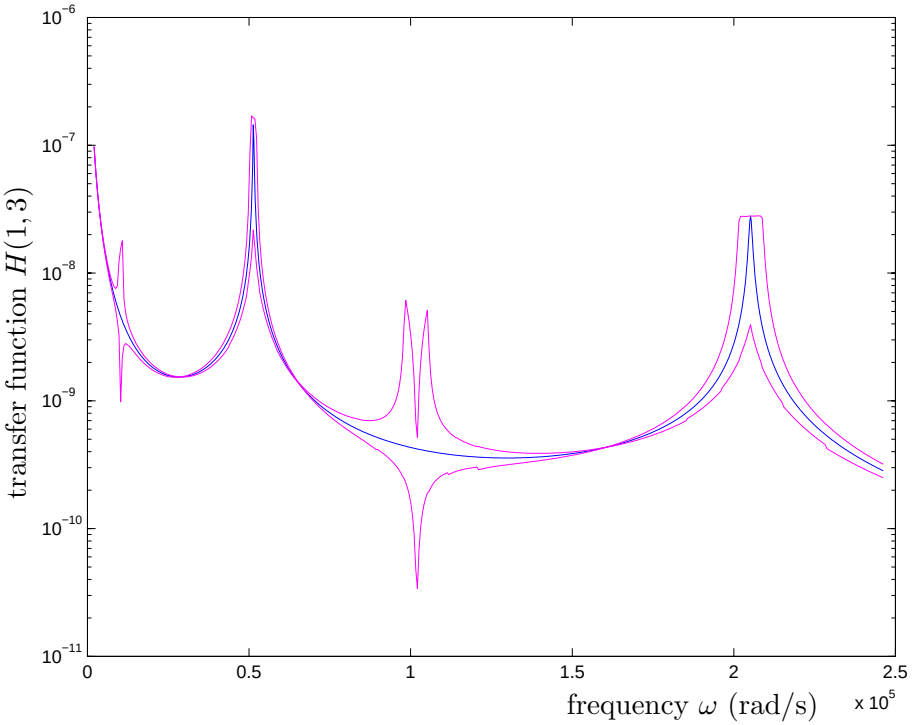

Figure 10. Modulus of the transfer function $H(1,3)$. The Young's modulus of the first blade is uncertain ($E = E_0 \pm 10\%$). Dashed line represent the deterministic case for which all the blades are identical, solid lines are the envelope calculated with the proposed algorithm.

For that kind of computation too, the modified algorithm gives accurate results, once again with the advantage of getting a robust envelope. This method can improve considerably the accuracy of prediction of the dynamic behavior of mechanical systems involving inaccurate parameters.

6. CONCLUSION

The vibrating systems are often modeled with a Finite Element Method. When they are depending on uncertain and bounded parameters, they can be studied thanks to the interval calculus. For the resolution of linear systems, in which some variables are intervals, one can find well suited algorithms, but they consider only full interval matrices, whereas this doesn't correspond to real physical problems. A new formulation is introduced in which the interval parameters are factorized when building Finite Elements matrices. Using this factorized formulation, a novel algorithm is presented. It corresponds to a reformulation of the iterative algorithm of S.M. Rump [9] adapted to the Finite Elements formulation. The convergence of this method has been studied, and a dichotomy scheme ensures the convergence of the algorithm. It is easy to notice on a simple example that the factorization and the proposed algorithm lead to better results than classical methods. On standard-sized Finite Elements Models, the classical methods wouldn't work, hence the novel method proposed is interesting. This method enables to find bounds of the transfer function of dynamic problems in which some of the parameters are uncertain and

bounded. The relevance of such an envelope is that one can be certain that all of the solutions corresponding to the bounded parameters are in this envelope. This is the robust aspect of the method.
If used in a design stage, this algorithm allows to take into account from the beginning of it's life the uncertainties in the physical parameters of a product. Furthermore, if the algorithm is used for analysis, it will be possible, as the bounds of the physical parameters are known, to find guaranteed bounds for the static and dynamic responses. Then safety zones can be defined, where a given level of the responses will never be reached.
The algorithm is based on Finite Elements Modeling, and the result is depending on the accuracy of the numerical model. Moreover, as for classical deterministic FEM, the refinement of the mesh has some kind of influence on the solution. It is also necessary to take into account the models errors in addition to the uncertainties on the parameters, but this is beyond the scope of this paper. The proposed algorithm can only handle a limited number of interval parameters. For working on industrial models, with tens or hundreds of uncertain parameters, the algorithm will have to be improved. But for a design stage, when few parameters are subject to important uncertainties, it should reveal really useful.

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7. APPENDIX: OPERATIONS ON INTERVALS

As the interval arithmetic is different from the classical arithmetic, we will define several arithmetic operations on intervals.
The four classical arithmetic operations are also defined:

$$\boldsymbol{x}+\boldsymbol{y}=\left[\underline{\boldsymbol{x}}+\underline{\boldsymbol{y}},\overline{\boldsymbol{x}}+\overline{\boldsymbol{y}}\right] \tag{41}$$

$$\boldsymbol{x}-\boldsymbol{y}=\left[\underline{\boldsymbol{x}}-\overline{\boldsymbol{y}},\overline{\boldsymbol{x}}-\underline{\boldsymbol{y}}\right] \tag{42}$$

$$\boldsymbol{x}*\boldsymbol{y}=\left[min(\underline{\boldsymbol{x}}\underline{\boldsymbol{y}},\underline{\boldsymbol{x}}\overline{\boldsymbol{y}},\overline{\boldsymbol{x}}\underline{\boldsymbol{y}},\overline{\boldsymbol{x}}\overline{\boldsymbol{y}}),max(\underline{\boldsymbol{x}}\underline{\boldsymbol{y}},\underline{\boldsymbol{x}}\overline{\boldsymbol{y}},\overline{\boldsymbol{x}}\underline{\boldsymbol{y}},\overline{\boldsymbol{x}}\overline{\boldsymbol{y}})\right] \tag{43}$$

$$1/\boldsymbol{x}=[1/\overline{\boldsymbol{x}},1/\underline{\boldsymbol{x}}]\quad(0\notin\boldsymbol{x}) \tag{44}$$

$$\boldsymbol{x}/\boldsymbol{y}=\boldsymbol{x}*(1/\boldsymbol{y})\quad(0\notin\boldsymbol{x}) \tag{45}$$

An interval vector $\boldsymbol{\{x\}}$ is a vector whose components are intervals.

$$\boldsymbol{\{x\}}=\begin{Bmatrix}\boldsymbol{x_1}\\ \vdots\\ \boldsymbol{x_n}\end{Bmatrix} \tag{46}$$

An interval matrix $\boldsymbol{[A]}$ is a matrix whose components are intervals.

$$\boldsymbol{[A]}=[\boldsymbol{A_{ij}}]\quad i=1..m,j=1..n \tag{47}$$

8. APPENDIX: ALGORITHM OF RUMP

The problem to be solved is:

$$[A]\{x\}=\{b\} \tag{48}$$

$[A]$ is a square matrix.
For an arbitrary non singular matrix $[R]$, and a vector $\{x_0\}$,

$$[A]\{x\}=\{b\} \tag{49}$$

is equivalent to

$$\{x^*\}=[G]\{x^*\}+\{g\} \tag{50}$$

with

$$[G]=[I]-[R][A] \tag{51}$$

$$\{g\}=[R](\{b\}-[A]\{x_0\}) \tag{52}$$

$$\{x\}=\{x_0\}+\{x^*\} \tag{53}$$

In practice, $[R]\approx[A^{-1}]$, and $\{x_0\}=[R]\{b\}$, so that $[G]$ and $\{g\}$ are of small norms, and $\{x^*\}$ is close to 0.

Let the interval vector $\boldsymbol{\{X\}}$ satisfy:

$$[G]\boldsymbol{\{X\}}+\{g\}\subset\{\boldsymbol{X^0}\} \tag{54}$$

where $[G]\boldsymbol{\{X\}}+\{g\}=\{[G]\{X\}+\{g\};\{X\}\in\boldsymbol{\{X\}}\}$, and $\{\boldsymbol{X^0}\}$ is the interior of $\boldsymbol{\{X\}}$. Then,

$$\{x^*\}=[G]\{x^*\}+\{g\} \tag{55}$$

has a unique solution $\{x^*\}\in[G]\boldsymbol{\{X\}}+\{g\}$ (Rump [9]).
The proof is true in the abstract, but the algorithm is used on computers that do

not always give true results (due to the mantissa truncation). The programs used to compute interval arithmetic have to take that problem into account (for example the package BIAS from Olaf Knuppel [27, 28]). If $\odot$ and $\oplus$ denote the computed interval multiplication and sum (they overestimate the true intervals), and

$$[G] \odot \{\boldsymbol{X}\} \oplus \{g\} \subset \{\boldsymbol{X^0}\} \tag{56}$$

is true in computed interval arithmetic, then we have also:

$$[G]\{\boldsymbol{X}\} + \{g\} \subset \{\boldsymbol{X^0}\} \tag{57}$$

since

$$[G]\{\boldsymbol{X}\} + \{g\} \subset [G] \odot \{\boldsymbol{X}\} \oplus \{g\} \tag{58}$$

The algorithm can be summarized as:

- First, an initialization stage
  $\boldsymbol{\epsilon} = [0.9, 1.1]$ is the so called inflation parameter.
  $[R] = inv(mid[\boldsymbol{A}])$ is an estimation of the inverse of $mid[\boldsymbol{A}]$.
  $\{\boldsymbol{x_0}\} = [R] * \{\boldsymbol{b}\}$ is an estimation of the solution.
  $\{\boldsymbol{g}\} = [R] * (\{\boldsymbol{b}\} - [\boldsymbol{A}] * \{\boldsymbol{x_0}\})$
  $\{\boldsymbol{x}\} = \{\boldsymbol{g}\}$ initialization of the solution $\{\boldsymbol{x^*}\}$
  $[\boldsymbol{G}] = I - [R] * [\boldsymbol{A}]$ is the iteration matrix in the equation
  $$\{\boldsymbol{x^*}\} = [\boldsymbol{G}]\{\boldsymbol{x^*}\} + \{\boldsymbol{g}\} \tag{59}$$
- second, iterative resolution
  $\{\boldsymbol{y}\} = \boldsymbol{\epsilon} * \{\boldsymbol{x}\}$
  $\{\boldsymbol{x}\} = \{\boldsymbol{g}\} + [\boldsymbol{G}] * \{\boldsymbol{y}\}$
  until $\{\boldsymbol{x}\} \subset \{\boldsymbol{y^0}\}$ or too many iterations.
  If the condition $\{\boldsymbol{x}\} \subset \{\boldsymbol{y^0}\}$ is satisfied, then $\{\boldsymbol{x}\}$ is a conservative solution of the equation $[\boldsymbol{A}]\{\boldsymbol{x}\} = \{\boldsymbol{b}\}$.

9. APPENDIX: MODULUS OF THE TRANSFER FUNCTION

The modulus of the dynamic compliance vector is normally calculated as:

$$|H| = \sqrt{H_r^2 + H_i^2} \tag{60}$$

To avoid the problem of overestimation due to the dependence of the real and imaginary parts of the dynamic compliance, we propose a method to compute the bounds of its modulus.
For a system with one interval parameter, the compliance vector could be written in the formalized way, after N iterations, according to the recurrent scheme proposed in 4:

$$\begin{Bmatrix} H_i^N \\ H_r^N \end{Bmatrix} = \begin{Bmatrix} H_i^{N-1} \\ H_r^{N-1} \end{Bmatrix} + \boldsymbol{\epsilon}^N \begin{Bmatrix} X_i^N \\ X_r^N \end{Bmatrix} \tag{61}$$

where H_i^N is the imaginary part of the vector H, computed at the loop N of the algorithm. The real and imaginary parts of H are both depending on the same

interval parameter $\boldsymbol{\epsilon}^N$, and applying directly the equation (60) would lead to large overestimations of $|H|$. The equation (61) can be written as:

$$\begin{Bmatrix} H_i^N \\ H_r^N \end{Bmatrix} = \sum_{n=1}^{N} \boldsymbol{\epsilon}^n \begin{Bmatrix} X_i^n \\ X_r^n \end{Bmatrix} \tag{62}$$

The modulus can then be calculated as

$$\boldsymbol{mod}(\{\boldsymbol{H}\}) = \sqrt{\sum_{n=1}^{N} {\boldsymbol{\epsilon_1}}^{2n} \left(X_i^{n2} + X_r^{n2} \right) + 2 \sum_{n=1}^{N-1} \sum_{p=n+1}^{N} {\boldsymbol{\epsilon_1}}^{n+p} \left(X_i^n X_i^p + X_r^n X_r^p \right)} \tag{63}$$

and the dependence between the real and imaginary parts is preserved in a better way than applying the equation (60) directly.