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Interval first-derivative-based iterative methods with high order of convergence for solving nonlinear systems of equations

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Abstract. In the present paper we propose technique of construction and investigation of new effective iterative interval methods for solving nonlinear systems of equations. Two methods of such type and their main features are presented and compared by efficiency. The fields of their preferable application are defined.

Keywords: nonlinear system of equation, interval iterative method, order of convergence.

The whole variety of application of the interval analysis can be conventionally shared into two basic directions: the analysis of mathematical models with uncertainties without taking into account distributions of probabilities of such data and with the known borders of their change and solving of the mathematical models which we receive as a result of approximation of the considered problem by another one, methods of receiving the solution for which are known, that makes the main problem of the applied and, in particular, computational mathematics. Computing algorithms thus frequently do not converge, converge not to the solution of a problem, require information about the unknown solution (about existence, multiplicity, bifurcation, etc.). Besides at realization even entirely determined algorithms on computer some difficulties are generated also by discrete-type structure of its memory. This demands rounding, which even after a small time period of work collects a huge amount.

The second direction of application of the interval analysis frequently is applied to construction of interval methods for solving the systems of nonlinear equations. However here, on the whole, the investigations are limited to construction of various updating of the interval analogues of the Newton method [1]. It is caused by that fact, that the interval estimation of derivatives of supreme orders demands great volume of calculations and decomposition algorithms of interval expansions of functions in Taylor's series are unknown.

In a basis of construction of new interval methods of the supreme orders of convergence for solving nonlinear systems of equations

$$f(x) = 0, \tag{1}$$

where $f\colon R^k \to R^k$, $k \in \mathbb{N}$, we put the next ideas [2, 3]:

- Idea of "immersing" of the given problem in the wider class of problems. Among solutions of such problem there are also all solutions of a problem (1)

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which it is necessary to allocate in special manner ("to make a filtration") with beforehand established accuracy or, at least with the greatest possible accuracy;

- Idea Runge of approximation with the greatest possible accuracy of supreme order derivatives of mapping $f(x)$ by linear combinations of values of its first derivative in corresponding points;
- Taking into account the behavior of "average" points of residual members in Laugrange form of the generalized Taylor series of mapping $f(x)$ in the case of compression of interval of decomposition into a point and ratio between these points at decomposition in Taylor's series of mapping $f(x)$ and its first derivative.

Let

$$g_m(x,y) = f(x_n) + (\alpha_1 f'(x_n) + \alpha_2 f'(x_n + \beta_2(x - x_n)) + \ldots + \alpha_m f'(x_n + \beta_m(x - x_n)))(y - x_n), \tag{2}$$

where α_i $(i = 1, 2, \ldots, m)$, β_j $(j = 1, 2, \ldots, m)$ – real coefficients, $m \geq 2$.

Mappings $f(x)$ and $g_m(x,y)$ satisfy to conditions of their decomposition in Taylor's series in the corresponding neighborhoods of a point x_n. Coefficients α_i, β_j we shall choose so that decompositions

$$f(x) = f(x_n) + f'(x_n)(x - x_n) + \frac{1}{2!}f''(x_n)(x - x_n)^2 + \ldots + \frac{1}{(p-1)!}f^{p-1}(x_n)(x - x_n)^{p-1} + \frac{1}{p!}f^p(x_n + \theta^{(0)}(x - x_n))(x - x_n)^p, \tag{3}$$

and

$$\begin{aligned} g_m(x,x) = f(x_n) &+ (\alpha_1 + \ldots + \alpha_m)f'(x_n)(x - x_n) + \\ &(\alpha_2\beta_2 + \ldots + \alpha_m\beta_m)f''(x_n)(x - x_n)^2 + \ldots + \\ &\frac{1}{(p-2)!}(\alpha_2\beta_2^{p-2} + \ldots + \alpha_m\beta_m^{p-2})f^{p-1}(x_n)(x - x_n)^{p-1} + \\ &\frac{1}{(p-1)!}\Big(\alpha_2\beta_2^{p-1}f^p(x_n + \theta_2^{(1)}\beta_2(x - x_n)) + \ldots + \\ &\alpha_m\beta_m^{p-1}f^p(x_n + \theta_m^{(1)}\beta_m(x - x_n))\Big)(x - x_n)^p, \end{aligned} \tag{4}$$

$$\theta^{(0)}, \theta_j^{(1)} \in (0,1), \qquad j = 2, \ldots, m,$$

would coincide with the greatest possible accuracy. It is easy to show [3], that they should be solutions of the system of equations

$$\sum_{i=1}^{m} \alpha_i = 1, \qquad \sum_{i=2}^{m} \alpha_i\beta_i^s = \frac{1}{s+1}, \quad s = 1, 2, \ldots, p-1, \tag{5}$$

$$\begin{cases} Y_n = x_n - \left(\frac{1}{4}f'\left(x_n\right) + \frac{3}{4}f'\left(x_n + \frac{2}{3}\left(X_n - x_n\right)\right)\right)^{-1} f\left(x_n\right), \\ X_{(n+1)} = X_{(n)} \cap Y_{(n)}, \qquad n = 0, 1, \ldots, \qquad x_n = \mathrm{mid}(X_n), \end{cases} \tag{6}$$

which, at carrying out corresponding enough common conditions [2] converges to solution of system (1) and has the order of convergence not less than 3.

If $m = 3$, then, according to the described above techniques for solving of system (1) we receive method [3]

$$\begin{cases} Y_n = x_n - (\alpha_1 f'(x_n) + \alpha_2 f'(x_n + \beta_2(X_n - x_n)) + \\ \qquad \alpha_3 f'(x_n + \beta_3(X_n - x_n)))^{-1} f(x_n), \\ X_{(n+1)} = X_{(n)} \cap Y_{(n)}, \quad n = 0, 1, \ldots, \end{cases} \tag{7}$$

where $x_n = \mathrm{mid}(X_n)$,

$$\alpha_1 = \frac{1}{9}; \quad \alpha_2 = \frac{16 - \sqrt{6}}{36}; \quad \beta_2 = \frac{6 + \sqrt{6}}{10}; \quad \alpha_3 = \frac{16 + \sqrt{6}}{36}; \quad \beta_3 = \frac{6 - \sqrt{6}}{10}.$$

Method (7) has some effective features, which we shall present in the next theorems.

Theorem 1. *Let mapping $f\colon R^k \to R^k$ is twice continuously differentiable and $x^* \in X_0$ where x^* is the solution of system* (1). *Then*

a) *every interval X_n, $n = 0, 1, 2, \ldots$, calculated by* (7), *contains the solution of system* (1)*;*

b) *if all matrices $F'\left(X_{(n)}\right)$, $n = 0, 1, 2, \ldots$, are not singular, then $\lim\limits_{n\to\infty} X_n = x^*$.*

Theorem 2. *Let mapping $f\colon R^k \to R^k$ is five times continuously differentiable and $x^* \in X_0$ where x^* is the solution of system* (1) *and matrices $F'\left(X_{(n)}\right)^{-1}$, $n = 0, 1, \ldots$, exist, then sequence of intervals $\left\{X_{(n)}\right\}_{n=0}^{\infty}$ calculated by* (7) *converges to x^*, moreover $\omega(X_{n+1}) \leq c \cdot \left(\omega(X_n)\right)^5$, where c is a positive constant.*

Here $F'(X) = \alpha_1 f'(x) + \alpha_2 f'(x + \beta_2(X - x)) + \alpha_3 f'(x + \beta_3(X - x))$, $x = \mathrm{mid}(X)$.

To prove those both theorems, we preliminary have proved the next lemma, which also has the independent importance.

Lemma. *Let mapping $f\colon R^k \to R^k$ is twice continuously differentiable and x^* is the real solution of system* (1), *$x^* \in X_0$ and $x_0 < x^*$. Then, if*

$$\Upsilon_0 \supseteq \left[x_0, x_0 + \frac{1}{k}\left(x^* - x_0\right)\right]$$

then

$$f''\big(x_0 + \theta_2^{(0)}(x^* - x_0)\big)(x^* - x_0)^2 \subset f''\big(x_0 + k[\theta^{(1)}](\Upsilon_0 - x_0)\big)(\Upsilon_0 - x_0)^2,$$

where $[\theta^{(1)}] \in [0, 1]$, $\theta_2^{(0)} \in (0, 1)$, $k > 0$ is a constant.

Method (7) has advantage over method (6) not only in the order of convergence, but also in that fact, that we must not analyze intermediate intervals if they contain solution x^* of system (1) and it is no necessity to expand them, if they do not contain x^*.

In [3], we have presented methods received by the described above technique for solving system (1) with the order of convergence not less than 7 and 9 correspondingly.

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