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DIFFERENTIAL INEQUALITIES

PREFACE

The simplest theorem on differential inequalities is the classical one on monotone functions, which reads as follows: *for a differentiable function* $\varphi(t)$ *satisfying the inequality* $\varphi'(t) \leqslant 0$ *in an interval* Δ *we have the inequality* $\varphi(t_1) \geqslant \varphi(t_2)$ *for any two points* t_1, t_2 *from the interval* Δ *such that* $t_1 < t_2$.

This theorem may be stated in a somewhat more sophisticated form. In order to do this, let us introduce the following definition. Consider a differential equation

$$y' = f(t, y) \tag{1}$$

with the right-hand side continuous in an open region D and suppose that for every point $(t_0, y_0) \in D$ the solution of (1) passing through this point is unique. Let us denote this solution by $\omega(t; t_0, y_0)$ and its maximal existence interval by $\Delta(t_0, y_0)$. Now, let $\varphi(t)$ be a continuous function on an interval Δ and assume its graph to lie in D. Under all these assumptions we say that the function $\varphi(t)$ is *decreasing with respect to equation* (1) if the following holds true: for every $t_0 \in \Delta$ the inequality $\varphi(t_0) \leqslant y_0$ implies the inequality $\varphi(t) \leqslant \omega(t; t_0, y_0)$ for all t's such that $t \geqslant t_0$ and $t \in \Delta \cap \cap \Delta(t_0, y_0)$.

Now, since for the particular equation

$$y' = 0 \tag{2}$$

we have $\omega(t; t_0, y_0) \equiv y_0$, the theorem on monotone functions may be restated as follows: *a differentiable function* $\varphi(t)$ *satisfying the inequality* $\varphi'(t) \leqslant 0$ *in an interval* Δ *is decreasing with respect to equation* (2).

The above statement is a particular case of the following general theorem: *under the preceding assumptions on equation* (1), *a differentiable function* $\varphi(t)$ *satisfying the inequality*

$$\varphi'(t) \leqslant f\big(t, \varphi(t)\big) \tag{3}$$

in an interval Δ *is decreasing with respect to equation* (1).

We state now, without insisting on a precise formulation, the problem covered by the above theorem: *an estimate for the initial value of a function* $\varphi(t)$ *and an estimate for its derivative being given, to find an adequate estimate for the function itself.* All theorems and their applications, presented in this book, concern problems of this type for functions of one or several variables. In case of several variables we will have, in general, to require that besides the initial estimates some boundary estimates be given in advance.

Differential inequalities treated in this book are the so-called *non-stationary* inequalities.

Chapters I-VIII of the book deal with the theory of ordinary differential inequalities and with its applications to ordinary differential equations and to first order and second order partial differential equations of parabolic and hyperbolic type. The theory of ordinary differential inequalities was originated by Chaplygin [6] and by Kamke [13] and then developed by Ważewski [60]. The main applications of the theory concern questions such as: estimates of solutions of differential equations, estimates of the existence domain of solutions, estimates of the difference between two solutions, criteria of the uniqueness of the solution, estimates of the error for an approximate solution, stability and Chaplygin's method.

Chapters IX-X concern partial differential inequalities of first and second order. First order partial differential inequalities were first treated by Haar [11] and by Nagumo [34]. Partial differential inequalities of second order, dealt with in this book, are of parabolic and hyperbolic type. First results on second order partial differential inequalities of parabolic type were obtained by Nagumo [35] and by Westphal [66].

Chapter XI deals with differential inequalities in linear spaces. This chapter as well as §§ 31, 32 in Chapter V and §§ 66, 67 in Chapter X are written by Włodzimierz Mlak.

We close these introductory remarks by the following one. From theorems that will be proved here on ordinary and partial differential inequalities, criteria of continuous dependence on initial values for solutions of corresponding equations can be derived. Now, since solutions of elliptic equations do not depend continuously on initial data, it is clear that theorems of the type described above cannot be expected to apply to partial differential equations or inequalities of elliptic type, i.e. to stationary equations or inequalities.

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CHAPTER I

MONOTONE FUNCTIONS

§ 1. Zygmund's lemma. We adopt the following terminology. A real function $\varphi(t)$ defined in an interval Δ is called *increasing* if for any two points t_1, t_2 from Δ such that

$$t_1 < t_2 \tag{1.1}$$

we have

$$\varphi(t_1) \leqslant \varphi(t_2) .$$

If for any two points of Δ inequality (1.1) implies

$$\varphi(t_1) < \varphi(t_2) ,$$

then $\varphi(t)$ is called *strictly increasing*. In a similar way we define a *decreasing* and a *strictly decreasing* function.

For a function $\varphi(t)$, defined in some neighborhood of the point t_0, we denote by $D^+\varphi(t_0)$, $D_+\varphi(t_0)$, $D^-\varphi(t_0)$, $D_-\varphi(t_0)$, respectively, its right-hand upper, right-hand lower, left-hand upper and left-hand lower *Dini's derivatives* at the point t_0, i.e.

$$D^+\varphi(t_0) = \limsup_{h\to 0+} \frac{\varphi(t_0+h)-\varphi(t_0)}{h} ,$$

$$D_+\varphi(t_0) = \liminf_{h\to 0+} \frac{\varphi(t_0+h)-\varphi(t_0)}{h} ,$$

$$D^-\varphi(t_0) = \limsup_{h\to 0-} \frac{\varphi(t_0+h)-\varphi(t_0)}{h} ,$$

$$D_-\varphi(t_0) = \liminf_{h\to 0-} \frac{\varphi(t_0+h)-\varphi(t_0)}{h} ,$$

(the values $+\infty$ and $-\infty$ being not excluded). Symbols $\varphi'_+(t_0)$ and $\varphi'_-(t_0)$ will stand for the right-hand and left-hand derivative respectively.

The inequality $a > 0$ will mean that either a is finite and positive or $a = +\infty$. The meaning of the inequalities $a \geqslant 0$, $a < 0$, $a \leqslant 0$ is defined in a similar way.

To begin with we will prove the following lemma.

ZYGMUND'S LEMMA. *Let $\varphi(t)$ be continuous in an interval Δ and write*

$$Z_+ = \{t \in \Delta : D_+\varphi(t) < 0\}.$$

Suppose that the set $\varphi(\Delta - Z_+)$ (1) does not contain any interval.

Under these assumptions $\varphi(t)$ is decreasing on Δ.

Proof. Suppose the contrary; then there would exist two points $t_1, t_2 \in \Delta$ satisfying (1.1) and such that $\varphi(t_1) < \varphi(t_2)$. Since, by our assumption, the set $\varphi(\Delta - Z_+)$ does not contain the interval $(\varphi(t_1), \varphi(t_2))$, there is a point $y_0 \in (\varphi(t_1), \varphi(t_2))$ such that

$$y_0 \notin \varphi(\Delta - Z_+). \tag{1.2}$$

By Darboux's property, the set

$$E = \{t \in (t_1, t_2) : \varphi(t) = y_0\}$$

is not empty. Let us denote by t_0 its least upper bound. Then we have $t_0 \in (t_1, t_2)$ and, by continuity,

$$\varphi(t_0) = y_0 \tag{1.3}$$

and

$$\varphi(t) > y_0 \quad \text{for} \quad t_0 < t < t_2. \tag{1.4}$$

Relations (1.2) and (1.3) imply that $t_0 \in Z_+$ and hence, by the definition of Z_+,

$$D_+\varphi(t_0) < 0. \tag{1.5}$$

On the other hand, by (1.3) and (1.4), it follows that

$$D_+\varphi(t_0) \geqslant 0,$$

which is a contradiction with (1.5). This completes the proof.

Remark 1.1. Since (1.3) and (1.4) imply $D^+\varphi(t_0) \geqslant 0$, it is obvious that the set Z_+ in Zygmund's lemma can be replaced by the set

$$Z^+ = \{t \in \Delta : D^+\varphi(t) < 0\}.$$

Remark 1.2. The set Z_+ can be replaced by the set

$$Z_- = \{t \in \Delta : D_-\varphi(t) < 0\}$$

or by the corresponding set Z^-. To prove Zygmund's lemma with Z_+ replaced by Z_- or Z^-, we have only to change the above argument by taking for t_0 the greatest lower bound of E.

Remark 1.3. A similar lemma holds true for increasing functions.

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(1) A being a subset of Δ, $\varphi(A)$ denotes the image of A by means of the mapping $y = \varphi(t)$.

§ 2. A necessary and sufficient condition for a continuous function to be monotone. As a consequence of Zygmund's lemma we get the following theorem.

THEOREM 2.1. *Let $\varphi(t)$ be continuous in an interval Δ. Then a necessary and sufficient condition for $\varphi(t)$ to be decreasing on Δ is that the set $\Delta - Q_+$, where*

$$Q_+ = \{t \in \Delta : D_+\varphi(t) \leqslant 0\},$$

be at most countable.

Proof. The necessity is obvious since for a decreasing function the set $\Delta - Q_+$ is empty. To prove the sufficiency of the condition, let $\varepsilon > 0$ be arbitrary and put

$$\psi(t) = \varphi(t) - \varepsilon t.$$

We have

$$D_+\psi(t) = D_+\varphi(t) - \varepsilon,$$

and, consequently,

$$D_+\psi(t) < 0 \quad \text{for} \quad t \in Q_+.$$

Hence it follows that for the set

$$Z_+ = \{t \in \Delta : D_+\psi(t) < 0\}$$

we have $Q_+ \subset Z_+$ and consequently $\Delta - Z_+ \subset \Delta - Q_+$. Therefore, the set $\Delta - Q_+$ being at most countable, the same holds true for the sets $\Delta - Z_+$ and $\psi(\Delta - Z_+)$. Hence the set $\psi(\Delta - Z_+)$ does not contain any interval and, by Zygmund's lemma, $\psi(t)$ is decreasing. Now, $\varepsilon > 0$ being arbitrary, it follows that $\varphi(t)$ is decreasing too.

COROLLARY 2.1. *Let $\varphi(t)$ be continuous in an interval Δ. Then a sufficient condition for $\varphi(t)$ to be strictly decreasing on Δ is that the set $\Delta - P_+$, where*

$$P_+ = \{t \in \Delta : D_+\varphi(t) < 0\},$$

be at most countable.

Proof. Let $\Delta - P_+$ be at most countable. By Theorem 2.1, $\varphi(t)$ is decreasing on Δ. If it were not strictly decreasing, we would have $\varphi(t_1) = \varphi(t_2)$ for some two points t_1, t_2 such that $t_1 < t_2$. Therefore, $\varphi(t)$ would be constant on the interval $[t_1, t_2]$ and consequently $\varphi'(t) \equiv 0$ on $[t_1, t_2]$, contrary to our assumption that $\Delta - P_+$ is at most countable.

Remark 2.1. Due to Remark 1.2, the set Q_+ in Theorem 2.1 can be replaced by the set

$$Q_- = \{t \in \Delta : D_-\varphi(t) \leqslant 0\}.$$

Remark 2.2. The results of this section can be summarized in a slightly less general form as follows: *if $\varphi(t)$ is continuous in an interval Δ and if*

$D_+\varphi(t) \leqslant 0$ for every $t \in \Delta$ or $D_-\varphi(t) \leqslant 0$ for every $t \in \Delta$, then $\varphi(t)$ is decreasing in Δ. Now, if we assume that for every $t \in \Delta$ we have either $D_+\varphi(t) \leqslant 0$ or $D_-\varphi(t) \leqslant 0$, then $\varphi(t)$ is not necessarily decreasing. Indeed, for Weierstrass's functions $\varphi(t)$ (a continuous function without finite derivative at any point) we have for every t either $D_+\varphi(t) = -\infty$ or $D_-\varphi(t) = -\infty$, and the function is not monotone.

Similar results for increasing functions follow from those concerning decreasing functions by considering $-\varphi(t)$ instead of $\varphi(t)$.

We close this paragraph by an important theorem due to Dini.

THEOREM 2.2. *For $\varphi(t)$ continuous in an interval Δ the following two propositions are true:*

1° *If any of its Dini's derivatives is $\leqslant a$ ($< a$) for $t \in Z \subset \Delta$, where $\Delta - Z$ is at most countable, then for any two different points t, s from Δ we have*

$$\frac{\varphi(t)-\varphi(s)}{t-s} \leqslant a \quad (< a)\,. \tag{2.1}$$

2° *If any of its Dini's derivatives is $\geqslant \beta$ ($> \beta$) for $t \in Z \subset \Delta$, where $\Delta - Z$ is at most countable, then for any two different points t, s of Δ we have*

$$\frac{\varphi(t)-\varphi(s)}{t-s} \geqslant \beta \quad (> \beta)\,.$$

Proof. Since 2° follows from 1° by taking $-\varphi(t)$ in place of $\varphi(t)$, we prove proposition 1°. Suppose then, for instance, that

$$D_+\varphi(t) \leqslant a \quad (< a) \quad \text{in} \quad Z \subset \Delta\,. \tag{2.2}$$

Fix s in Δ and put

$$\psi(t) = \varphi(t) - \varphi(s) - at \quad \text{for} \quad t \in \Delta\,.$$

$\psi(t)$ is then continuous in Δ and, by (2.2),

$$D_+\psi(t) = D_+\varphi(t) - a \leqslant 0 \quad (< 0) \quad \text{in} \quad Z\,.$$

Since $\Delta - Z$ is at most countable, it follows, by Theorem 2.1 (Corollary 2.1), that $\psi(t)$ is decreasing (strictly decreasing) in Δ and consequently

$$\psi(t) \leqslant \psi(s) \quad (\psi(t) < \psi(s)) \quad \text{for} \quad t > s\,.$$

Hence we get (2.1) for $t > s$. Since s and $t > s$ were arbitrary points in the interval Δ, we conclude that (2.1) holds true for any two different points t, s of Δ.

Next theorem is an immediate consequence of the preceding one.

THEOREM 2.3. *Let $\varphi(t)$ be continuous in an open interval Δ. Assume that one of its Dini's derivatives is finite and continuous at $t_0 \in \Delta$. Then $\varphi'(t_0)$ exists.*

Proof. Suppose, for instance, that $D_+\varphi(t)$ is finite and continuous at t_0. Put $D_+\varphi(t_0) = l$ and take an arbitrary $\varepsilon > 0$. Then there is a $\delta > 0$ so that

$$l-\varepsilon < D_+\varphi(t) < l+\varepsilon \quad \text{for} \quad t \in (t_0-\delta, t_0+\delta) .$$

Hence, by Theorem 2.2, we get

$$(2.3) \qquad l-\varepsilon < \frac{\varphi(t)-\varphi(t_0)}{t-t_0} < l+\varepsilon \quad \text{for} \quad t \in (t_0-\delta, t_0+\delta) , t \neq t_0 .$$

$\varepsilon > 0$ being arbitrary, inequality (2.3) implies the conclusion of our theorem.

COROLLARY 2.2. *For $\varphi(t)$ continuous in an open interval Δ assume that one of its Dini's derivatives is finite and continuous on Δ. Then $\varphi'(t)$ exists and is continuous on Δ.*

§ 3. A sufficient condition for a function to be monotone. As a further consequence of Zygmund's lemma we prove the following theorem.

THEOREM 3.1. *Let $\varphi(t)$ be absolutely continuous in an interval Δ and assume that*

$$(3.1) \qquad \varphi'(t) \leqslant 0 \textit{ for almost every } t \in \Delta .$$

Then $\varphi(t)$ is decreasing in Δ.

Proof. Let $\varepsilon > 0$ be arbitrary and put

$$\psi(t) = \varphi(t) - \varepsilon t .$$

$\psi(t)$ is absolutely continuous in Δ and

$$\psi'(t) = \varphi'(t) - \varepsilon \quad \text{for almost every} \quad t \in \Delta .$$

Therefore, by (3.1), we have $\psi'(t) < 0$ for almost every $t \in \Delta$ and hence the set $\Delta - Z_+$, where

$$Z_+ = \{t \in \Delta : D_+\psi(t) < 0\} ,$$

is of measure 0. $\psi(t)$ being absolutely continuous the set $\psi(\Delta - Z_+)$ is of measure 0 too, and consequently does not contain any interval. Hence, by Zygmund's lemma, $\psi(t)$ is decreasing in Δ and $\varepsilon > 0$ being arbitrary the same holds true for $\varphi(t)$.

Remark 3.1. A similar theorem is true for increasing functions.

Remark 3.2. By an argument similar to that used in the proof of Theorem 3.1 we show the following result: *If $\varphi(t)$ is a generalized absolutely continuous function* (see [45]) *in an interval Δ and if its approximative derivative* (see [45]) *is non-positive almost everywhere in Δ, then $\varphi(t)$ is decreasing in Δ.*

CHAPTER II

MAXIMUM AND MINIMUM SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

§ 4. Some notations and definitions. Let $Y=(y_1,\ldots,y_n)$, $\widetilde{Y}=(\widetilde{y}_1,\ldots,\widetilde{y}_n)$ be two points of the n-dimensional space. We will write

$$Y\leqslant\widetilde{Y}\quad\text{if}\quad y_j\leqslant\widetilde{y}_j\qquad(j=1,2,\ldots,n)$$

and

$$Y<\widetilde{Y}\quad\text{if}\quad y_j<\widetilde{y}_j\qquad(j=1,2,\ldots,n)\,.$$

The index i being fixed we write

$$Y\overset{i}{\leqslant}\widetilde{Y}\quad\text{if}\quad y_j\leqslant\widetilde{y}_j\quad(j=1,2,\ldots,n)\qquad\text{and}\qquad y_i=\widetilde{y}_i\,.$$

Let a system of functions $f_j(X,Y)=f_j(x_1,\ldots,x_p,y_1,\ldots,y_n)$ $(j=1,2,\ldots,n)$ be defined in a region D.

CONDITION V_+ (V_-). System $f_j(X,Y)$ $(j=1,2,\ldots,n)$ is said to satisfy condition V_+ (V_-) with regard to Y in D if for every fixed index i the function $f_i(X,Y)$ is increasing (decreasing) with respect to each variable $y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_n$ separately.

CONDITION W_+ (W_-). System $f_j(X,Y)$ $(j=1,2,\ldots,n)$ is said to satisfy condition W_+ (W_-) with respect to Y in D if for every fixed index i the following implication holds true:

$$Y\overset{i}{\leqslant}\widetilde{Y}\,,\quad(X,Y)\in D\,,\quad(X,\widetilde{Y})\in D\Rightarrow f_i(X,Y)\leqslant f_i(X,\widetilde{Y})$$

$$(Y\overset{i}{\leqslant}\widetilde{Y}\,,\quad(X,Y)\in D\,,\quad(X,\widetilde{Y})\in D\Rightarrow f_i(X,Y)\geqslant f_i(X,\widetilde{Y}))\,.$$

CONDITION $\overline{\mathrm{W}}_+$ ($\overline{\mathrm{W}}_-$). System $f_j(X,Y)$ $(j=1,2,\ldots,n)$ is said to satisfy condition $\overline{\mathrm{W}}_+$ ($\overline{\mathrm{W}}_-$) with respect to Y in D if the following implication holds true:

$$Y\leqslant\widetilde{Y}\,,\quad(X,Y)\in D\,,\quad(X,\widetilde{Y})\in D\Rightarrow f_j(X,Y)\leqslant f_j(X,\widetilde{Y})\qquad(j=1,2,\ldots,n)$$

$$(Y\leqslant\widetilde{Y}\,,\quad(X,Y)\in D\,,\quad(X,\widetilde{Y})\in D\Rightarrow f_j(X,Y)\geqslant f_j(X,\widetilde{Y})\qquad(j=1,2,\ldots,n))\,.$$

It is obvious that condition W_+ (W_-) implies condition V_+ (V_-) and that for $n = 1$ all four conditions are trivially satisfied. It is also clear that for $n = 2$ condition W_+ (W_-) and condition V_+ (V_-) are equivalent. This equivalence is—in general—no more valid for $n > 2$, as may be shown by a suitable counter-example. However, the above equivalence holds true in special regions without any restriction on the dimension. For instance, it is easy to check the equivalence of the conditions W_+ (W_-) and V_+ (V_-) in the case when the projection of the region D on the space $(y_1, \ldots, y_n)$ is a parallelepipede

$$-\infty \leqslant a_j < y_j < b_j \leqslant +\infty \qquad (j = 1, 2, \ldots, n).$$

For $Y = (y_1, \ldots, y_n)$ we write

$$-Y = (-y_1, \ldots, -y_n), \qquad |Y| = (|y_1|, \ldots, |y_n|).$$

For $\Phi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$ we write

$$D_-\Phi(t) = (D_-\varphi_1(t), \ldots, D_-\varphi_n(t))$$

and similarly for D^-, D_+ and D^+.

§ 5. Definition of the maximum (minimum) solution. Let a system of ordinary differential equations

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, \ldots, y_n) \qquad (i = 1, 2, \ldots, n) \tag{5.1}$$

be defined in a region D and let $(t_0, Y_0) \in D$.

A solution $\Omega(t) = (\omega_1(t), \ldots, \omega_n(t))$ of system (5.1), passing through the point (t_0, Y_0) and defined in some interval $\Delta^+ = [t_0, a)$ (¹), is called *right-hand maximum* (*minimum*) *solution* of system (5.1) in the interval Δ^+, passing through the point (t_0, Y_0), if for every solution $Y(t) = (y_1(t), \ldots, y_n(t))$ of (5.1), passing through (t_0, Y_0) and defined in an interval $\tilde{\Delta}^+ = [t_0, \tilde{a})$ (¹), we have

$$Y(t) \leqslant \Omega(t) \qquad (Y(t) \geqslant \Omega(t)) \qquad \text{for} \qquad t \in \Delta^+ \cap \tilde{\Delta}^+.$$

We define in a similar way the *left-hand maximum* (*minimum*) *solution* passing through (t_0, Y_0). It is clear that the maximum (minimum) solution in some interval, passing through a given point, is uniquely determined (whenever it exists) in that interval. It is also evident that if the solution of system (5.1), passing through (t_0, Y_0) to right (left) is unique in some interval, then it is both right-hand (left-hand) maximum and minimum solution in this interval.

(¹) In Δ^+ resp. $\tilde{\Delta}^+$ stands a resp. $\tilde{a}$ for a finite number or $+\infty$.

Now, the following two propositions are easy to check.

PROPOSITION 5.1. *By the mapping*

$$\tau = -t\,, \quad \eta_j = y_j \quad (j = 1, 2, \ldots, n) \tag{5.2}$$

the right-hand maximum (minimum) solution of system (5.1), *passing through* (t_0, Y_0), *is transformed into the left-hand maximum (minimum) solution of system*

$$\frac{d\eta_i}{d\tau} = -\sigma_i(-\tau, \eta_1, \ldots, \eta_n) \quad (i = 1, 2, \ldots, n)\,, \tag{5.3}$$

passing through $(-t_0, Y_0)$.

PROPOSITION 5.2. *By the mapping*

$$\tau = t\,, \quad \eta_j = -y_j \quad (j = 1, 2, \ldots, n) \tag{5.4}$$

the right-hand maximum (minimum) solution of system (5.1), *passing through* (t_0, Y_0), *is transformed into the right-hand minimum (maximum) solution of system*

$$\frac{d\eta_i}{d\tau} = -\sigma_i(\tau, -\eta_1, \ldots, -\eta_n) \quad (i = 1, 2, \ldots, n)\,, \tag{5.5}$$

passing through $(t_0, -Y_0)$.

A similar proposition holds true for the left-hand maximum (minimum) solution. Sufficient conditions for the existence of the right-hand (left-hand) maximum and minimum solution will be given in further paragraphs.

§ 6. Basic lemmas on strong ordinary differential inequalities. We prove

LEMMA 6.1. *Let the right-hand sides of system* (5.1) *be defined in some open region* D *and satisfy in* D *condition* W_+ *with respect to* Y (see § 4). *Let* $(t_0, Y_0) \in D$. *Assume that* $\Phi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$ *is continuous in* $\tilde{\Delta}_+ = [t_0, \tilde{a})$ *and that the curve* $Y = \Phi(t)$ *lies in* D. *Let* $Y(t) = (y_1(t), \ldots, y_n(t))$ *be an arbitrary solution of system* (5.1), *passing through* (t_0, Y_0) *and defined in some interval* $\Delta_+ = [t_0, a)$.

Under these assumptions, if

$$\Phi(t_0) < Y_0 \tag{6.1}$$

and

$$D_-\varphi_i(t) < \sigma_i(t, \varphi_1(t), \ldots, \varphi_n(t)) \quad (i = 1, 2, \ldots, n) \tag{6.2}$$

for $t \in (t_0, \tilde{a})$, *then we have the inequality*

$$\Phi(t) < Y(t) \quad \textit{for} \quad t \in \Delta_+ \cap \tilde{\Delta}_+\,.$$

Proof. Since $Y(t_0) = Y_0$, by (6.1) and by the continuity, the set

$$E = \{\tilde{t} : t_0 < \tilde{t} < \min(a, \tilde{a}),\, \Phi(t) < Y(t) \text{ for } t_0 \leqslant t < \tilde{t}\}$$

is non-void. Denote by t^* its least upper bound [1]. We have to prove that $t^* = \min(a, \tilde{a})$. Suppose that $t^* < \min(a, \tilde{a})$. Then, by the definition of t^*, we have

$$\Phi(t) < Y(t) \quad \text{for} \quad t_0 \leqslant t < t^* \tag{6.3}$$

and, by the continuity, for at least one index j

$$\Phi(t^*) \overset{j}{\leqslant} Y(t^*) \tag{6.4}$$

(see § 4). From (6.3) and (6.4) we get, in particular,

$$\varphi_j(t) < y_j(t) \quad \text{for} \quad t_0 \leqslant t < t^*\,, \ \varphi_j(t^*) = y_j(t^*)\,.$$

Hence

$$D_-\varphi_j(t^*) \geqslant y_j'(t^*)\,. \tag{6.5}$$

On the other hand, from (6.2) and (6.4) we deduce, due to the condition W_+ (see § 4),

$$D_-\varphi_j(t^*) < \sigma_j\big(t^*, \Phi(t^*)\big) \leqslant \sigma_j\big(t^*, Y(t^*)\big)\,.$$

Since

$$y_j'(t^*) = \sigma_j\big(t^*, Y(t^*)\big)\,,$$

it follows that

$$D_-\varphi_j(t^*) < y_j'(t^*)\,,$$

which gives a contradiction with (6.5). Therefore, we have $t^* = \min(a, \tilde{a})$ and this completes the proof of our lemma.

Remark 6.1. It is possible to construct a counter-example showing that—in general—Lemma 6.1 is not true if the left-hand derivative in (6.2) is replaced by the right-hand one.

Next we state two easy to check propositions.

PROPOSITION 6.1. *If the right-hand sides of system* (5.1) *satisfy condition* W_+ (see § 4) *with respect to* Y, *then the right-hand sides of the transformed system* (5.3) (see Proposition 5.1) *satisfy condition* W_- (see § 4) *with regard to* Y.

By mapping (5.2) (*denoting* $\psi_i(\tau) = \varphi_i(-\tau)$) *the system of differential inequalities* (6.2) *is transformed into the system*

$$D^+\psi_i(\tau) > -\sigma_i\big(-\tau, \psi_1(\tau), \ldots, \psi_n(\tau)\big) \quad (i = 1, 2, \ldots, n)\,.$$

PROPOSITION 6.2. *If the right-hand sides of system* (5.1) *satisfy condition* W_+ (see § 4) *with respect to* Y, *then the right-hand sides of the transformed system* (5.5) (see Proposition 5.2) *satisfy the same condition.*

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[1] By the least upper bound of a set which is unbounded from above we mean $+\infty$.

By mapping (5.4) (*putting* $\psi_i(\tau) = -\varphi_i(\tau)$) *the system of differential inequalities* (6.2) *is transformed into the system*

$$D^-\psi_i(\tau) > -\sigma_i\big(\tau, -\psi_1(\tau), \ldots, -\psi_n(\tau)\big) \qquad (i = 1, 2, \ldots, n).$$

Applying mapping (5.4) we get from Lemma 6.1, by Proposition 6.2, the following lemma:

LEMMA 6.2. *Under the assumptions of Lemma* 6.1, *if*

$$\Phi(t_0) > Y_0$$

and

$$D^-\varphi_i(t) > \sigma_i\big(t, \varphi_1(t), \ldots, \varphi_n(t)\big) \qquad (i = 1, 2, \ldots, n)$$

for $t \in (t_0, \tilde{\alpha})$, *then we have the inequality*

$$\Phi(t) > Y(t) \quad \textit{for} \quad t \in \Delta_+ \cap \tilde{\Delta}_+ .$$

Similarly, applying mapping (5.2) and using Proposition 6.1 we derive from Lemmas 6.1 and 6.2 the next lemma.

LEMMA 6.3. *Let the right-hand sides of system* (5.1) *be defined in some open region* D *and satisfy in* D *condition* W_-(see § 4) *with respect to* Y. *Let* $(t_0, Y_0) \in D$. *Assume that* $\Phi(t) = \big(\varphi_1(t), \ldots, \varphi_n(t)\big)$ *is continuous in* $\tilde{\Delta}_- = (\tilde{\beta}, t_0]$ (¹) *and that the curve* $Y = \Phi(t)$ *lies in* D. *Let* $Y(t) = \big(y_1(t), \ldots, y_n(t)\big)$ *be an arbitrary solution of system* (5.1), *passing through* (t_0, Y_0) *and defined in some interval* $\Delta_- = (\beta, t_0]$ (¹).

Under these assumptions, if

$$\Phi(t_0) < Y_0 \qquad (\Phi(t_0) > Y_0)$$

and

$$D^+\varphi_i(t) > \sigma_i\big(t, \varphi_1(t), \ldots, \varphi_n(t)\big) \qquad \Big(D_+\varphi_i(t) < \sigma_i\big(t, \varphi_1(t), \ldots, \varphi_n(t)\big)\Big) \qquad (i = 1, 2, \ldots, n)$$

for $t \in (\tilde{\beta}, t_0)$, *then we have the inequality*

$$\Phi(t) < Y(t) \qquad (\Phi(t) > Y(t))$$

for $t \in \Delta_- \cap \tilde{\Delta}_-$.

§ 7. Some notions and theorems on ordinary differential equations. Let the right-hand sides of system (5.1) be continuous in some open region D and let $\Phi(t) = \big(\varphi_1(t), \ldots, \varphi_n(t)\big)$ and $\Psi(t) = \big(\psi_1(t), \ldots, \psi_n(t)\big)$ be two solutions defined on $\Delta_+ = [t_0, \alpha)$ and $\tilde{\Delta}_+ = [t_0, \tilde{\alpha})$ respectively. Suppose that $\Delta_+ \subset \tilde{\Delta}_+$. The solution $\Psi(t)$ is called *right-hand continuation of the solution* $\Phi(t)$ if

$$\Psi(t) = \Phi(t) \quad \text{for} \quad t \in \Delta_+ .$$

(¹) In Δ_- resp. $\tilde{\Delta}_-$ is β resp. $\tilde{\beta}$ a finite number or $-\infty$.

In a similar way we define the *left-hand continuation of a solution.* A solution, which is both a right-hand and left-hand continuation of another one, is called simply *continuation.*

A solution $\Phi(t)$ defined in $\Delta_+ = [t_0, a)$ is said *to reach the boundary of the open region D by its right-hand extremity* if the corresponding solution-path $Y = \Phi(t)$ is not contained in any compact subset of D. In this case the interval $[t_0, a)$ is called the *right-hand maximal interval of existence of the solution $\Phi(t)$.*

It is obvious that for a solution $\Phi(t)$ reaching the boundary of D by its right-hand extremity there is no right-hand continuation different from $\Phi(t)$.

A solution reaching the boundary of D by its left-hand extremity and the left-hand maximal interval of existence are defined similarly.

Now the following theorem holds true (see [14], p. 135).

THEOREM 7.1. *Every solution of system* (5.1) *with continuous right-hand sides in an open region D admits at least one continuation reaching the boundary of D by its both extremities.*

The last theorem can be restated in a less precise way as follows: *Every solution can be continued to the boundary of D in both directions.*

Remark 7.1. The above continuation is, in general, not unique. In case of uniqueness, Theorem 7.1 is an almost immediate consequence of the next theorem (see [64]).

THEOREM 7.2. *Assume the right-hand sides of system* (5.1) *to be continuous in an open region D. Let $\Phi(t)$ be a solution defined in a bounded interval $\Delta_+ = [t_0, a)$ $(\Delta_- = (\beta, t_0])$ and suppose that for some sequence t_ν we have*

$$\lim_{\nu\to\infty} \big(t_\nu, \Phi(t_\nu)\big) = (a, Y_0) \qquad [(\beta, Y_0)]$$

and $(a, Y_0) \in D$ $[(\beta, Y_0) \in D]$. *Then the limit*

$$\lim_{t\to a} \Phi(t) = Y_0\ ^{(1)} \qquad (\lim_{t\to\beta} \Phi(t) = Y_0)$$

exists and

$$\Psi(t) = \begin{cases} \Phi(t) & \textit{for} \quad t \in [t_0, a) \ (t \in (\beta, t_0]), \\ Y_0 & \textit{for} \quad t = a \ (t = \beta) \end{cases}$$

is a solution of system (5.1) *in the closed interval* $[t_0, a]$ $([\beta, t_0])$.

Next, for the convenience of the reader, we prove a theorem giving a rough estimate of the interval of existence of a solution.

(1) $\lim_{t\to a} \Phi(t) = \big(\lim_{t\to a} \varphi_1(t), \ldots, \lim_{t\to a} \varphi_n(t)\big)$.

THEOREM 7.3. *Let the right-hand sides of system* (5.1) *be continuous in a cube*

$$Q: |t-t_0| < a\,, \quad |y_i - \mathring{y}_i| < a \qquad (i = 1, 2, \ldots, n)$$

and satisfy the inequalities

$$|\sigma_i(t, Y)| \leqslant M \qquad (i = 1, 2, \ldots, n)\,. \tag{7.1}$$

Suppose that

$$|\widetilde{y}_i - \mathring{y}_i| < \frac{a}{3} \qquad (i = 1, 2, \ldots, n) \tag{7.2}$$

and take an arbitrary solution $Y(t) = \big(y_1(t), \ldots, y_n(t)\big)$ *of system* (5.1), *reaching the boundary of* Q *by its both extremities and passing through the point* $(t_0, \widetilde{Y}) = (t_0, \widetilde{y}_1, \ldots, \widetilde{y}_n)$. *Denote its maximal interval of existence by* $\Delta = (\alpha, \beta)$ *and put*

$$\delta = (t_0 - h, t_0 + h)\,,$$

where

$$h = \min\left(a, \frac{a}{3M}\right). \tag{7.3}$$

Under these assumptions we have

$$\delta \subset \Delta\,. \tag{7.4}$$

Proof. Suppose that (7.4) is not true and, for instance,

$$t_0 < \beta < t_0 + h\,. \tag{7.5}$$

Choose b so that

$$\beta < b < t_0 + h\,. \tag{7.6}$$

The solution $Y(t)$ reaching the boundary of Q by its right-hand extremity the solution-path $Y = Y(t)$, $t \in [t_0, \beta)$, is not contained in the compact subset of Q

$$t_0 \leqslant t \leqslant b\,, \qquad |y_i - \mathring{y}_i| \leqslant \tfrac{2}{3}a \qquad (i = 1, 2, \ldots, n)\,.$$

Hence, since $\beta < b$, there is a $t^* \in (t_0, \beta)$ and an index j such that

$$|y_j(t^*) - \mathring{y}_j| > \tfrac{2}{3}a\,. \tag{7.7}$$

From (7.2) and (7.7) it follows that

$$|y_j(t^*) - \widetilde{y}_j| > \frac{a}{3}\,. \tag{7.8}$$

On the other hand, there is a $\tau \in (t_0, t^*)$ so that

$$|y_j(t^*) - \widetilde{y}_j| = |y_j(t^*) - y_j(t_0)| = |t^* - t_0|\,|y_j'(\tau)| = |t^* - t_0|\,\big|\sigma_j\big(\tau, Y(\tau)\big)\big|\,. \tag{7.9}$$

Since $t^* \in (t_0, \beta)$, we get from (7.3) and (7.5)

$$|t^* - t_0| < \frac{a}{3M}.$$

Hence, by (7.1) and (7.9), we have

$$|y_j(t^*) - \tilde{y}_j| \leqslant \frac{a}{3},$$

which contradicts (7.8). Thus the proof is completed.

§ 8. Local existence of the right-hand maximum solution. We first prove a theorem giving, among others, sufficient conditions for the local existence of the right-hand maximum solution.

THEOREM 8.1. *Suppose that the right-hand sides of system* (5.1) *are continuous and satisfy condition* W_+ *with respect to* Y (see § 4) *in an open region* D. *Let* $(t_0, Y_0) \in D$ *and take an arbitrary sequence of points* $(t_0, Y^\nu) \in D$ *such that*

$$(8.1) \qquad Y_0 < Y^{\nu+1} < Y^\nu, \quad \lim_{\nu\to\infty} Y^\nu = Y_0.$$

For every positive integer ν *consider the system of ordinary differential equations*

$$(8.2) \qquad \frac{dy_i}{dt} = \sigma_i(t, y_1, \ldots, y_n) + \frac{1}{\nu} \quad (i = 1, 2, \ldots, n)$$

and let $Y^\nu(t) = (y_1^\nu(t), \ldots, y_n^\nu(t))$ *be an arbitrary solution of* (8.2), *passing through* (t_0, Y^ν) *and reaching the boundary of* D *by its both extremities* (*such solution exists by Theorem* 7.1).

Under these assumptions, there is a positive number h *so that*

1° *For indices* ν *sufficiently large* $Y^\nu(t)$ *is defined in* $\Delta_h = [t_0, t_0 + h)$ *and*

$$Y^{\nu+1}(t) < Y^\nu(t) \quad \text{for} \quad t \in \Delta_h.$$

2° *The sequence* $Y^\nu(t)$ *is uniformly convergent in the interval* Δ_h *to the right-hand maximum solution* $\Omega(t) = (\omega_1(t), \ldots, \omega_n(t))$ *of system* (5.1) *in* Δ_h, *passing through* (t_0, Y_0), *and*

$$Y^\nu(t) > \Omega(t).$$

3° *If* $Y = \Phi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$ *is an arbitrary continuous curve for* $t \in \Delta_{\tilde{h}} = [t_0, t_0 + \tilde{h})$, *contained in* D *and satisfying the initial inequality*

$$(8.3) \qquad \Phi(t_0) \leqslant Y_0$$

and the differential inequalities

$$D_-\varphi_i(t) \leqslant \sigma_i\big(t, \varphi_1(t), \dots, \varphi_n(t)\big) \quad \textit{for} \quad t_0 < t < t_0 + \tilde{h} \quad (i = 1, 2, \dots, n), \tag{8.4}$$

then

$$\Phi(t) \leqslant \Omega(t) \quad \textit{for} \quad t \in \Delta_h \cap \Delta_{\tilde{h}}. \tag{8.5}$$

Proof. There is a positive number a, so that the closure of the cube

$$Q : |t - t_0| < a, \ |y_i - \mathring{y}_i| < a \quad (i = 1, 2, \dots, n),$$

where $Y_0 = (\mathring{y}_1, \dots, \mathring{y}_n)$, is contained in D. The functions $\sigma_i(t, Y)$ being continuous in $\bar{Q}$, we have for some M

$$\left|\sigma_i(t, Y) + \frac{1}{\nu}\right| \leqslant M \quad \text{for} \quad (t, Y) \in \bar{Q} \quad (i = 1, 2, \dots, n;\ \nu = 1, 2, \dots).$$

Put

$$h = \min\left(a, \frac{a}{3M}\right).$$

Since, by (8.1), there is

$$|y_i^\nu - \mathring{y}_i| < \frac{a}{3} \quad (i = 1, 2, \dots, n)$$

from a certain index ν_0 on, we see, by Theorem 7.3, that $Y^\nu(t)$ are defined in $\Delta_h = [t_0, t_0 + h)$ for $\nu > \nu_0$. In what follows, we consider only indices $\nu > \nu_0$. Since the right-hand sides of system (8.2) satisfy condition W_+ with respect to Y in D and because of the inequalities

$$Y^{\nu+1}(t_0) = Y^{\nu+1} < Y^\nu = Y^\nu(t_0),$$

$$\frac{dy_i^{\nu+1}}{dt} = \sigma_i\big(t, Y^{\nu+1}(t)\big) + \frac{1}{\nu+1} < \sigma_i\big(t, Y^{\nu+1}(t)\big) + \frac{1}{\nu} \quad (i = 1, 2, \dots, n),$$

we have, by Lemma 6.1,

$$Y^{\nu+1}(t) < Y^\nu(t) \quad \text{for} \quad t \in \Delta_h.$$

By a similar argument we prove that the sequence $Y^\nu(t)$ is bounded from below by any solution of system (5.1), passing through the point (t_0, Y_0). Hence and from the last inequalities it follows that there exists the limit

$$\lim_{\nu\to\infty} Y^\nu(t) = \Omega(t) \quad \text{for} \quad t \in \Delta_h \tag{8.6}$$

and, by a standard argument, we get that $\Omega(t)$ is a solution of system (5.1), passing through (t_0, Y_0) and that the convergence in (8.6) is uniform. By (8.3) and (8.4), we have

$$\Phi(t_0) < Y^\nu$$

and

$$D_-\varphi_i(t) < \sigma_i(t, \Phi(t)) + \frac{1}{\nu} \quad \text{for} \quad t_0 < t < t_0 + \tilde{h} \quad (i = 1, 2, \ldots, n).$$

Hence, by Lemma 6.1,

$$\Phi(t) < Y^{\nu}(t) \quad \text{for} \quad t \in \Delta_h \cap \Delta_{\tilde{h}}. \tag{8.7}$$

From (8.6) and (8.7) follows (8.5). In particular, (8.5) holds true for $\Phi(t)$ being an arbitrary solution of system (5.1), passing through (t_0, Y_0). Therefore, $\Omega(t)$ is the right-hand maximum solution through (t_0, Y_0) of system (5.1) in the interval Δ_h. Thus the proof of 1°, 2° and 3° is completed.

§ 9. Global existence of the maximum and minimum solution. Now we prove

THEOREM 9.1. *Assume the right-hand sides of system* (5.1) *to be continuous and to satisfy condition* W_+ *with respect to* Y (see § 4) *in an open region* D. *Then, through every point* $(t_0, Y_0) \in D$ *there exists the right-hand maximum and the right-hand minimum solution reaching the boundary of* D *by its right-hand extremity.*

Proof. We first prove the part of theorem concerning the right-hand maximum solution. By Theorem 8.1, for $(t_0, Y_0) \in D$ there is a positive h, so that the right-hand maximum solution through (t_0, Y_0) exists in the interval $\Delta_h = [t_0, t_0 + h)$. Denote by h_0 the least upper bound of such numbers h. Now notice that if we have the right-hand maximum solution in some interval $\Delta_{\tilde{h}}$, then its restriction to any interval Δ_h, where $h < \tilde{h}$, is the right-hand maximum solution in Δ_h. Hence it follows that for every positive $h < h_0$ there is the right-hand maximum solution in Δ_h, say $\Omega_h(t)$. Next, we conclude that if $0 < h_1 < h_2 < h_0$, then—by the uniqueness (see § 5)—the right-hand maximum solution in Δ_{h_2} is the right-hand continuation (see § 7) of the one defined in Δ_{h_1}. Now, for $t \in [t_0, t_0 + h)$ choose h so that $t < t_0 + h < t_0 + h_0$ and put

$$\Omega(t) = \Omega_h(t). \tag{9.1}$$

By our preceding remark, the value of $\Omega(t)$ is independent of the choice of h. Hence, formula (9.1) defines a function in the interval $\Delta_{h_0} = [t_0, t_0 + h_0)$. It is clear that $\Omega(t)$ is the right-hand maximum solution through (t_0, Y_0) in Δ_{h_0}. Next, we will prove that $\Omega(t)$ reaches the boundary of D by its right-hand extremity. Indeed, if it were not so, the corresponding solution-path $Y = \Omega(t)$ would be contained in some compact subset of D (see § 7). Therefore, there would exist a sequence t_ν $(t_0 < t_\nu < t_0 + h_0)$, so that

$$\lim_{\nu \to \infty} (t_\nu, \Omega(t_\nu)) = (t_0 + h_0, \tilde{Y}) \in D.$$

Hence, by Theorem 7.2, we would have

$$\lim_{t\to t_0+h_0} \big(t, \Omega(t)\big) = (t_0+h_0,\, \widetilde{Y})$$

and

$$\widetilde{\Omega}(t) = \begin{cases} \Omega(t) & \text{for} \quad t \in \Delta_{h_0}\,, \\ \widetilde{Y} & \text{for} \quad t = t_0+h_0 \end{cases}$$

would be a solution of (5.1) in the closed interval $[t_0, t_0+h_0]$. Since $(t_0+h_0, \widetilde{Y}) \in D$, we can apply Theorem 8.1 to the point $(t_0+h_0, \widetilde{Y})$ and hence we get that there is a positive $\widetilde{h}$, so that the right-hand maximum solution through $(t_0+h_0, \widetilde{Y})$ exists in the interval $[t_0+h_0, t_0+h_0+\widetilde{h})$. Denote it by $\widetilde{\widetilde{\Omega}}(t)$. Then $\Omega^*(t)$ defined by the formula

$$\Omega^*(t) = \begin{cases} \Omega(t) & \text{for} \quad t \in \Delta_{h_0}\,, \\ \widetilde{\widetilde{\Omega}}(t) & \text{for} \quad t \in [t_0+h_0, t_0+h_0+\widetilde{h}) \end{cases}$$

is clearly a solution of system (5.1), passing through (t_0, Y_0) and defined in the interval $\Delta_{h_0+\tilde{h}} = [t_0, t_0+h_0+\widetilde{h})$.

We will prove now that:

(α) $\Omega^*(t)$ is the right-hand maximum solution through (t_0, Y_0) in the interval $\Delta_{h_0+\tilde{h}}$.

To prove (α), we have to show that if $Y(t)$ is an arbitrary solution through (t_0, Y_0) defined in some interval $\Delta_h = [t_0, t_0+h)$, then

$$Y(t) \leqslant \Omega^*(t) \quad \text{for} \quad t \in \Delta_h \cap \Delta_{h_0+\tilde{h}}\,. \tag{9.2}$$

Inequality (9.2) is true if $h \leqslant h_0$ because $\Omega^*(t) = \Omega(t)$ in Δ_{h_0} and $\Omega(t)$ is the right-hand maximum solution through (t_0, Y_0) in Δ_{h_0}. If $h > h_0$, then, by the preceding argument, we have (9.2) in Δ_{h_0} and, by continuity, $Y(t_0+h_0) \leqslant \Omega^*(t_0+h_0) = \widetilde{\widetilde{\Omega}}(t_0+h_0)$. Hence, due to the definition of $\widetilde{\widetilde{\Omega}}(t)$ and by Theorem 8.1, 3°, it follows that

$$Y(t) \leqslant \widetilde{\widetilde{\Omega}}(t) = \Omega^*(t) \quad \text{for} \quad t \in [t_0+h_0, t_0+h) \cap [t_0+h_0, t_0+h_0+\widetilde{h})\,,$$

which completes the proof of (α). But, proposition (α) contradicts the definition of h_0 and consequently the first part of Theorem 9.1 is proved. Now applying the mapping (5.4) and using Proposition 5.2 and Proposition 6.2 we get the second part of our theorem, concerning the minimum solution, as an immediate consequence of the first part.

THEOREM 9.2. *Assume the right-hand sides of system* (5.1) *to be continuous and to satisfy condition* W_- *with respect to* Y (see § 4) *in an open region* D. *Then, through every point* $(t_0, Y_0) \in D$ *there is the left-hand maximum and the left-hand minimum solution reaching the boundary of* D *by its left-hand extremity.*

Proof. Theorem 9.2 follows from Theorem 9.1 by applying the mapping (5.2) and by Proposition 5.1 and 6.1.

Remark 9.1. In case $n = 1$, i.e. when system (5.1) reduces to a single equation, both conditions W_+ and W_- are trivially satisfied (see § 4). Hence we have the following result: *For a single first order differential equation with a right-hand side continuous in an open region D there is, through every point $(t_0, Y_0) \in D$, the right-hand (left-hand) maximum and minimum solution reaching the boundary of D by its right-hand (left-hand) extremity.*

Remark 9.2. In case $n = 2$ condition W_+ (W_-) in Theorem 9.1 (Theorem 9.2) can be substituted by the equivalent condition V_+ (V_-) (see § 4). However, in case $n > 2$ condition W_+ in Theorem 9.1 cannot be replaced by the essentially weaker condition V_+. Indeed, it is possible to construct a suitable counter-example (see [60]) showing that for a system of three equations, with right-hand sides continuous and satisfying condition V_+ in an open region D, it may happen that the right-hand maximum solution—which exists locally—cannot be continued so as to reach the boundary of D by its right-hand extremity.

The theorem we are going to prove next is a generalization of 3° in Theorem 8.1, which was of local character.

THEOREM 9.3. *Assume the right-hand sides of system* (5.1) *to be continuous and to satisfy condition* W_+ *with respect to* Y (see § 4) *in an open region D. Let $(t_0, Y_0) \in D$ and denote by $\Omega^+(t)$ the right-hand maximum solution through (t_0, Y_0), reaching the boundary of D by its right-hand extremity. Let $\Delta = [t_0, a_0)$ be its existence interval.*

Under these assumptions, if $Y = \Phi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$ is an arbitrary continuous curve for $t \in \tilde{\Delta} = [t_0, \tilde{a}_0)$, contained in D and satisfying the initial inequality

$$\Phi(t_0) \leqslant Y_0$$

and the differential inequalities

$$D_-\varphi_i(t) \leqslant \sigma_i(t, \varphi_1(t), \ldots, \varphi_n(t)) \quad \textit{for} \quad t_0 < t < \tilde{a}_0 \quad (i = 1, 2, \ldots, n),$$

then

$$\Phi(t) \leqslant \Omega^+(t) \quad \textit{for} \quad t \in \Delta \cap \tilde{\Delta}. \tag{9.3}$$

Proof. By 3° of Theorem 8.1, inequality (9.3) holds true in the interval $[t_0, a)$ for some $a > t_0$ and sufficiently close to t_0. Let a^* be the least upper bound of such numbers a. We have to show that $a^* = \min(a_0, \tilde{a}_0)$. Suppose that $a^* < \min(a_0, \tilde{a}_0)$; then $a^* \in \Delta \cap \tilde{\Delta}$ and since—by the definition of a^*—(9.3) holds in $[t_0, a^*)$, we have by continuity

$$\Phi(a^*) \leqslant \Omega^+(a^*).$$

Hence we can apply 3° of Theorem 8.1 to the point $(a^*, \Omega^+(a^*))$ and—noticing that $\Omega^+(t)$ is the right-hand maximum solution through $(a^*, \Omega^+(a^*))$

in the interval $[\alpha^*, a_0)$—we get that inequality (9.3) holds in some interval $[\alpha^*, \alpha^{**})$, where $\alpha^{**} < \min(a_0, \tilde{a}_0)$ is sufficiently close to α^*. Therefore, inequality (9.3) is satisfied in $[t_0, \alpha^{**})$, contrary to the definition of α^*, since $\alpha^{**} > \alpha^*$. This contradiction completes the proof.

Remark. For $n > 2$ condition W_+ in Theorem 9.3 cannot be substituted by the weaker condition V_+ (see § 4). Indeed, the subsequent counter-example (see [60]) shows that with the condition V_+ it may occur that inequality (9.3) does not hold in any right-hand neighborhood of t_0.

Let $D = D_1 \cup D_2 \subset (t, y_1, \dots, y_n)$, where

$$D_1 : -\infty < t < +\infty \,, \; y_1^2 + y_2^2 < 1 \,, \; -\infty < y_3 < +\infty \,,$$

$$D_2 : -\infty < t < +\infty \,, \; (y_1 - 3)^2 + (y_2 - 3)^2 < 1 \,, \; -\infty < y_3 < +\infty \,,$$

and put

$$\sigma_i(t, y_1, y_2, y_3) = \begin{cases} 1 & \text{in} \quad D_1 \,, \\ -1 & \text{in} \quad D_2 \end{cases} \qquad (i = 1, 2, 3) \,.$$

It is easy to check that the functions σ_i $(i = 1, 2, 3)$, thus defined, satisfy in D condition V_+. Now, for $\varphi_i(t) \equiv 0$ $(i = 1, 2, 3)$ we have

$$\varphi_i(0) < 3 \quad (i = 1, 2) \,, \qquad \varphi_3(0) \leqslant 0$$

and

$$\varphi_i'(t) \leqslant \sigma_i\big(t, \varphi_1(t), \varphi_2(t), \varphi_3(t)\big) \quad \text{for} \quad t \geqslant 0 \quad (i = 1, 2, 3) \,.$$

The unique solution of the system

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, y_2, y_3) \qquad (i = 1, 2, 3) \,,$$

passing through $(0, 3, 3, 0)$, and consequently its right-hand maximum solution through $(0, 3, 3, 0)$ is obviously

$$\omega_i^+(t) = 3 - t \quad (i = 1, 2) \,, \qquad \omega_3^+(t) = -t \text{ for } t \geqslant 0 \,.$$

However, we have

$$\varphi_3(t) > \omega_3^+(t) \quad \text{for} \quad t > 0 \,.$$

It is also possible to construct a similar example with D and its intersections by planes $t = \text{const}$ being connected.

By mapping (5.4) and by Propositions 5.2 and 6.2 we get from Theorem 9.3 the following one:

THEOREM 9.4. *Under the assumptions of Theorem 9.3 denote by $\Omega_+(t)$ the right-hand mimimum solution through (t_0, Y_0), reaching the boundary of D by its right-hand extremity. Let $\Delta = [t_0, a_0)$ be the existence interval of $\Omega_+(t)$. This being assumed, if $Y = \Phi(t) = \big(\varphi_1(t), \dots, \varphi_n(t)\big)$ is an arbitrary continuous curve for $t \in \tilde{\Delta} = [t_0, \tilde{a}_0)$, contained in D and satisfying the initial inequality*

$$\Phi(t_0) \geqslant Y_0 \,,$$

and the differential inequalities

$$D^-\varphi_i(t) \geqslant \sigma_i(t, \varphi_1(t), \ldots, \varphi_n(t)) \quad \textit{for} \quad t_0 < t < \tilde{a}_0 \quad (i = 1, 2, \ldots, n),$$

then

$$\Phi(t) \geqslant \Omega_+(t) \quad \textit{for} \quad t \in \Delta \cap \tilde{\Delta}.$$

Using the mapping (5.2) and Propositions 5.1 and 6.1 it is easy to derive from the above theorems similar theorems concerning the situation to the left from the initial point.

Since in the case of a single equation conditions W_+ and W_- are trivially satisfied, we get—as corollaries of the above theorems—the following two theorems.

THEOREM 9.5. *Assume the right-hand side of equation*

$$\frac{dy}{dt} = \sigma(t, y) \tag{9.4}$$

to be continuous in an open region D. Let $(t_0, y_0) \in D$ and denote by $\omega^+(t)$ ($\omega_+(t)$) the right-hand maximum (minimum) solution through (t_0, y_0), reaching the boundary of D by its right-hand extremity, and defined in the interval $\Delta_+ = [t_0, a_0)$. Let $y = \varphi(t)$ be a continuous curve for $t \in \tilde{\Delta}_+ = [t_0, \tilde{a}_0)$, contained in D and satisfying the initial inequality

$$\varphi(t_0) \leqslant y_0 \quad (\varphi(t_0) \geqslant y_0)$$

and the differential inequality

$$D_-\varphi(t) \leqslant \sigma(t, \varphi(t)) \quad (D^-\varphi(t) \geqslant \sigma(t, \varphi(t))) \quad \textit{for} \quad t_0 < t < \tilde{a}_0.$$

Under these assumptions we have

$$\varphi(t) \leqslant \omega^+(t) \quad (\varphi(t) \geqslant \omega_+(t)) \quad \textit{for} \quad t \in \Delta_+ \cap \tilde{\Delta}_+.$$

THEOREM 9.6. *Suppose the right-hand side of equation (9.4) to be continuous in an open region D. Let $(t_0, y_0) \in D$ and denote by $\omega^-(t)$ ($\omega_-(t)$) the left-hand maximum (minimum) solution through (t_0, y_0), reaching the boundary of D by its left-hand extremity and defined in an interval $\Delta_- = (\beta, t_0]$. Let $y = \varphi(t)$ be a continuous curve for $t \in \tilde{\Delta}_- = (\tilde{\beta}, t_0]$, contained in D and satisfying the initial inequality*

$$\varphi(t_0) \leqslant y_0 \quad (\varphi(t_0) \geqslant y_0)$$

and the differential inequality

$$D^+\varphi(t) \geqslant \sigma(t, \varphi(t)) \quad (D_+\varphi(t) \leqslant \sigma(t, \varphi(t))) \quad \textit{for} \quad \tilde{\beta} < t < t_0.$$

Under these assumptions we have

$$\varphi(t) \leqslant \omega^-(t) \quad (\varphi(t) \geqslant \omega_-(t)) \quad \textit{for} \quad t \in \Delta_- \cap \tilde{\Delta}_-.$$

Remark 9.3. We will see in § 13 that Theorems 9.3-9.6 hold true with any of the four Dini's derivatives.

EXAMPLE 9.1. Let $\varphi(t)$ be continuous in $[t_0, \tilde{a}_0)$ and suppose that $\varphi(t_0) \leqslant y_0$ and

$$D_-\varphi(t) \leqslant a(t)\varphi(t) + b(t) \quad \text{for} \quad t \in (t_0, \tilde{a}_0),$$

where $a(t)$ and $b(t)$ are continuous in some open interval Δ containing t_0. Here equation (9.4) has the form

$$\frac{dy}{dt} = a(t)y + b(t),$$

and its unique solution through (t_0, y_0) is

$$\omega(t;\, t_0, y_0) = \exp\Big(\int_{t_0}^{t} a(\tau)d\tau\Big)\Big\{y_0 + \int_{t_0}^{t} b(\sigma)\exp\Big(-\int_{t_0}^{\sigma} a(\tau)d\tau\Big)d\sigma\Big\}.$$

Hence, by Theorem 9.5, we have

$$\varphi(t) \leqslant \exp\Big(\int_{t_0}^{t} a(\tau)d\tau\Big)\Big\{y_0 + \int_{t_0}^{t} b(\sigma)\exp\Big(-\int_{t_0}^{\sigma} a(\tau)d\tau\Big)d\sigma\Big\} \quad \text{for} \quad t \in \Delta \cap [t_0, \tilde{a}_0).$$

EXAMPLE 9.2. Consider a system of differential equations

$$\frac{dy_i}{dt} = f_i(t, y_1, \ldots, y_n) \qquad (i = 1, 2, \ldots, n) \tag{9.5}$$

with right-hand sides continuous in the region

$$D: 0 < t < +\infty, \qquad \sum_{i=1}^{n} y_i^2 < h^2$$

and satisfying the inequality

$$2\sum_{i=1}^{n} y_i f_i(t, y_1, \ldots, y_n) \leqslant -c\sum_{i=1}^{n} y_i^2, \tag{9.6}$$

where c is a positive constant. Under these assumptions every solution of system (9.5) exists in an infinite interval and tends to zero as t goes to $+\infty$.

Indeed, let $y_i(t)$ $(i = 1, 2, \ldots, n)$ be a solution of (9.5) starting at some $t_0 > 0$ and let $[t_0, \gamma)$ be its right-hand maximal interval of existence. Consider the function

$$\varphi(t) = \sum_{i=1}^{n} [y_i(t)]^2,$$

for which we have

$$\varphi(t_0) = \sum_{i=1}^{n} [y_i(t_0)]^2 < h^2$$

and, by (9.6),

$$\varphi'(t) = 2\sum_{i=1}^{n} y_i(t)y_i'(t) = 2\sum_{i=1}^{n} y_i(t)f_i\big(t, y_1(t), \dots, y_n(t)\big)$$

$$\leqslant -c\sum_{i=1}^{n} [y_i(t)]^2 = -c\varphi(t)$$

in the interval $[t_0, \gamma)$. Hence, putting $y_0 = \varphi(t_0)$ we have, by Theorem 9.5,

$$\varphi(t) \leqslant y_0 e^{-c(t-t_0)} \quad \text{for} \quad t \in [t_0, \gamma)\,. \tag{9.7}$$

Since $y_0 < h^2$, it follows that

$$\varphi(t) = \sum_{i=1}^{n} [y_i(t)]^2 \leqslant y_0 < h^2$$

on the interval $[t_0, \gamma)$. Hence we must have $\gamma = +\infty$, because otherwise the solution would not reach the boundary of the region D by its right-hand extremity. On the other hand, from (9.7) it follows that the solution tends to zero as $t \to +\infty$.

§ 10. Continuity of the maximum and minimum solution on the initial point and on the right-hand sides of system. We begin this section by proving a lemma generalizing parts 1° and 2° of Theorem 8.1, which were of local character.

LEMMA 10.1. *Under the assumptions of Theorem* 8.1 *let* $\Omega(t)$ *be, in the interval* $[t_0, a_0)$, *the right-hand maximum solution through* (t_0, Y_0), *reaching the boundary of* D *by its right-hand extremity* (*such solution exists by Theorem* 9.1). *Then, for every* $a \in (t_0, a_0)$ *there is an index* ν_0 *such that*

1° *for* $\nu \geqslant \nu_0$, $Y^\nu(t)$ *exists in the interval* $[t_0, a)$ *and*

$$\Omega(t) < Y^{\nu+1}(t) < Y^\nu(t)\,,$$

2° $\lim\limits_{Y\to\infty} Y^\nu(t) = \Omega(t)$ *uniformly in* $[t_0, a)$.

Proof. By Theorem 8.1, the set of numbers $a \in (t_0, a_0)$, such that 1° and 2° hold true for some ν_0, is non-void. Let a^* be its least upper bound. We have to show that $a^* = a_0$. Suppose that $a^* < a_0$ and consider the point $\big(a^*, \Omega(a^*)\big) \in D$. Let Q^* be a cube centered at $\big(a^*, \Omega(a^*)\big)$ such that $\overline{Q^*}$ is contained in D. By the continuity, there is a positive M such that

$$\left|\sigma_i(t, Y) + \frac{1}{\nu}\right| \leqslant M \quad (i = 1, 2, \dots, n;\ \nu = 1, 2, \dots) \tag{10.1}$$

for $(t, Y) \in Q^*$. Choose α^{**} and $a > 0$ so that

$$t_0 < \alpha^{**} < \alpha^* , \tag{10.2}$$

$$\alpha^* - \alpha^{**} < \min\left(a, \frac{a}{3M}\right) = h \tag{10.3}$$

and that the cube

$$Q : |t - \alpha^{**}| < a , \ |y_i - \omega_i(\alpha^{**})| < a \qquad (i = 1, 2, \ldots, n)$$

be contained in Q^*. Such a choice is obviously possible. Since $Q \subset Q^*$, inequalities (10.1) hold true in Q and since $\alpha^{**} < \alpha^*$, 1° and 2° are satisfied in $[t_0, \alpha^{**}]$ for some ν_0. Hence, in particular,

$$\lim_{\nu\to\infty} Y^\nu(\alpha^{**}) = \Omega(\alpha^{**}) , \qquad \Omega(\alpha^{**}) < Y^{\nu+1}(\alpha^{**}) < Y^\nu(\alpha^{**}) \qquad \text{for} \qquad \nu > \nu_0 ,$$

and consequently we see, by the choice of h (compare (10.3)) and by the proof of Theorem 8.1 applied to the point $(\alpha^{**}, \Omega(\alpha^{**}))$, that 1° and 2° are satisfied in the interval $[\alpha^{**}, \alpha^{**} + h)$ for indices ν sufficiently large. Therefore, 1° and 2° hold true in the interval $[t_0, \alpha^{**} + h)$ from a certain ν on. But, in view of the definition of α^*, this is impossible because, by (10.3), $\alpha^* < \alpha^{**} + h$. This contradiction completes the proof.

Let us denote by $\Omega^+(t; t_0, Y_0)$ the right-hand maximum solution through (t_0, Y_0), reaching the boundary of D by its right-hand extremity and let $\Delta^+(t_0, Y_0)$ be its existence interval. We define in a similar obvious way the symbols $\Omega_+(t; t_0, Y_0)$, $\Omega^-(t; t_0, Y_0)$, $\Omega_-(t; t_0, Y_0)$, $\Delta_+(t_0, Y_0)$, $\Delta^-(t_0, Y_0)$, $\Delta_-(t_0, Y_0)$.

We will show the right-hand sided (left-hand sided) continuity of $\Omega^+(t; t_0, Y_0)$ $(\Omega_+(t; t_0, Y_0))$ on the initial point (t_0, Y_0), i.e. we will prove

$$\lim_{\substack{Y\to Y_0\\ Y\geqslant Y_0}} \Omega^+(t; t_0, Y) = \Omega^+(t; t_0, Y_0) , \qquad \lim_{\substack{Y\to Y_0\\ Y\leqslant Y_0}} \Omega_+(t; t_0, Y) = \Omega_+(t; t_0, Y_0) .$$

More generally and more precisely we have the following theorem.

THEOREM 10.1. *Let the right-hand sides of system* (5.1) *be continuous and satisfy condition* W_+ *with respect to* Y (see § 4) *in an open region* D. *Let* $(t_0, Y_0) \in D$. *Consider the right-hand maximum (minimum) solution* $\Omega^+(t; t_0, Y_0)$ $(\Omega_+(t; t_0, Y_0))$ *through* (t_0, Y_0), *reaching the boundary of* D *by its right-hand extremity and let* $\Delta^+(t_0, Y_0)$ $(\Delta_+(t_0, Y_0))$ *be its existence interval. For* $E = (\varepsilon_1, \ldots, \varepsilon_n)$, *where* $\varepsilon_i \geqslant 0$ $(\varepsilon_i \leqslant 0)$ $(i = 1, 2, \ldots, n)$, *denote by* $\Omega^+(t; t_0, Y, E)$ $(\Omega_+(t; t_0, Y, E))$ *the right-hand maximum (minimum) solution through* (t_0, Y) *of the system*

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, \ldots, y_n) + \varepsilon_i \qquad (i = 1, 2, \ldots, n) , \tag{10.4}$$

reaching the boundary of D *by its right-hand extremity and defined in the interval* $\Delta^+(t_0, Y, E)$ $(\Delta_+(t_0, Y, E))$. *Then,* 1° *To every* $a \in \Delta^+(t_0, Y_0)$ $(\Delta_+(t_0, Y_0))$ *there is a* $\delta(a) > 0$ *such that* $\Omega^+(t; t_0, Y, E)$ $(\Omega_+(t; t_0, Y, E))$ *is defined in* $[t_0, a)$, *whenever* $0 \leqslant \varepsilon_i < \delta(a)$ $(-\delta(a) \leqslant \varepsilon_i \leqslant 0)$ $(i = 1, 2, \ldots, n)$ *and*

$$|Y - Y_0| < \delta(a)\ (^1), \qquad Y_0 \leqslant Y\ (Y_0 \geqslant Y).$$

2° *We have uniformly in* $[t_0, a)$

$$\lim_{\substack{Y \to Y_0, E \to 0\\ Y \geqslant Y_0, E \geqslant 0}} \Omega^+(t; t_0, Y, E) = \Omega^+(t; t_0, Y_0)$$

$$\Big(\lim_{\substack{Y \to Y_0, E \to 0\\ Y \leqslant Y_0, E \leqslant 0}} \Omega_+(t; t_0, Y, E) = \Omega_+(t; t_0, Y_0)\Big).$$

Proof. We first prove the part of theorem concerning the right-hand maximum solution. Take a sequence of points Y^ν, so that

$$(t_0, Y^\nu) \in D, \qquad Y_0 < Y^{\nu+1} < Y^\nu, \qquad \lim_{\nu \to \infty} Y^\nu = Y_0,$$

and let $Y^\nu(t)$ be a solution of system (8.2), passing through (t_0, Y^ν) and continued to the boundary of D in both directions. For fixed $a \in \Delta^+(t_0, Y_0)$ there is, by Lemma 10.1, an index ν_0 such that $Y^{\nu_0}(t)$ exists in $[t_0, a]$ and

$$Y^{\nu_0}(t) > \Omega^+(t; t_0, Y_0).$$

Because of the uniform convergence of $Y^\nu(t)$ to $\Omega^+(t; t_0, Y_0)$ in $[t_0, a]$ we can assume that ν_0 is chosen sufficiently large so that the compact set

$$\{(t, Z) : t \in [t_0, a],\ \Omega^+(t; t_0, Y_0) \leqslant Z \leqslant Y^{\nu_0}(t)\} \tag{10.5}$$

be contained in D. On the other hand, since $Y_0 < Y^{\nu_0}$, there is a $\delta(a) > 0$ such that if

$$Y_0 \leqslant Y, \qquad |Y - Y_0| < \delta(a), \qquad 0 \leqslant \varepsilon_i < \delta(a) \qquad (i = 1, 2, \ldots, n), \tag{10.6}$$

then

$$Y < Y^{\nu_0}, \qquad 0 \leqslant \varepsilon_i < \frac{1}{\nu_0} \qquad (i = 1, 2, \ldots, n). \tag{10.7}$$

Let Y and ε_i satisfy (10.6). Then, since $Y_0 \leqslant Y$, we have by Theorem 9.3, applied to the point (t_0, Y) and to system (10.4),

$$\Omega^+(t; t_0, Y_0) \leqslant \Omega^+(t; t_0, Y, E) \qquad \text{for} \qquad t \in [t_0, a) \cap \Delta^+(t_0, Y, E). \tag{10.8}$$

In view of the inequalities

$$\frac{d\omega_i^+(t; t_0, Y, E)}{dt} = \sigma_i\big(t, \Omega^+(t; t_0, Y, E)\big) + \varepsilon_i < \sigma_i\big(t, \Omega^+(t; t_0, Y, E)\big) + \frac{1}{\nu_0}$$

---

(1) For two points A and B, $|A - B|$ denotes their Euclidean distance.

and of (10.7) we get, by Lemma 6.1 applied to the point (t_0, Y^{ν_0}), and to system (8.2),

$$\Omega^+(t; t_0, Y, E) < Y^{\nu_0}(t) \quad \text{for} \quad t \in [t_0, a) \cap \Delta^+(t_0, Y, E)\,. \tag{10.9}$$

From (10.8) and (10.9) it follows that $[t_0, a) \subset \Delta^+(t_0, Y, E)$. Indeed, otherwise we would have $\Delta^+(t_0, Y, E) \subset [t_0, a)$ and the solution-path corresponding to $\Omega^+(t; t_0, Y, E)$ would be contained in the compact subset (10.5) of D, which is impossible, since $\Omega^+(t; t_0, Y, E)$ reaches the boundary of D by its right-hand extremity. Thus we have proved 1°.

Now, to prove 2°, let ε be an arbitrary positive number. Since, by Lemma 10.1,

$$\lim_{\nu\to\infty} Y^\nu(t) = \Omega^+(t; t_0, Y_0)$$

uniformly on $[t_0, a)$, there is a ν_1 such that

$$|Y^{\nu_1}(t) - \Omega^+(t; t_0, Y_0)| < \varepsilon \quad \text{for} \quad t \in [t_0, a)\,. \tag{10.10}$$

Because of the inequality $Y_0 < Y^{\nu_1}$, there exists a positive $\delta(\varepsilon) < \delta(a)$ such that

$$Y < Y^{\nu_1}\,, \qquad 0 \leqslant \varepsilon_i < \frac{1}{\nu_1} \qquad (i = 1, 2, \ldots, n)\,,$$

whenever

$$Y_0 \leqslant Y\,, \quad |Y - Y_0| < \delta(\varepsilon)\,, \quad 0 \leqslant \varepsilon_i < \delta(\varepsilon) \quad (i = 1, 2, \ldots, n)\,. \tag{10.11}$$

Let Y and ε_i satisfy (10.11); then, by the same argument as in the first part of the proof, we conclude that

$$\Omega^+(t; t_0, Y_0) \leqslant \Omega^+(t; t_0, Y, E) < Y^{\nu_1}(t) \quad \text{for} \quad t \in [t_0, a)\,. \tag{10.12}$$

From (10.10) and (10.12) follows

$$|\Omega^+(t; t_0, Y, E) - \Omega^+(t; t_0, Y_0)| < \varepsilon \quad \text{in} \quad [t_0, a)$$

for Y and ε_i satisfying (10.11). This completes the proof of 2°. Applying the mapping (5.4) we obtain that part of our theorem which refers to the right-hand minimum solution as an immediate consequence of the just proved result referring to the right-hand maximum solution.

By mapping (5.2) we derive from Theorem 10.1 the following theorem:

THEOREM 10.2. *Let the right-hand sides of system* (5.1) *be continuous and satisfy condition* W_- *with respect to* Y (see § 4) *in an open region* D. *Consider the left-hand maximum (minimum) solution* $\Omega^-(t; t_0, Y_0)$ $(\Omega_-(t; t_0, Y_0))$ *through* $(t_0, Y_0) \in D$, *reaching the boundary of* D *by its left-hand extremity and defined in the interval* $\Delta^-(t_0, Y_0)$ $(\Delta_-(t_0, Y_0))$. *For* $E = (\varepsilon_1, \ldots, \varepsilon_n)$, *where* $\varepsilon_i \leqslant 0$ $(\varepsilon_i \geqslant 0)$ $(i = 1, 2, \ldots, n)$, *denote by* $\Omega^-(t; t_0, Y, E)$ $(\Omega_-(t; t_0, Y, E))$ *the left-hand maximum (minimum) solution* *through* (t_0, Y) *of system* (10.4), *reaching the boundary of* D *by its left-hand extremity and defined in the interval* $\Delta^-(t_0, Y, E)$ $(\Delta_-(t_0, Y, E))$. *Then*

1° *To every* $\beta \in \Delta^-(t_0, Y_0)$ $(\Delta_-(t_0, Y_0))$ *there is a* $\delta(\beta) > 0$ *such that* $\Omega^-(t; t_0, Y, E)$ $(\Omega_-(t; t_0, Y, E))$ *is defined in* $(\beta, t_0]$, *whenever*

$$|Y - Y_0| < \delta(\beta)\,, \qquad Y_0 \leqslant Y \ (Y_0 \geqslant Y)\,,$$
$$-\delta(\beta) < \varepsilon_i \leqslant 0 \ \ (0 \leqslant \varepsilon_i < \delta(\beta)) \qquad (i = 1, 2, \ldots, n)\,.$$

2° *We have uniformly in* $(\beta, t_0]$

$$\lim_{\substack{Y \to Y_0,\, E \to 0\\ Y \geqslant Y_0,\, E \leqslant 0}} \Omega^-(t; t_0, Y, E) = \Omega^-(t; t_0, Y_0)\,,$$
$$\Big(\lim_{\substack{Y \to Y_0,\, E \to E_0\\ Y \leqslant Y_0,\, E \geqslant 0}} \Omega_-(t; t_0, Y, E) = \Omega_-(t; t_0, Y_0)\Big)\,.$$

We close this section by the following example (see [4]).

Example. Consider the differential equation

$$\frac{dy}{dt} = \sigma(t, y)\,, \tag{10.13}$$

where

$$\sigma(t, y) = \begin{cases} 2Ly + 2M\sqrt{y} & \text{for} \quad y \geqslant 0\,, \\ 0 & \text{for} \quad y < 0; \end{cases}$$

$L > 0$, $M > 0$ are some constants.

We will prove that for each point (t_0, y_0), where $y_0 \geqslant 0$, the right-hand maximum solution of (10.13) through (t_0, y_0) is

$$\omega(t; t_0, y_0) = \left[\sqrt{y_0}\, e^{L(t-t_0)} + \frac{M}{L}\left(e^{L(t-t_0)} - 1\right)\right]^2. \tag{10.14}$$

Suppose first that $y_0 > 0$; then, since $\sigma(t, y) \geqslant 0$, we have for any solution $y(t)$ of (10.13) through (t_0, y_0)

$$y(t) \geqslant y_0 > 0 \quad \text{for} \quad t \geqslant t_0\,.$$

Therefore, putting $u(t) = \sqrt{y(t)}$, we find that $u(t)$ satisfies, for $t \geqslant t_0$, the linear equation

$$\frac{du}{dt} = Lu + M$$

and consequently is of the form

$$u(t) = \sqrt{y_0}\, e^{L(t-t_0)} + \frac{M}{L}\left(e^{L(t-t_0)} - 1\right).$$

Hence it follows that, for $y_0 > 0$, function (10.14) is, in the interval $t \geqslant t_0$, the unique solution of equation (10.13) through (t_0, y_0) and consequently its right-hand maximum solution through (t_0, y_0). Our assertion for $y_0 = 0$ follows now from Theorem 10.1 if we let $y_0 > 0$ tend to 0. Notice that for $y_0 = 0$ we do not have uniqueness.

By Theorem 9.5, we get the following result. Let $\varphi(t)$ be continuous and non-negative for $t \in [t_0, a)$. Suppose it satisfies the initial inequality

$$\varphi(t_0) \leqslant y_0$$

and the differential inequality

$$D_- \varphi(t) \leqslant 2L\varphi(t) + 2M\sqrt{\varphi(t)}\,.$$

This being assumed, we have for $t \in [t_0, a)$

$$\varphi(t) \leqslant \omega(t;\, t_0, y_0)\,,$$

where $\omega(t;\, t_0, y_0)$ is given by formula (10.14).

CHAPTER III

FIRST ORDER ORDINARY DIFFERENTIAL INEQUALITIES

§ 11. Basic theorems on first order ordinary differential inequalities. In this section we give theorems generalizing Theorems 9.3 and 9.4 in the direction that will be briefly explained here (see [22] and [61]). In Theorem 9.3 we assumed the system of differential inequalities to be satisfied in the whole interval where the curve $Y = \Phi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$ was defined. This assumption will be substituted by a less restrictive one; we will require only that for every index i the i-th differential inequality be satisfied at such points t where $\varphi_i(t)$ is greater than the i-th component of the maximum solution. As we will see (Example 11.1, Remark 48.1), such a weakening of assumptions is very useful in applications of the theory of ordinary differential inequalities.

THEOREM 11.1. *Suppose the right-hand sides of system* (5.1) *are continuous and satisfy condition* W_+ *with respect to* Y (see § 4) *in an open region* D. *Let* $(t_0, Y_0) \in D$ *and consider the right-hand maximum solution* $\Omega^+(t; t_0, Y_0) = (\omega_1^+(t), \ldots, \omega_n^+(t))$ *through* (t_0, Y_0), *defined in the interval* $[t_0, a_0)$ *and reaching the boundary of* D *by its right-hand extremity. Let* $Y = \Phi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$ *be a continuous curve on the interval* $[t_0, \gamma)$ *and suppose that* $(t, \Phi(t)) \in D$. *Write* $a_1 = \min(a_0, \gamma)$ *and*

$$\overset{+}{E}_i = \{t \in (t_0, a_1) : \varphi_i(t) > \omega_i^+(t)\} \qquad (i = 1, 2, \ldots, n).$$

Under these assumptions, if

$$\Phi(t_0) \leqslant Y_0, \tag{11.1}$$

$$D_-\varphi_i(t) \leqslant \sigma_i(t, \Phi(t)) \quad \textit{for} \quad t \in \overset{+}{E}_i \quad (i = 1, 2, \ldots, n), \tag{11.2}$$

then the sets $\overset{+}{E}_i$ $(i = 1, 2, \ldots, n)$ *are empty, i.e.*

$$\Phi(t) \leqslant \Omega^+(t; t_0, Y_0) \quad \textit{for} \quad t \in [t_0, a_1). \tag{11.3}$$

Proof. Take a sequence of points Y^ν such that $(t_0, Y^\nu) \in D$, $Y_0 < Y^{\nu+1} < Y^\nu$ and $\lim_{\nu\to\infty} Y^\nu = Y_0$. Let $Y^\nu(t) = (y_1^\nu(t), \ldots, y_n^\nu(t))$ be a solution of system (8.2), passing through (t_0, Y^ν) and continued to the boundary of D in both directions. Take an arbitrary $a \in (t_0, a_1)$. By Lemma 10.1, there is an index ν_0 (depending on a) such that, for $\nu \geqslant \nu_0$, $Y^\nu(t)$ exists on $[t_0, a)$ and

$$(11.4) \qquad \Omega^+(t; t_0, Y_0) < Y^{\nu+1}(t) \leqslant Y^\nu(t) \quad \text{in} \quad [t_0, a),$$

$$(11.5) \qquad \lim_{\nu\to\infty} Y^\nu(t) = \Omega^+(t; t_0, Y_0) \quad \text{in} \quad [t_0, a).$$

In view of (11.5), to prove (11.3) it is sufficient to show that for $\nu \geqslant \nu_0$ we have

$$(11.6) \qquad \Phi(t) < Y^\nu(t) \quad \text{in} \quad [t_0, a).$$

Take a fixed $\nu \geqslant \nu_0$. Since $\Phi(t_0) \leqslant Y_0 < Y^\nu = Y^\nu(t_0)$, inequality (11.6) holds, by continuity, in some interval $[t_0, \tilde{a})$. Denote by a^* the least upper bound of $\tilde{a} \in (t_0, a)$ such that (11.6) is satisfied in $[t_0, \tilde{a})$. We have to show that $a^* = a$. Suppose $a^* < a$; then, by the definition of a^* and by the continuity, we have

$$(11.7) \qquad \Phi(t) < Y^\nu(t) \quad \text{on} \quad [t_0, a^*),$$

and for at least one index j

$$(11.8) \qquad \Phi(a^*) \overset{j}{\leqslant} Y^\nu(a^*)$$

(see § 4). Hence, in particular,

$$(11.9) \qquad \varphi_j(a^*) = y_j^\nu(a^*), \quad \varphi_j(t) < y_j^\nu(t) \quad \text{for} \quad t \in (t_0, a^*).$$

From (11.9) it follows that

$$(11.10) \qquad D_-\varphi_j(a^*) \geqslant y_j^{\nu\prime}(a^*) = \sigma_j\big(a^*, Y^\nu(a^*)\big) + \frac{1}{\nu}.$$

On the other hand, since by (11.4) we have $\omega_j^+(a^*) < y_j^\nu(a^*)$, we get from (11.9) that $\omega_j^+(a^*) < \varphi_j(a^*)$ and consequently $a^* \in \overset{+}{E}_j$. Therefore, by (11.2), (11.8) and by condition W_+ (see § 4), we have

$$D_-\varphi_j(a^*) \leqslant \sigma_j\big(a^*, \Phi(a^*)\big) \leqslant \sigma_j\big(a^*, Y^\nu(a^*)\big) < \sigma_j\big(a^*, Y^\nu(a^*)\big) + \frac{1}{\nu},$$

contrary to inequality (11.10). Hence, $a^* < a$ is impossible and this completes the proof.

Example 11.1 (see [59]). Consider a linear equation

$$\frac{dy}{dt} = a(t)y + b(t),$$

where $a(t)$ and $b(t)$ are continuous, complex-valued functions on $[0, a)$. Put $s(t) = \operatorname{Re} a(t)$ and suppose that $|b(t)| \leqslant \varrho(t)$, where $\varrho(t)$ is continuous.

Let $y(t)$ satisfy the above equation in $[0, a)$. Under these assumptions we have in $[0, a)$

$$|y(t)| \leqslant \omega(t) ,$$

where

$$\omega(t) = |y(0)| \exp\Big(\int_0^t s(\tau)d\tau\Big) + \int_0^t \exp\Big(\int_u^t s(\tau)d\tau\Big)\varrho(u)du .$$

Indeed, put

$$E = \{t \in (0, a) : |y(t)| > \omega(t)\} .$$

For $t \in E$ we have obviously $|y(t)| > 0$, and consequently

$$\frac{d}{dt}|y(t)| = \frac{1}{2}\,\frac{y(t)\overline{y'(t)} + \overline{y(t)}y'(t)}{|y(t)|} .$$

Since

$$y'(t)\overline{y(t)} = a(t)|y(t)|^2 + b(t)\overline{y(t)} ,$$

$$\overline{y'(t)}y(t) = \overline{a(t)}|y(t)|^2 + \overline{b(t)}y(t) ,$$

$$b(t)\overline{y(t)} + \overline{b(t)}y(t) \leqslant 2\varrho(t)|y(t)| ,$$

we get

$$\frac{d}{dt}|y(t)| \leqslant \frac{a(t) + \overline{a(t)}}{2}|y(t)| + \varrho(t) .$$

Thus we have shown that $t \in E$ implies

$$\frac{d}{dt}|y(t)| \leqslant s(t)|y(t)| + \varrho(t) .$$

Now, since $\omega(t)$ is the unique solution of the linear equation

$$\frac{dy}{dt} = s(t)y + \varrho(t) ,$$

satisfying the initial condition $\omega(0) = |y(0)|$, our assertion follows from Theorem 11.1. Observe that we were able to check the differential inequality only for t such that $|y(t)| > 0$.

By means of the mapping (5.4) we get from Theorem 11.1 the following theorem.

THEOREM 11.2. *Suppose the right-hand members of system* (5.1) *are continuous and satisfy condition* W_+ *with respect to* Y (see § 4) *in* D. *Let* $(t_0, Y_0) \in D$ *and consider the right-hand minimum solution* $\Omega_+(t; t_0, Y_0) = \big(\omega_+^1(t), \ldots, \omega_+^n(t)\big)$ *through* (t_0, Y_0), *defined in* $[t_0, a_0)$ *and reaching the boundary of* D *by its right-hand extremity. Let* $Y = \Phi(t) = \big(\varphi_1(t), \ldots, \varphi_n(t)\big)$ *be a continuous curve on* $[t_0, \gamma)$ *and assume that* $\big(t, \Phi(t)\big) \in D$. *Put* $a_1 = \min(a_0, \gamma)$ *and*

$$\underset{+}{E_i} = \{t \in (t_0, a_1) : \varphi_i(t) < \omega_+^i(t)\} \qquad (i = 1, 2, \ldots, n) .$$

Under these assumptions, if

$$\Phi(t_0) \geqslant Y_0\,,$$

$$D^-\varphi_i(t) \geqslant \sigma_i\big(t, \Phi(t)\big) \quad \textit{for} \quad t \in \underset{+}{E_i} \quad (i = 1, 2, \dots, n)\,,$$

then

$$\Phi(t) \geqslant \Omega_+(t; t_0, Y_0) \quad \textit{for} \quad t \in [t_0, a_1)\,.$$

Using the mapping (5.2) we get, as an immediate consequence of Theorems 11.1 and 11.2, the following theorem (see Propositions 5.1 and 6.1).

THEOREM 11.3. *Suppose the right-hand sides of system* (5.1) *are continuous and satisfy condition* W_- *with respect to* Y (see § 4) *in an open region* D. *Let* $(t_0, Y_0) \in D$ *and consider the left-hand maximum* (*minimum*) *solution* $\Omega^-(t; t_0, Y_0) = \big(\omega_1^-(t), \dots, \omega_n^-(t)\big)$ $\big(\Omega_-(t; t_0, Y_0) = \big(\omega_-^1(t), \dots, \omega_-^n(t)\big)\big)$ *through* (t_0, Y_0), *defined in the interval* $(\beta_0, t_0]$ *and reaching the boundary of* D *by its left-hand extremity. Let* $Y = \Phi(t) = \big(\varphi_1(t), \dots, \varphi_n(t)\big)$ *be continuous on* $(\delta, t_0]$ *and assume that* $\big(t, \Phi(t)\big) \in D$. *Write* $\beta_1 = \max(\beta_0, \delta)$ *and*

$$\bar{E}_i = \{t \in (\beta_1, t_0)\colon \varphi_i(t) > \omega_i^-(t)\} \quad (i = 1, 2, \dots, n)$$

$$(\underline{E}_i = \{t \in (\beta_1, t_0)\colon \varphi_i(t) < \omega_-^i(t)\})\,.$$

Under these assumptions, if

$$\Phi(t_0) \leqslant Y_0 \quad (\Phi(t_0) \geqslant Y_0)\,,$$

and

$$D^+\varphi_i(t) \geqslant \sigma_i\big(t, \Phi(t)\big) \quad \textit{for} \quad t \in \bar{E}_i \quad (i = 1, 2, \dots, n)$$

$$(D_+\varphi_i(t) \leqslant \sigma_i\big(t, \Phi(t)\big) \quad \textit{for} \quad t \in \underline{E}_i)\,,$$

then

$$\Phi(t) \leqslant \Omega^-(t; t_0, Y_0) \quad \textit{for} \quad t \in (\beta_1, t_0]$$

$$(\Phi(t) \geqslant \Omega_-(t; t_0, Y_0) \quad \textit{for} \quad t \in (\beta_1, t_0])\,.$$

§ 12. Necessity of condition V_+ (V_-) in theorems on differential inequalities. Let the right-hand members of system (5.1), with $n > 1$, be continuous in a parallelepipede

$$D: -\infty \leqslant a < t < b \leqslant +\infty\,, \quad -\infty \leqslant a_i < y_i < b_i \leqslant +\infty \quad (i = 1, 2, \dots, n)\,.$$

Since conditions W_+ and V_+ are equivalent in D (see § 4), we get from Theorems 11.1 and 11.2, as a particular conclusion, the following result:

If the right-hand sides of system (5.1) *satisfy condition* V_+ *with respect to* Y (*see* § 4), *then*

(α_+) *to every point* $(t_0, Y_0) \in D$ *there is a solution* $\Omega^+(t; t_0, Y_0)$ $(\Omega_+(t; t_0, Y_0))$ *through* (t_0, Y_0) *such that for any solution* $Y(t)$ *satisfying the initial inequality* $Y(t_0) \leqslant Y_0$ $(Y(t_0) \geqslant Y_0)$ *we have* $Y(t_0) \leqslant \Omega^+(t; t_0, Y_0)$ $(Y(t_0) \geqslant \Omega_+(t; t_0, Y_0))$ *in some right-hand neighborhood of* t_0.

The above result can be inverted; in fact, we have the following theorem for an arbitrary open region D:

THEOREM 12.1. *Let the right-hand sides of system* (5.1), *with* $n > 1$, *be continuous in an open region* D. *Then the condition* V_- *with respect to* Y *is a necessary one for the property* (α_+) *to hold true.*

Proof. It is sufficient to prove the part of theorem referring to Ω^+. The part of theorem concerning Ω_+ will follow then by the mapping (5.4).

Let the indices i and $j \neq i$ be fixed and consider two points

$$(t_0, Y_0) = (t_0, \mathring{y}_1, \ldots, \mathring{y}_n) \in D; \ (t_0, \widetilde{Y}) = (t_0, \mathring{y}_1, \ldots, \mathring{y}_{j-1}, \widetilde{y}_j, \mathring{y}_{j+1}, \ldots, \mathring{y}_n) \in D$$

such that $\widetilde{y}_j < \mathring{y}_j$. Let $\widetilde{Y}(t)$ be a solution through $(t_0, \widetilde{Y})$. Since $\widetilde{Y}(t_0) = \widetilde{Y} \leqslant Y_0$, we have, by ($\alpha_+$),

$$\widetilde{Y}(t) \leqslant \Omega^+(t; t_0, Y_0)$$

in some right-hand neighborhood of t_0. In particular, $\widetilde{y}_i(t_0) = \mathring{y}_i = \omega_i^+(t_0)$, $\widetilde{y}_i(t) \leqslant \omega_i^+(t)$ in a right-hand neighborhood of t_0. Hence, it follows that

$$\sigma_i(t_0, \widetilde{Y}) = \sigma_i\big(t_0, \widetilde{Y}(t_0)\big) = \widetilde{y}_i'(t_0) \leqslant \omega_i^+(t_0) = \sigma_i\big(t_0, \Omega^+(t_0; t_0, Y_0)\big) = \sigma_i(t_0, Y_0),$$

and thus the proof is completed.

By mapping (5.2) we obtain from Theorem 12.1 a similar theorem concerning condition V_- and the property:

(α_-) *To every point* $(t_0, Y_0) \in D$ *there is a solution* $\Omega^-(t; t_0, Y_0)$ ($\Omega_-(t; t_0, Y_0)$) *through* (t_0, Y_0) *such that for any solution* $Y(t)$ *satisfying the initial inequality* $Y(t_0) \leqslant Y_0$ ($Y(t_0) \geqslant Y_0$) *we have* $Y(t) \leqslant \Omega^-(t; t_0, Y_0)$ ($Y(t) \geqslant \Omega_-(t; t_0, Y_0)$) *in some left-hand neighborhood of* t_0.

From the last remark and from Theorem 12.1 follows the next theorem.

THEOREM 12.2. *The only systems* (5.1) *with right-hand members continuous in an open region* D, *for which both properties* (α_+) *and* (α_-) *hold true, are those of the form*

$$\frac{dy_i}{dt} = \sigma_i(t, y_i) \quad (i = 1, 2, \ldots, n), \tag{12.1}$$

i.e. systems of independent equations, each containing only one unknown function.

Proof. The right-hand sides of system (5.1), having both properties (α_+) and (α_-), satisfy necessarily conditions V_+ and V_-. This means that the function $\sigma_i(t, Y)$ is both increasing and decreasing with respect to the variables $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n$ and hence depends only on the variable y_i.

Less precisely, Theorem 12.2 may be summarized in the following way:

Systems (12.1) *are the only ones that can be used for estimates from above* (*from below*) *both to right and to left from the initial point.*

§ 13. Some variants of theorems on differential inequalities. To begin with we will show that Theorem 11.1 holds true if the derivative D_- in (11.2) is replaced by D^- or D_+ or D^+. We do it for D^+, for instance. Obviously it is sufficient to prove that if (11.2) is satisfied with D^+ instead of D_-, then it is satisfied with D_- too. Suppose then that

$$(13.1)\qquad D^+\varphi_i(t) \leqslant \sigma_i\big(t, \Phi(t)\big) \quad \text{for} \quad t \in \overset{+}{E}_i \quad (i = 1, 2, \ldots, n)\,.$$

The set $\overset{+}{E}_i$ is open and, therefore, is the union of a sequence (finite or infinite) of open intervals. Take any of these intervals, say Δ_i^ν, and consider the Picard's transform $\Psi(t) = \big(\psi_1(t), \ldots, \psi_n(t)\big)$ of $\Phi(t)$, defined by the formula

$$(13.2)\qquad \psi_i(t) = \varphi_i(t) - \int_{\tau_0}^{t} \sigma_i\big(\tau, \Phi(\tau)\big)\,d\tau \qquad (i = 1, 2, \ldots, n)\,,$$

where τ_0 is fixed in Δ_i^ν. By (13.1) and (13.2), we have

$$D^+\psi_i(t) = D^+\varphi_i(t) - \sigma_i\big(t, \Phi(t)\big) \leqslant 0 \quad \text{for} \quad t \in \Delta_i^\nu$$
$$(i = 1, 2, \ldots, n;\ \nu = 1, 2, \ldots)\,.$$

Hence, $\psi_i(t)$ being continuous in the interval Δ_i^ν, we get, by Theorem 2.1, that $\psi_i(t)$ is decreasing in Δ_i^ν. Therefore,

$$0 \geqslant D_-\psi_i(t) = D_-\varphi_i(t) - \sigma_i\big(t, \Phi(t)\big) \quad \text{in} \quad \Delta_i^\nu$$
$$(i = 1, 2, \ldots, n;\ \nu = 1, 2, \ldots)\,,$$

what was to be proved.

By a similar argument we show that Theorems 11.2 and 11.3 hold true with any of the four Dini's derivatives appearing in the system of differential inequalities.

All theorems of this chapter will be formulated, from now on, with the D_- derivative; but, due to the preceding remarks, they will be true with any of the four remaining derivatives, and in our subsequent considerations we will remember this fact without pointing it explicitly.

Applying Picard's transform (13.2) we obtain, by the argument used in our preceding remarks, the following theorem.

THEOREM 13.1. *Theorems* 11.1, 11.2 *and* 11.3 *are true if the corresponding differential inequalities are supposed to be satisfied in the sets* $E_i - C_i$, *where* $C_i \subset E_i$ *is an arbitrary countable set.*

A much stronger result is obtained if we additionally assume that $\Phi(t)$ is absolutely continuous. In fact, we have the following theorem.

THEOREM 13.2. *Under the assumptions of Theorem* 11.1, *let* $\Phi(t) = \big(\varphi_1(t), \ldots, \varphi_n(t)\big)$ *be absolutely continuous in* $[t_0, \gamma)$. *This being assumed, if*

$$\Phi(t_0) \leqslant Y_0\,,$$

and

(13.3) $\varphi_i'(t) \leqslant \sigma_i(t, \Phi(t))$ *almost everywhere in* $\overset{+}{E}_i$ $(i = 1, 2, \dots, n)$,

then

$$\Phi(t) \leqslant \Omega^+(t; t_0, Y_0) \quad \textit{in} \quad [t_0, a_1).$$

Proof. In view of Theorem 11.1, it is sufficient to show that (13.3) implies (11.2). Like in our considerations at the beginning of this paragraph, let $\overset{+}{E}_i = \bigcup_{\nu=1}^{\infty} \Delta_i^\nu$, where Δ_i^ν are open intervals, and introduce the Picard's transform (13.2). The function $\psi_i(t)$ is absolutely continuous in Δ_i^ν because so is $\varphi_i(t)$. By (13.3), we have $\psi_i'(t) = \varphi_i'(t) - \sigma_i(t, \Phi(t)) \leqslant 0$ almost everywhere in Δ_i^ν. Hence, by Theorem 3.1, the function $\psi_i(t)$ is decreasing in Δ_i^ν, and therefore

$$D_-\varphi_i(t) - \sigma_i(t, \Phi(t)) = D_-\psi_i(t) \leqslant 0 \quad \text{in} \quad \Delta_i^\nu \quad (\nu = 1, 2, \dots),$$

what was to be proved.

Similar theorems, corresponding to Theorems 11.2 and 11.3, can be stated in an obvious way.

Using Remark 3.2, we show similarly that Theorem 13.2 holds true if $\Phi(t)$ is a generalized absolutely continuous function and (13.3) is satisfied with $\varphi_i'(t)$ substituted by the approximative derivative of $\varphi_i(t)$ (see [22] and [50]).

§ 14. Comparison systems. In this section we introduce systems of first order ordinary differential equations having some special properties. These systems, called *comparison systems*, will be used in applications of the theory of differential inequalities.

A system of differential equations

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, \dots, y_n) \quad (i = 1, 2, \dots, n) \tag{14.1}$$

will be called *comparison system of type* I if its right-hand sides are continuous and non-negative and satisfy condition W_+ with respect to Y (see § 4) in the closed region

$$\bar{Q}: t \geqslant 0, \; y_i \geqslant 0 \quad (i = 1, 2, \dots, n).$$

Example 14.1. The linear system

$$\frac{dy_i}{dt} = \sum_{i,j=1}^{n} a_{ij}(t) y_j + b_i(t) \quad (i = 1, 2, \dots, n),$$

with $a_{ij}(t), b_i(t)$ continuous and non-negative for $t \geqslant 0$, is a comparison system of type I.

Since the region $\bar{Q}$ is not open, we are not able here to apply directly the results of § 9 on the maximum solution of system (14.1). Nevertheless, we will show that the following proposition holds true:

PROPOSITION 14.1. *Through every point* $(0, H) = (0, \eta_1, \ldots, \eta_n) \in \bar{Q}$ *there is the right-hand maximum solution of the comparison system of type* I, *which will be denoted by* $\Omega(t; H) = (\omega_1(t; H), \ldots, \omega_n(t; H))$ *and its maximal interval of existence by* $\Delta(H) = [0, a_0(H))$. *Moreover, we have either* $a_0(H) = +\infty$, *or* $a_0(H)$ *is finite and* [1]

$$\lim_{t \to a_0} |\Omega(t; H)| = +\infty .$$

Proof. It is easy to see that there exists an extension $\tilde{\sigma}_i(t, Y)$ of $\sigma_i(t, Y)$, so that $\tilde{\sigma}_i(t, Y)$ are continuous and non-negative and satisfy condition W_+ with respect to Y in the whole space of points (t, Y). Now, by Theorem 9.1, applied to the extended system

$$\frac{dy_i}{dt} = \tilde{\sigma}_i(t, y_1, \ldots, y_n) \quad (i = 1, 2, \ldots, n), \tag{14.2}$$

there is the right-hand maximum solution $\Omega(t; H)$ of (14.2) in an interval $\Delta(H)$, passing through $(0, H)$ and reaching the boundary of the space by its right-hand extremity. For this solution, since $\tilde{\sigma}_i(t, Y)$ are non-negative and since $\Omega(0; H) = H \geqslant 0$, we have $\Omega(t; H) \geqslant 0$ in $\Delta(H)$. Hence, $(t, \Omega(t; H)) \in \bar{Q}$ for $t \in \Delta(H)$, and consequently $\Omega(t; H)$ is the solution of the original system (14.1) with required properties. The existence of the limit

$$\lim_{t \to a_0} \sqrt{\sum_{i=1}^{n} [\omega_i(t; H)]^2}$$

follows from the fact that $\omega_i(t; H)$ are increasing functions since $\sigma_i(t, Y) \geqslant 0$.

Remark 14.1. Taking advantage of the extended system (14.2) we can prove that Theorem 10.1 holds true for a comparison system of type I.

Using the extended system (14.2) we derive from Theorem 11.1 the next theorem.

FIRST COMPARISON THEOREM. *A comparison system* (14.1) *of type* I *being given, let* $(0; H) \in \bar{Q}$ *and denote by* $\Omega(t; H) = (\omega_1(t; H), \ldots, \omega_n(t; H))$ *its right-hand maximum solution through* $(0, H)$, *defined in* $[0, a_0)$. *Let* $\Phi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$ *be continuous and non-negative in some interval* $[0, \gamma)$. *Put* $a_1 = \min(a_0, \gamma)$ *and*

$$E_i = \{t \in (0, a_1) : \varphi_i(t) > \omega_i(t; H)\} \quad (i = 1, 2, \ldots, n) .$$

Under these assumptions, if

$$\Phi(0) \leqslant H ,$$

[1] For a point $A = (a_1, \ldots, a_n)$, $|A|$ denotes $\sqrt{\sum_{i=1}^{n} a_i^2}$.

and

$$D_-\varphi_i(t) \leqslant \sigma_i(t, \Phi(t)) \quad \text{for} \quad t \in E_i \quad (i = 1, 2, \ldots, n),$$

then

$$\Phi(t) \leqslant \Omega(t; H) \quad \text{for} \quad t \in [0, a_1).$$

For $n = 1$ we introduce two special types of comparison equations; but, first we prove a lemma.

LEMMA 14.1. *Let the right-hand side of the differential equation*

$$\frac{dy}{dt} = \sigma(t, y) \tag{14.3}$$

be continuous and non-negative in the region

$$Q : t > 0 , \; y \geqslant 0$$

and suppose that

$$\sigma(t, 0) \equiv 0 . \tag{14.4}$$

Under these assumptions, for every point $(t_0, y_0) \in Q$ there is the left-hand minimum solution $\omega_-(t; t_0, y_0)$ through (t_0, y_0), and its maximal interval of existence is $(0, t_0]$. Moreover, we have

$$\omega_-(t; t_0, 0) \equiv 0 .$$

Proof. We consider the auxiliary equation

$$\frac{dy}{dt} = \tilde{\sigma}(t, y) , \tag{14.5}$$

where

$$\tilde{\sigma}(t, y) = \begin{cases} \sigma(t, y) & \text{for} \quad t > 0 , \; y > 0 , \\ 0 & \text{for} \quad t > 0 , \; y \leqslant 0 . \end{cases}$$

By (14.4), the right-hand side of equation (14.5) is continuous in the open half-plane $t > 0$. Hence, by Remark 9.1, there is the left-hand miminum solution $\omega_-(t; t_0, y_0)$ of (14.5) through (t_0, y_0), reaching the boundary of the positive half-plane by its left-hand extremity. Denote its existence interval by $(\beta, t_0]$. We will show that

1° $\omega_-(t; t_0, y_0) \geqslant 0$ for $t \in (\beta, t_0]$,

2° $\beta = 0$.

To prove 1°, observe that the unique solution of (14.5) issued from a point (t^*, y^*), where $y^* < 0$, is $y(t) \equiv y^* < 0$. Hence it follows that 1° holds true since $\omega_-(t_0; t_0, y_0) = y_0 \geqslant 0$. Now, we must have $\beta = 0$; otherwise, since $\omega_-'(t; t_0, y_0) \geqslant 0$ and by 1°, the solution path $y = \omega_-(t; t_0, y_0)$ would be contained in the compact subset $0 < \beta \leqslant t \leqslant t_0$, $0 \leqslant y \leqslant y_0$ of the positive half-plane, which is impossible because $\omega_-(t; t_0, y_0)$ reaches the boundary of the positive half-plane by its left-hand extremity. From 1° and 2° it follows that $\omega_-(t; t_0, y_0)$ is the left-hand minimum solution of the original equation (14.3) with required properties; $\omega_-(t; t_0, 0) \equiv 0$ is obvious.

Equation (14.3) with the right-hand member continuous and non-negative for $t > 0$, $y \geqslant 0$, and satisfying (14.4), will be called *comparison equation of type* II if $y(t) \equiv 0$ is in every interval $(0, \gamma)$ the only solution satisfying the condition

$$\lim_{t\to 0} y(t) = 0 .$$

EXAMPLE 14.2. We give three examples of comparison equations of type II:

(α) $\dfrac{dy}{dt} = a(t)y$ with $a(t) \geqslant 0$ continuous for $t \geqslant 0$;

(β) $\dfrac{dy}{dt} = \sigma(y)$ with $\sigma(y) > 0$ for $y > 0$, $\sigma(0) = 0$, $\displaystyle\int_0^\delta \frac{dy}{\sigma(y)} = +\infty$;

(γ) $\dfrac{dy}{dt} = |\ln t| y$.

SECOND COMPARISON THEOREM. *Let a comparison equation* (14.3) *of type* II *be given and let* $\varphi(t)$ *be continuous in an interval* $[0, a)$ *and satisfy the condition*

$$\varphi(0) \leqslant 0 . \tag{14.6}$$

Write

$$E = \{t \in (0, a) : \varphi(t) > 0\}$$

and suppose that

$$D_-\varphi(t) \leqslant \sigma\big(t, \varphi(t)\big) \quad \textit{for} \quad t \in E . \tag{14.7}$$

Under these assumptions

$$\varphi(t) \leqslant 0 \quad \textit{in} \quad [0, a) .$$

Proof. Suppose that for some $t_0 \in (0, a)$ we have

$$\varphi(t_0) = y_0 > 0 .$$

By Lemma 14.1, the left-hand minimum solution of (14.3) $\omega_-(t; t_0, y_0)$, issued from (t_0, y_0), is defined in $(0, t_0]$. Since $\varphi(0) \leqslant 0$ and $\varphi(t_0) > 0$, there is the first t_1 to left from t_0, such that $\varphi(t_1) = 0$. We have obviously $t_1 \geqslant 0$ and

$$0 < \varphi(t) \quad \text{for} \quad t_1 < t \leqslant t_0 .$$

Hence, applying Theorem 9.6 (compare Remark 9.3) to equation (14.3) (considered in the open region $t > 0$, $y > 0$) we see, by (14.7), that

$$0 \leqslant \omega_-(t; t_0, y_0) \leqslant \varphi(t) \quad \text{for} \quad t_1 < t \leqslant t_0 . \tag{14.8}$$

If $t_1 = 0$, then from (14.6) and (14.8) it follows that

$$\lim_{t\to 0} \omega_-(t;\, t_0,\, y_0) = 0\,. \tag{14.9}$$

If $t_1 > 0$, then since $\varphi(t_1) = 0$, we have $\omega_-(t_1;\, t_0,\, y_0) = 0$, by (14.8); hence, by Lemma 14.1, we get $\omega_-(t;\, t_0,\, y_0) \equiv \omega_-(t;\, t_1,\, 0) \equiv 0$ for $0 < t \leqslant t_1$ and, consequently, (14.9) holds true in this case too. Therefore, $\omega_-(t;\, t_0,\, y_0)$ would be a solution of (14.3) tending to zero as t goes to zero and different from $y(t) \equiv 0$ since $\omega_-(t_0;\, t_0,\, y_0) = y_0 > 0$. But, this is impossible in view of the definition of a comparison equation of type II. This contradiction completes the proof.

Remark 14.2. A comparison equation (14.3) of type II is not—in general—one of type I, because $\sigma(t, y)$ is not supposed to be continuous for $t = 0$. If $\sigma(t, y)$ is continuous for $t = 0$, then the second comparison theorem is a corollary of the first one.

Equation (14.3) with the right-hand side continuous and non-negative for $t > 0$, $y \geqslant 0$, and satisfying (14.4), will be called *comparison equation of type* III if the following property holds true:

(α_1) *In every interval* $(0, \gamma)$ *the function* $y(t) \equiv 0$ *is the only solution satisfying the conditions*

$$\lim_{t\to 0} y(t) = \lim_{t\to 0} \frac{y(t)}{t} = 0\,. \tag{14.10}$$

A comparison equation of type II is obviously one of type III too. But, a comparison equation of type III may not be one of type II. This is shown by the following example.

Example 14.3. Let

$$\frac{dy}{dt} = \frac{y}{t}\,.$$

The general solution of this equation is $y = Ct$ ($C = \text{const}$) and hence the equation is of type III, but not of type II.

THIRD COMPARISON THEOREM. *Let a comparison equation* (14.3) *of type* III *be given and let* $\varphi(t)$ *be continuous in an interval* $[0, a)$ *and satisfy the conditions*

$$\varphi(0) \leqslant 0\,, \qquad D^+\varphi(0) \leqslant 0\,. \tag{14.11}$$

Put

$$E = \{t \in (0,\, a) : \varphi(t) > 0\}$$

and suppose that

$$D_-\varphi(t) \leqslant \sigma\big(t,\, \varphi(t)\big) \qquad \textit{for} \qquad t \in E\,.$$

Under these assumptions

$$\varphi(t) \leqslant 0 \qquad \textit{in} \qquad [0,\, a)\,.$$

Proof. We proceed just like in the proof of the second comparison theorem and find that if the thesis were not true, then for some $t_0 \in (0, a)$ and $0 \leqslant t_1 < t_0$ we would have $\varphi(t_1) = 0$ and (14.8) with $y_0 = \varphi(t_0)$. Hence, if $t_1 = 0$, it would follow from (14.8) and (14.11) that

$$\lim_{t\to 0} \omega_-(t; t_0, y_0) = \lim_{t\to 0} \frac{\omega_-(t; t_0, y_0)}{t} = 0 . \tag{14.12}$$

If $t_1 > 0$, then—like in the proof of the second comparison theorem—we have $\omega_-(t; t_0, y_0) \equiv 0$ for $0 < t \leqslant t_1$ and consequently (14.12) would hold in this case too. Therefore, $\omega_-(t; t_0, y_0)$ would be a solution satisfying conditions (14.10) and different from $y(t) \equiv 0$ (because $\omega_-(t_0; t_0, y_0) = y_0 > 0$), contrary to the definition of a comparison equation of type III.

Remark 14.3. It is obvious that property (α_1) in the definition of the comparison equation of type III implies the following one:

(α_2) *In every interval* $(0, \gamma)$ *the function* $y(t) \equiv 0$ *is the only solution of* (14.3) *satisfying the conditions*

$$\lim_{t\to 0} y(t) = \lim_{t\to 0} y'(t) = 0 .$$

Now we will construct an example showing that

1° property (α_2) is essentially weaker than property (α_1),

2° if property (α_1) is replaced by property (α_2), then the third comparison theorem is—in general—false.

Indeed, let $\varphi(t)$ be differentiable for $t \geqslant 0$ and satisfy the conditions

1) $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$,

2) $\varphi'_+(0) = 0$, $\varphi'(t) \geqslant 0$ for $t > 0$,

3) $\varphi'(t)$ is continuous for $t > 0$,

4) $\lim_{t\to 0} \varphi'(t)$ does not exist.

It is not difficult to construct such a function. Consider the linear equation

$$\frac{dy}{dt} = \frac{\varphi'(t)}{\varphi(t)} y . \tag{14.13}$$

Its right-hand side is continuous and non-negative for $t > 0$, $y \geqslant 0$ and its general solution is $y = C\varphi(t)$. Hence, by 1) and 2), every solution of (14.13) satisfies conditions (14.10) and consequently equation (14.13) does not have property (α_1). On the other hand, by 4), property (α_2) holds true. Moreover, the function $\varphi(t)$ satisfies, with respect to equation (14.13), all the assumptions of the third comparison theorem and, by 1), is not $\leqslant 0$.

§ 15. Absolute value estimates. This section deals with a theorem that enables us to obtain estimates of absolute value of functions both to right and to left from the initial point.

Before stating the theorem we first prove a proposition on Dini's derivatives of the absolute value of a function.

PROPOSITION 15.1. *For a function* $\varphi(t)$ *defined in the neighborhood of* t_0 *we have the inequalities*

$$D_-|\varphi(t_0)| \leqslant |D_-\varphi(t_0)|\,, \tag{15.1}$$

$$D_+|\varphi(t_0)| \leqslant |D_+\varphi(t_0)|\,. \tag{15.2}$$

Proof. We prove, for instance, (15.1). Let t_ν be a sequence such that $t_\nu < t_0$, $t_\nu \to t_0$ and

$$\lim_{\nu\to\infty}\frac{\varphi(t_\nu)-\varphi(t_0)}{t_\nu - t_0} = D_-\varphi(t_0)\,, \qquad \lim_{\nu\to\infty}\frac{|\varphi(t_\nu)|-|\varphi(t_0)|}{t_\nu - t_0} \geqslant D_-|\varphi(t_0)|\,. \tag{15.3}$$

Since

$$\left|\frac{\varphi(t_\nu)-\varphi(t_0)}{t_\nu - t_0}\right| \geqslant \frac{\big||\varphi(t_\nu)|-|\varphi(t_0)|\big|}{|t_\nu - t_0|} = \left|\frac{|\varphi(t_\nu)|-|\varphi(t_0)|}{t_\nu - t_0}\right| \geqslant \frac{|\varphi(t_\nu)|-|\varphi(t_0)|}{t_\nu - t_0}\,,$$

inequality (15.1) follows from (15.3).

THEOREM 15.1. *Let a comparison system* (14.1) *of type* I (see § 14) *be given and let* $\Phi(x) = \big(\varphi_1(x), \ldots, \varphi_n(x)\big)$ *be continuous in the interval* $|x-x_0| < \gamma$. *Assume that* (1)

$$|\Phi(x_0)| \leqslant H\,, \tag{15.4}$$

where $H = (\eta_1, \ldots, \eta_n)$ *and put*

$$E_i = \{x\colon |x-x_0| < \min\big(\gamma, a_0(H)\big)\,,\ |\varphi_i(x)| > \omega_i(|x-x_0|;\, H)\} \qquad (i = 1, 2, \ldots, n)\,,$$

where $\Omega(t;\, H) = \big(\omega_1(t;\, H), \ldots, \omega_n(t;\, H)\big)$ *is the right-hand maximum solution of the comparison system through* $(0, H)$, *defined in the interval* $[0, a_0(H))$. *Suppose finally that*

$$|D_-\varphi_i(x)| \leqslant \sigma_i\big(|x-x_0|\,, |\Phi(x)|\big) \quad \text{for} \quad x \in E_i \ (i = 1, 2, \ldots, n)\,. \tag{15.5}$$

This being assumed, we have

$$|\Phi(x)| \leqslant \Omega(|x-x_0|;\, H) \quad \text{for} \quad |x-x_0| < \min\big(\gamma, a_0(H)\big)\,. \tag{15.6}$$

Proof. Since the assumptions of our theorem are invariant under the mapping $\xi = -x+2x_0$ (2), it is enough to prove (15.6) for the interval

$$0 \leqslant x - x_0 < \min(\gamma, a_0)\,. \tag{15.7}$$

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(1) For the definition of the symbol $|\;|$, see § 4.

(2) It should be remarked that the mapping $\xi = -x + 2x_0$ transforms the derivative D_- in (15.5) into D^+; however, Theorem 15.1 is true with D_- substituted by D^+. This explains how the invariance of assumptions is to be understood.

For this purpose, put

$$\psi_i(t) = |\varphi_i(x_0+t)| \quad \text{for} \quad 0 \leqslant t < \gamma \quad (i = 1, 2, \ldots, n),$$

$$\tilde{E}_i = \{t\colon 0 < t < \min(\gamma, a_0),\ \psi_i(t) > \omega_i(t; H)\} \quad (i = 1, 2, \ldots, n).$$

Then $\psi_i(t)$ are continuous in $[0, \gamma)$ and, by (15.4),

$$\Psi(0) \leqslant H.$$

Moreover, in view of Proposition 15.1 we have, by (15.5),

$$D_-\psi_i(t) \leqslant \sigma_i(t, \Psi(t)) \quad \text{for} \quad t \in \tilde{E}_i \quad (i = 1, 2, \ldots, n).$$

Hence, by the first comparison theorem, we get

$$\Psi(t) \leqslant \Omega(t; H) \quad \text{for} \quad 0 \leqslant t < \min(\gamma, a_0). \tag{15.8}$$

From (15.8) follows (15.6) in the interval (15.7), what completes the proof.

If $\Phi(x_0) = H > 0$ $(-H < 0)$, then it is useful to have some better estimate of $\Phi(x)$ from below (from above) in the neighborhood of x_0. Such an estimate is given in the following theorem:

THEOREM 15.2. *Under the assumptions of Theorem* 15.1 *suppose additionally that the right-hand members of the comparison system satisfy condition* $\overline{\mathrm{W}}_+$ (*i.e. are increasing with respect to all variables* y_j). *Assume that*

$$\Phi(x_0) = H > 0 \quad (\Phi(x_0) = -H < 0), \tag{15.9}$$

and

$$|D_-\varphi_i(t)| \leqslant \sigma_i(|x-x_0|, |\Phi(x)|) \quad \textit{for} \quad |x-x_0| < \min(\gamma, a_0) \quad (i = 1, 2, \ldots, n). \tag{15.10}$$

This being supposed, we have

$$\Phi(x) \geqslant 2H - \Omega(|x-x_0|; H) \quad (\Phi(x) \leqslant -2H + \Omega(|x-x_0|; H)) \tag{15.11}$$

in the interval

$$|x-x_0| < \min(\gamma, a_0). \tag{15.12}$$

Proof. We restrict ourselves to the case $\Phi(x_0) = H > 0$. Like in Theorem 15.1 it is sufficient to prove (15.11) in the interval (15.7). By Theorem 15.1, the inequalities

$$|\Phi(x)| \leqslant \Omega(x-x_0; H)$$

hold true in the interval (15.7). Hence, by (15.10) and by condition $\overline{\mathrm{W}}_+$, we get in (15.7)

$$D_-\varphi_i(t) \geqslant -\sigma_i(x-x_0, \Omega(x-x_0; H)) \quad (i = 1, 2, \ldots, n). \tag{15.13}$$

Put

$$\psi_i(x) = \varphi_i(x) - 2\eta_i + \omega_i(x-x_0; H) \quad (i = 1, 2, \ldots, n).$$

The functions $\psi_i(x)$ are continuous in (15.7) and, by (15.13), we have

$$D_-\psi_i(x) = D_-\varphi_i(x) + \omega_i'(x-x_0; H) = D_-\varphi_i(x) + \sigma_i(x-x_0, \Omega) \geqslant 0 \qquad (i = 1, 2, \ldots, n).$$

Hence, by Theorem 2.1, $\psi_i(x)$ are increasing in the interval (15.7) and since $\Psi(x_0) = 0$ by (15.9), we get $\Psi(x) \geqslant 0$, i.e.

$$\Phi(x) \geqslant 2H - \Omega(x-x_0; H)$$

in (15.7), what was to be proved.

As an immediate corollary of Theorem 15.2 we get the next theorem.

THEOREM 15.3. *Under the assumptions of Theorem* 15.2 *suppose that*

(15.14) $$H > \widetilde{H} \geqslant 0 \qquad (-H < -\widetilde{H} \leqslant 0),$$

where $\widetilde{H} = (\widetilde{\eta}_1, \ldots, \widetilde{\eta}_n)$. *Denote by* t_i *the least root of the equation in* t

(15.15) $$2\eta_i - \omega_i(t; H) = \widetilde{\eta}_i \qquad (-2\eta_i + \omega_i(t; H) = -\widetilde{\eta}_i)$$

if such a root exists in the interval $0 < t < a_0$; *if it does not exist, put* $t_i = +\infty$.

This being assumed, we have

(15.16) $$\Phi(x) > \widetilde{H} \qquad (\Phi(x) < -\widetilde{H})$$

in the interval

(15.17) $$|x-x_0| < \min(\gamma, a_0, t_1, \ldots, t_n).$$

Proof. Since $2\eta_i - \omega_i(0; H) = \eta_i > \widetilde{\eta}_i$, we have, by the definition of t_i,

(15.18) $$2\eta_i - \omega_i(t; H) > \widetilde{\eta}_i \qquad (i = 1, 2, \ldots, n)$$

in the interval

$$0 \leqslant t < \min(\gamma, a_0, t_1, \ldots, t_n).$$

Hence, by (15.11), we obtain (15.16) in the interval (15.17).

EXAMPLE. Let $\varphi_i(x)$ $(i = 1, 2, \ldots, n)$ be continuous in the interval

(15.19) $$|x-x_0| < \gamma$$

and satisfy differential inequalities

$$|D_-\varphi_i(x)| \leqslant K \sum_{j=1}^{n} |\varphi_j(x)| + L \qquad (i = 1, 2, \ldots, n;\ K \geqslant 0;\ L \geqslant 0).$$

The comparison system is here of the form

$$\frac{dy_i}{dt} = K \sum_{j=1}^{n} y_j + L \qquad (i = 1, 2, \ldots, n)$$

and its unique solution through the point $(0, \eta_1, \ldots, \eta_n)$ is

$$y_i = \begin{cases} \eta_i + Lt & \text{for} \quad K = 0 \quad (i = 1, 2, \ldots, n), \\ \left(\eta_i + \dfrac{L}{nK}\right)e^{nKt} - \dfrac{L}{nK} & \text{for} \quad K > 0 . \end{cases}$$

Hence, if $|\varphi_i(x_0)| \leqslant \eta_i$ $(i = 1, 2, \ldots, n)$, then, by Theorem 15.1,

$$|\varphi_i(x)| \leqslant \begin{cases} \eta_i + L|x - x_0| & \text{for} \quad K = 0 \quad (i = 1, 2, \ldots, n), \\ \left(\eta_i + \dfrac{L}{nK}\right)e^{nK|x-x_0|} - \dfrac{L}{nK} & \text{for} \quad K > 0 \end{cases}$$

in the interval (15.19). If, moreover, $\varphi_i(x_0) = \eta_i > 0$ $(i = 1, 2, \ldots, n)$, then, by Theorem 15.2,

$$\varphi_i(x) \geqslant \begin{cases} \eta_i - L|x - x_0| & \text{for} \quad K = 0 \quad (i = 1, 2, \ldots, n), \\ 2\eta_i - \left(\eta_i + \dfrac{L}{nK}\right)e^{nK|x-x_0|} + \dfrac{L}{nK} & \text{for} \quad K > 0 . \end{cases}$$

Let $K > 0$ and $\varphi_i(x_0) = \eta_i > \tilde{\eta}_i \geqslant 0$ $(i = 1, 2, \ldots, n)$. Equation (15.15) is now

$$2\eta_i - \left(\eta_i + \frac{L}{nK}\right)e^{nKt} + \frac{L}{nK} = \tilde{\eta}_i$$

and its only root is

$$t_i = \frac{1}{nK}\ln\left[1 + (\eta_i - \tilde{\eta}_i)\left(\eta_i + \frac{L}{nK}\right)^{-1}\right]. \tag{15.20}$$

Therefore, by Theorem 15.3, we have

$$\varphi_i(x) > \tilde{\eta}_i \quad (i = 1, 2, \ldots, n)$$

in the interval $|x - x_0| < \min(\gamma, t_1, \ldots, t_n)$, where t_i are given by formula (15.20).

Now, let $\varphi(t)$ be a vector-valued function of the real variable t, its values belonging to a normed linear space $\mathfrak{L}$ with the norm $\| \ \|$. Suppose $\varphi(t)$ is strongly differentiable at a point t_0. Then, using the properties of the norm we check that

$$\left|D_-\|\varphi(t_0)\|\right| \leqslant \|\varphi'(t_0)\| . \tag{15.21}$$

For vector-valued functions we can prove the following theorem.

THEOREM 15.4. *Let a comparison system* (14.1) *of type* I (see § 14) *be given and let* $\psi_i(x)$ $(i = 1, 2, \ldots, n)$ *be strongly continuous vector-valued functions of the real variable* x *on the interval* $|x - x_0| < \gamma$. *Assume that*

$$\|\psi_i(x_0)\| \leqslant \eta_i \quad (i = 1, 2, \ldots, n)$$

and put

$$\widetilde{E}_i = \{x : |x-x_0| < \min(\gamma, a_0(H)), \|\psi_i(x)\| > \omega_i(|x-x_0|; H)\}$$
$$(i = 1, 2, \ldots, n),$$

where $\Omega(t; H) = (\omega_1(t; H), \ldots, \omega_n(t; H))$ *is the right-hand maximum solution of the comparison system, issued from* $(0, H) = (0, \eta_1, \ldots, \eta_n)$ *and defined in* $[0, a_0(H))$. *Suppose finally that* $\psi_i(x)$ *is strongly differentiable in* $\widetilde{E}_i$ *and*

$$\|\psi_i'(x)\| \leqslant \sigma_i(|x-x_0|, \|\psi_1(x)\|, \ldots, \|\psi_n(x)\|) \quad \textit{for} \quad x \in \widetilde{E}_i \quad (i = 1, 2, \ldots, n).$$

This being assumed, we have

$$\|\psi_i(x)\| \leqslant \omega_i(|x-x_0|; H) \quad \textit{for} \quad |x-x_0| < \min(\gamma, a_0(H)) \quad (i = 1, 2, \ldots, n).$$

Proof. If we put

$$\varphi_i(x) = \|\psi_i(x)\| \qquad (i = 1, 2, \ldots, n)$$

and use (15.21), then all assumptions of Theorem 15.1 are satisfied.

Example. Suppose the real functions $\psi_1(x), \ldots, \psi_k(x)$ are differentiable on the interval $|x-x_0| < \gamma$ and satisfy the following initial inequality

$$\sqrt{\sum_{i=1}^{k} [\psi_i(x_0)]^2} \leqslant \eta,$$

and differential inequality

$$\sqrt{\sum_{i=1}^{k} [\psi_i'(x)]^2} \leqslant K \sqrt{\sum_{i=1}^{k} [\psi_i(x)]^2} + L \qquad (K > 0, L \geqslant 0)$$

in the interval $|x-x_0| < \gamma$. Then we have

$$\sqrt{\sum_{i=1}^{k} [\psi_i(x)]^2} \leqslant \left(\eta + \frac{L}{K}\right) e^{K|x-x_0|} - \frac{L}{K} \quad \text{in} \quad |x-x_0| < \gamma.$$

Indeed, the sequence of functions $\psi_1(x), \ldots, \psi_k(x)$ can be considered as a vector-valued function $\Psi(x)$ with values in the Euclidean space. The above initial and differential inequalities can now be rewritten in the form

$$\|\Psi(x_0)\| \leqslant \eta, \qquad \|\Psi'(x)\| \leqslant K\|\Psi(x)\| + L,$$

where $\|\ \|$ is the Euclidean norm. Hence, by Theorem 15.4 (in our case we have $n = 1$), we get in the interval $|x-x_0| < \gamma$

$$\|\Psi(x)\| \leqslant \omega(|x-x_0|; \eta),$$

where $\omega(t;\eta)$ is the unique solution through $(0,\eta)$ of the linear equation

$$\frac{dy}{dt} = Ky + L .$$

The last inequality is nothing else but the inequality that was to be proved.

§ 16. Infinite systems of ordinary differential inequalities and systems satisfying Carathéodory's conditions. This paragraph deals with analogues of Theorems 9.1 and 9.3 for countable systems of first order differential equations and inequalities.

The method of proving Theorems 9.1 and 9.3 for both finite and infinite systems, due to W. Mlak and C. Olech, which we use here is based on the validity of Theorems 9.1 and 9.5 for a single differential equation resp. inequality (see [30]).

We also discuss Theorems 9.1 and 9.3 for systems satisfying Carathéodory's conditions.

Consider a finite or countable system of ordinary differential equations

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, y_2, \dots) \qquad (i = 1, 2, \dots) . \tag{16.1}$$

By a *solution* of system (16.1) we mean a sequence of differentiable functions $y_i(t)$ $(i = 1, 2, \dots)$ in some interval Δ satisfying (16.1) for $t \in \Delta$. The right-hand maximum solution of (16.1) through a point $(t_0, \mathring{y}_1, \mathring{y}_2, \dots)$ is defined in a similar way like that of a finite system of differential equations.

Concerning the right-hand sides of system (16.1) we introduce the following assumptions:

Assumptions H. *The functions $\sigma_i(t, y_1, y_2, \dots)$ $(i = 1, 2, \dots)$ are defined and bounded in the region*

$$D : a < t < b \; , \; y_1, y_2, \dots \; \textit{arbitrary} .$$

For every fixed i, the function $\sigma_i(t, y_1, y_2, \dots)$ is increasing in the variables $y_1, \dots, y_{i-1}, y_{i+1}, \dots,$ and is continuous in D in the following sense: for any point $(t_0, Y_0) = (t_0, \mathring{y}_1, \mathring{y}_2, \dots) \in D$, if $t \to t_0$, $y_k \to \mathring{y}_k$ $(k = 1, 2, \dots)$, then $\sigma_i(t, Y) \to \sigma_i(t_0, Y_0)$.

Theorem 16.1. *Let the right-hand sides of system* (16.1) *satisfy Assumptions* H *and $(t_0, Y_0) = (t_0, \mathring{y}_1, \mathring{y}_2, \dots)$ be an arbitrary point of D. Then*

1° *there is the right-hand maximum solution $\omega_i(t)$ $(i = 1, 2, \dots)$ of* (16.1) *through (t_0, Y_0) in the interval*

$$t_0 \leqslant t < b \; , \tag{16.2}$$

2° *for any sequence* $\Phi(t) = (\varphi_1(t), \varphi_2(t), \ldots)$ *of continuous functions in the interval* (16.2), *such that*

$$\varphi_i(t_0) \leqslant \mathring{y}_i \quad (i = 1, 2, \ldots), \tag{16.3}$$

$$D_-\varphi_i(t) \leqslant \sigma_i(t, \varphi_1(t), \varphi_2(t), \ldots) \quad (i = 1, 2, \ldots) \tag{16.4}$$

in the interval $t_0 < t < b$, *we have*

$$\varphi_i(t) \leqslant \omega_i(t) \quad (i = 1, 2, \ldots) \tag{16.5}$$

in the interval (16.2).

Proof. Denote by $\mathcal{F}$ the family of sequences of continuous functions in the interval (16.2). Take an arbitrary sequence $\Phi(t) = (\varphi_1(t), \varphi_2(t), \ldots) \in \mathcal{F}$ and put

$$\sigma_i(t, y; \Phi) = \sigma_i(t, \varphi_1(t), \ldots, \varphi_{i-1}(t), y, \varphi_{i+1}(t), \ldots)$$

in the region

$$D^* : t_0 \leqslant t < b, \; y \text{ arbitrary}.$$

The function $\sigma_i(t, y; \Phi)$ is obviously continuous in D^*. Hence, by Theorem 9.1 (see Remark 9.1), there is the right-hand maximum solution of the single equation

$$\frac{dy}{dt} = \sigma_i(t, y; \Phi) \tag{16.6}$$

through the point $(t_0, \mathring{y}_i)$, reaching the boundary of D^* by its right-hand extremity. We denote it by $\omega_i(t; \Phi)$ and we claim that it exists in the interval (16.2). Indeed, the right-hand side of equation (16.6) is bounded and hence every solution of (16.6) is bounded in every bounded subinterval of (16.2). Therefore, if $\omega_i(t, y; \Phi)$ did not exist in the whole interval (16.2), it would be bounded and consequently it would not reach the boundary of D^* by its right-hand extremity. Now, denote by $\mathcal{F}_1$ the subfamily of $\mathcal{F}$, consisting of sequences satisfying (16.3) and (16.4). This family is not empty since, for instance, the sequence $\varphi_i(t) = \mathring{y}_i + \mu_i(t - t_0)$ $(i = 1, 2, \ldots)$, where

$$\mu_i = \inf_{(t,Y) \in D} \sigma_i(t, Y),$$

belongs obviously to $\mathcal{F}_1$. Let $\Phi(t)$ be an arbitrary sequence in $\mathcal{F}_1$; then, by (16.4), we have

$$D_-\varphi_i(t) \leqslant \sigma_i(t, \varphi_i(t); \Phi)$$

in the interval (16.2). Hence, by Theorem 9.5 applied to the single equation (16.6), it follows that for every fixed i

$$\varphi_i(t) \leqslant \omega_i(t; \Phi) \tag{16.7}$$

in the interval (16.2). Since the function $\sigma_i(t, y; \Phi)$ is bounded in D^*, uniformly with respect to $\Phi \in \mathcal{F}_1$, it follows that for every i the family of functions $\omega_i(t;\Phi)$ is bounded from above at every point $t \in [t_0, b)$ and equicontinuous in this interval. Hence

$$\omega_i(t) = \sup_{\Phi\in\mathcal{F}_1} \omega_i(t;\Phi)$$

exists in the interval (16.2) and is a continuous function. Moreover, it satisfies obviously the initial condition

$$\omega_i(t_0) = \mathring{y}_i .$$

By (16.7), inequalities (16.5) hold true for any sequence $\Phi(t)\in\mathcal{F}_1$. Hence points 1° and 2° of our theorem will be proved if we show that $\omega_i(t)$ $(i = 1, 2, \ldots)$ is a solution of system (16.1). To do this, we first observe that for two sequences $\Phi(t) = \big(\varphi_1(t), \varphi_2(t), \ldots\big) \in \mathcal{F}$ and $\widetilde{\Phi}(t) = \big(\widetilde{\varphi}_1(t), \widetilde{\varphi}_2(t), \ldots\big) \in \mathcal{F}$ such that

$$\varphi_i(t) \leqslant \widetilde{\varphi}_i(t) \qquad (i = 1, 2, \ldots) \tag{16.8}$$

we have

$$\omega_i(t;\Phi) \leqslant \omega_i(t;\widetilde{\Phi}) \qquad (i = 1, 2, \ldots) \tag{16.9}$$

in the interval (16.2). Indeed, by (16.8) and by the monotonicity conditions imposed on the functions $\sigma_i(t, Y)$, we get

$$\begin{aligned}\frac{d\omega_i(t;\Phi)}{dt} &= \sigma_i\big(t, \omega_i(t;\Phi); \Phi\big) = \sigma_i\big(t, \varphi_1(t), \ldots, \varphi_{i-1}(t), \omega_i(t;\Phi), \varphi_{i+1}(t), \ldots\big) \\ &\leqslant \sigma_i\big(t, \widetilde{\varphi}_1(t), \ldots, \widetilde{\varphi}_{i-1}(t), \omega_i(t;\Phi), \widetilde{\varphi}_{i+1}(t), \ldots\big) = \sigma_i\big(t, \omega_i(t;\Phi); \widetilde{\Phi}\big) .\end{aligned}$$

Hence, $\omega_i(t;\widetilde{\Phi})$ being the right-hand maximum solution of

$$\frac{dy}{dt} = \sigma_i(t, y; \widetilde{\Phi})$$

through $(t_0, \mathring{y}_i)$, we obtain (16.9) by Theorem 9.5. In particular, if $\Phi(t)$ is any sequence in $\mathcal{F}_1$ and $\widetilde{\Phi}(t) = \Omega(t) = \big(\omega_1(t), \omega_2(t), \ldots\big)$, it follows from (16.5) and (16.9) that

$$\omega_i(t;\Phi) \leqslant \omega_i(t;\Omega) \quad \text{for} \quad \Phi\in\mathcal{F}_1 \quad (i = 1, 2, \ldots) .$$

Therefore,

$$\omega_i(t) = \sup_{\Phi\in\mathcal{F}_1} \omega_i(t;\Phi) \leqslant \omega_i(t;\Omega) \qquad (i = 1, 2, \ldots) \tag{16.10}$$

and consequently, putting $\widetilde{\Omega}(t) = \big(\omega_1(t;\Omega), \omega_2(t;\Omega), \ldots\big)$, we get

$$\omega_i(t;\Omega) \leqslant \omega_i(t;\widetilde{\Omega}) \qquad (i = 1, 2, \ldots) . \tag{16.11}$$

On the other hand, we have

$$(16.12)\qquad \frac{d\omega_i(t;\,\Omega)}{dt} = \sigma_i\big(t,\,\omega_i(t;\,\Omega);\,\Omega\big)$$
$$= \sigma_i\big(t,\,\omega_1(t),\,\dots,\,\omega_{i-1}(t),\,\omega_i(t;\,\Omega),\,\omega_{i+1}(t),\,\dots\big)\,.$$

Hence, by (16.10) and by the monotonicity conditions, we conclude that

$$\frac{d\omega_i(t;\,\Omega)}{dt} \leqslant \sigma_i\big(t,\,\omega_1(t;\,\Omega),\,\omega_2(t;\,\Omega),\,\dots\big) \qquad (i = 1, 2, \dots)\,.$$

The last inequalities, together with the relations

$$\omega_i(t_0;\,\Omega) = \mathring{y}_i \qquad (i = 1, 2, \dots)\,,$$

mean that the sequence $\widetilde{\Omega}(t)$ satisfies (16.3) and (16.4) and consequently belongs to $\mathcal{F}_1$. Hence it follows that

$$(16.13)\qquad \omega_i(t;\,\widetilde{\Omega}) \leqslant \sup_{\Phi\in\mathcal{F}_1} \omega_i(t;\,\Phi) = \omega_i(t) \qquad (i = 1, 2, \dots)\,.$$

Inequalities (16.10), (16.11) and (16.13) imply that

$$\omega_i(t) = \omega_i(t;\,\Omega) \qquad (i = 1, 2, \dots)$$

in the interval (16.2) and consequently, by (16.12), it follows that $\omega_i(t)$ $(i = 1, 2, \dots)$ is a solution of system (16.1) in the interval (16.2), what was to be proved.

We introduce now Carathéodory's conditions. We say that the right-hand sides of the finite or countable system (16.1), defined in the region

$$D:\, a < t < b\,,\ y_1, y_2, \dots \text{ arbitrary}\,,$$

satisfy *Carathéodory's conditions* if

(α) for every fixed t, $\sigma_i(t, y_1, y_2, \dots)$ $(i = 1, 2, \dots)$ are continuous in the variables $y_1, y_2, \dots$ (in the sense specified in Assumptions H),

(β) for fixed $y_1, y_2, \dots,$ $\sigma_i(t,\,Y)$ $(i = 1, 2, \dots)$ are measurable in t and there exist functions $m_i(t)$ $(i = 1, 2, \dots)$, Lebesgue integrable on every bounded subinterval of (a, b), such that

$$|\sigma_i(t,\,Y)| \leqslant m_i(t) \qquad (i = 1, 2, \dots)\,.$$

By a *solution* of system (16.1), satisfying Caratheodory's conditions, we mean a sequence of functions $y_i(t)$ $(i = 1, 2, \dots)$ which are absolutely continuous on some interval Δ and satisfy (16.1) almost everywhere on Δ.

It is a well-known theorem, due to Carathéodory (see for instance [7]), that *under the above conditions in case of a single equation there is a solution of* (16.1) *through every point* $(t_0,\,Y_0)\in D$, *defined on the interval* (a, b).

The right-hand maximum solution is defined as usually. Now we have the following theorem.

THEOREM 16.2. *Let the right-hand sides of the finite or countable system* (16.1) *satisfy Carathéodory's conditions in the region* D. *Suppose that, for every fixed* i, *the function* $\sigma_i(t, Y)$ *is increasing in the variables* $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots$, *and let* $(t_0, Y_0) = (t_0, \mathring{y}_1, \mathring{y}_2, \ldots) \in D$. *Under the above assumptions the following propositions hold true*:

1° *there is the right-hand maximum solution* $\omega_i(t)$ $(i = 1, 2, \ldots)$ *of* (16.1) *through* (t_0, Y_0) *in the interval* (16.2),

2° *for any sequence* $(\varphi_1(t), \varphi_2(t), \ldots)$ *of absolutely continuous functions on* (16.2), *satisfying initial inequalities*

$$\varphi_i(t_0) \leqslant \mathring{y}_i \quad (i = 1, 2, \ldots)$$

and differential inequalities

$$\varphi_i'(t) \leqslant \sigma_i\big(t, \varphi_1(t), \varphi_2(t), \ldots\big) \quad (i = 1, 2, \ldots)$$

almost everywhere on the interval $t_0 < t < b$, *we have*

$$\varphi_i(t) \leqslant \omega_i(t) \quad (i = 1, 2, \ldots) \quad \textit{on } (16.2).$$

Proof. It is sufficient to prove Theorem 16.2 in the case when the system (16.1) reduces to a single equation with one unknown function. Indeed, it is not difficult to check that adequately modified arguments used in the proof of Theorem 16.1 permit to derive the validity od Theorem 16.2 from its validity in the case of one equation.

Let us then consider one equation

$$\frac{dy}{dt} = \sigma(t, y) \tag{16.14}$$

and assume its right-hand side to satisfy Carathéodory's conditions in the region

$$\widetilde{D}\colon\ a < t < b,\ y \text{ arbitrary}.$$

Let $(t_0, y_0) \in \widetilde{D}$. What concerns the existence of the right-hand maximum solution $\omega(t)$ of (16.14) through (t_0, y_0) on the interval (16.2) we refer to [7] and we restrict ourselves to the proof of point 2°. Let $\varphi(t)$ be an absolutely continuous function on (16.2) and suppose that

$$\varphi(t_0) \leqslant y_0\,, \tag{16.15}$$

$$\varphi'(t) \leqslant \sigma\big(t, \varphi(t)\big) \text{ almost everywhere on } (t_0, b)\,. \tag{16.16}$$

We have to prove that

$$\varphi(t) \leqslant \omega(t) \quad \text{on} \quad [t_0, b)\,. \tag{16.17}$$

To this purpose, consider an auxiliary equation

$$\frac{dy}{dt} = \tau(t, y)\,, \tag{16.18}$$

where

$$\tau(t, y) = \begin{cases} \sigma(t, y) & \text{for} \quad y \geqslant \varphi(t), \\ \sigma\big(t, \varphi(t)\big) & \text{for} \quad y \leqslant \varphi(t). \end{cases}$$

It may be checked that the right-hand side of (16.18) satisfies Caratheodory's conditions. Denote by $y(t)$ a solution of (16.18) through (t_0, y_0), defined in the interval $[t_0, b)$. We will show that

$$\varphi(t) \leqslant y(t) \quad \text{on} \quad [t_0, b). \tag{16.19}$$

Suppose the contrary, i.e. $\varphi(t_2) > y(t_2)$ for some $t_2 \in (t_0, b)$. Then, since, by (16.15), $\varphi(t_0) \leqslant y_0 = y(t_0)$, there would exist a t_1, $t_0 < t_1 < t_2$, such that

$$\varphi(t_1) = y(t_1), \tag{16.20}$$

$$\varphi(t) > y(t) \quad \text{on} \quad (t_1, t_2). \tag{16.21}$$

On the other hand, by (16.16) and (16.21) and by the definition of $\tau(t, y)$, we have almost everywhere in the interval (t_1, t_2)

$$\varphi'(t) - y'(t) \leqslant \sigma\big(t, \varphi(t)\big) - \tau\big(t, y(t)\big) = \sigma\big(t, \varphi(t)\big) - \sigma\big(t, \varphi(t)\big) = 0.$$

Hence, both functions $\varphi(t)$ and $y(t)$ being absolutely continuous, the function $\varphi(t) - y(t)$ is, by Theorem 3.1, decreasing on the interval $[t_1, t_2]$ and consequently we have, by (16.20),

$$\varphi(t) \leqslant y(t) \quad \text{on} \quad (t_1, t_2),$$

what contradicts (16.21). Thus inequality (16.19) is proved. But, from this inequality and from the definition of $\tau(t, y)$ it follows that $y(t)$ is a solution of the equation (16.14) through (t_0, y_0). Hence, $\omega(t)$ being its right-hand maximum solution through (t_0, y_0), we get

$$y(t) \leqslant \omega(t) \quad \text{on} \quad [t_0, b).$$

The last inequality together with (16.19) implies (16.17).

CHAPTER IV

ORDINARY DIFFERENTIAL INEQUALITIES OF HIGHER ORDER AND SOME INTEGRAL INEQUALITIES

§ 17. Preliminary remarks and definitions. Consider an ordinary differential equation of order $n \geqslant 2$

$$y^{(n)}(t) = \sigma\big(t, y(t), y'(t), \dots, y^{(n-1)}(t)\big)\,, \tag{17.1}$$

with the right-hand member $\sigma(t, y_0, y_1, \dots, y_{n-1})$ continuous in an open region D of the space $(t, y_0, y_1, \dots, y_{n-1})$. Let $(t_0, Y_0) = (t_0, \mathring{y}_0, \mathring{y}_1, \dots, \mathring{y}_{n-1})$ and introduce Cauchy initial conditions

$$y^{(j)}(t_0) = \mathring{y}_j \quad (j = 0, 1, \dots, n-1)\,. \tag{17.2}$$

It is a well-known fact that the Cauchy problem for equation (17.1) with initial conditions (17.2) is equivalent to the Cauchy problem for the system of first order differential equations

$$\begin{aligned} \frac{dy_i}{dt} &= y_{i+1} \quad (i = 0, 1, \dots, n-2)\,, \\ \frac{dy_{n-1}}{dt} &= \sigma(t, y_0, y_1, \dots, y_{n-1}) \end{aligned} \tag{17.3}$$

with initial values

$$y_j(t_0) = \mathring{y}_j \quad (j = 0, 1, \dots, n-1)\,. \tag{17.4}$$

This equivalence is understood in the following sense. If $y(t)$ is a solution of problem (17.1), (17.2), then $\big(y_0(t), \dots, y_{n-1}(t)\big)$ defined by the formulas

$$y_j(t) = y^{(j)}(t) \quad (j = 0, 1, \dots, n-1) \tag{17.5}$$

is a solution of problem (17.3), (17.4). Vice versa, if $\big(y_0(t), \dots, y_{n-1}(t)\big)$ is a solution of problem (17.3), (17.4), then $y(t) = y_0(t)$ is that of problem (17.1), (17.2).

A solution of equation (17.1) is said *to reach the boundary of D by its right-hand (left-hand) extremity* if the same is true for the corresponding solution of system (17.3) (see § 7).

By the mapping

$$\tau = -t\,, \quad \eta = y\,, \tag{17.6}$$

a function $y(t)$ of class C^n is transformed into the function $\eta(\tau) = y(-\tau)$ so that

$$\eta^{(j)}(\tau) = (-1)^j y^{(j)}(-\tau) \quad (j = 0, 1, \ldots, n)\,. \tag{17.7}$$

Hence, the mapping (17.6) transforms equation (17.1) into

$$\eta^{(n)}(\tau) = (-1)^n \sigma\big(-\tau, \eta(\tau), -\eta'(\tau), \ldots, (-1)^{n-1}\eta^{(n-1)}(\tau)\big)\,. \tag{17.8}$$

The corresponding system (17.3) is now

$$\begin{aligned} \frac{d\eta_i}{d\tau} &= \eta_{i+1} \quad (i = 0, 1, \ldots, n-2)\,, \\ \frac{d\eta_{n-1}}{d\tau} &= (-1)^n \sigma\big(-\tau, \eta_0, -\eta_1, \ldots, (-1)^{n-1}\eta_{n-1}\big)\,. \end{aligned} \tag{17.9}$$

Condition W_+. The right-hand member of equation (17.1) will be said to satisfy condition W_+ with respect to $Y = (y_0, y_1, \ldots, y_{n-1})$ in D if the right-hand sides of the corresponding system (17.3) satisfy condition W_+ with regard to Y (see § 4). This condition obviously means that for any two points $(t, Y) = (t, y_0, \ldots, y_{n-2}, y_{n-1}) \in D$ and $(t, \tilde{Y}) = (t, \tilde{y}_0, \ldots, \tilde{y}_{n-2}, y_{n-1}) \in D$ such that $y_i \leqslant \tilde{y}_i$ $(i = 0, 1, \ldots, n-2)$, we have

$$\sigma(t, Y) \leqslant \sigma(t, \tilde{Y})\,. \tag{17.10}$$

Condition $\overline{W}_+$. If inequality (17.10) is satisfied for any two points $(t, Y) = (t, y_0, \ldots, y_{n-1}) \in D$ and $(t, \tilde{Y}) = (t, \tilde{y}_1, \ldots, \tilde{y}_{n-1}) \in D$ such that $y_j \leqslant \tilde{y}_j$ $(j = 0, 1, \ldots, n-1)$, then the right-hand member of equation (17.1) is said to satisfy condition $\overline{W}_+$ with respect to Y in D.

It is obvious that in this case the right-hand sides of the corresponding system (17.3) satisfy condition $\overline{W}_+$ (see § 4).

Condition W_-. The right-hand member of equation (17.1) will be said to satisfy condition W_- if the right-hand side of the transformed equation (17.8) satisfies condition W_+.

This is equivalent to saying that for any two points $(t, Y) = (t, y_0, \ldots, y_{n-2}, y_{n-1}) \in D$ and $(t, \tilde{Y}) = (t, \tilde{y}_0, \ldots, \tilde{y}_{n-2}, y_{n-1}) \in D$ such that

$$(-1)^i y_i \leqslant (-1)^i \tilde{y}_i \quad (i = 0, 1, \ldots, n-2)$$

the inequality

$$(-1)^n \sigma(t, Y) \leqslant (-1)^n \sigma(t, \tilde{Y})$$

holds true.

§ 18. Maximum and minimum solution of an nth order ordinary differential equation. A solution $\omega^+(t; t_0, Y_0) = \omega^+(t; t_0, \mathring{y}_0, \ldots, \mathring{y}_{n-1})$ $(\omega_+(t; t_0, Y_0))$ of equation (17.1), satisfying initial conditions (17.2) and defined in an interval $\Delta_+ = [t_0, a)$, is called a *right-hand maximum* (*minimum*) *solution of* (17.1) *through* (t_0, Y_0) if the corresponding solution of system (17.3) with initial data (17.4) is the right-hand maximum (minimum) solution of system (17.3) through (t_0, Y_0) (see § 5). This comes to saying that for any solution $y(t)$ of (17.1), satisfying initial conditions (17.2) and defined in some interval $\tilde{\Delta}_+ = [t_0, \tilde{\alpha})$, the inequalities

$$y^{(j)}(t) \leqslant [\omega^+(t; t_0, Y_0)]^{(j)} \quad (y^{(j)}(t) \geqslant [\omega_+(t; t_0, Y_0)]^{(j)}) \quad (j = 0, 1, \dots, n-1)$$

hold true for $t \in \Delta_+ \cap \tilde{\Delta}_+$.

A solution $\omega^-(t; t_0, Y_0)$ $(\omega_-(t; t_0, Y_0))$ of equation (17.1), satisfying (17.2) and defined in an interval $\Delta_- = (\beta, t_0)$, is called a *left-hand maximum (minimum) solution of* (17.1) *through* (t_0, Y_0) if it is transformed by the mapping (17.6) into the right-hand maximum (minimum) solution of the transformed equation (17.8) through $(-t_0, \mathring{y}_0, -\mathring{y}_1, \dots, (-1)^{n-1}\mathring{y}_{n-1})$. This is equivalent to saying that for any solution $y(t)$ of (17.1), satisfying (17.2) and defined in some interval $\tilde{\Delta}_- = (\tilde{\beta}, t_0]$, the inequalities

$$(-1)^j y^{(j)}(t) \leqslant (-1)^j [\omega^-(t; t_0, Y_0)]^{(j)}, \quad ((-1)^j y^{(j)}(t) \geqslant (-1)^j [\omega_-(t; t_0, Y_0)]^{(j)}) \quad (j = 0, 1, \dots, n-1)$$

are true for $t \in \Delta_- \cap \tilde{\Delta}_-$.

By Theorem 9.1, the following theorem is an immediate consequence of the definition of the right-hand maximum (minimum) solution.

THEOREM 18.1. *Let the right-hand member* $\sigma(t, y_0, \dots, y_{n-1})$ *of equation* (17.1) *be continuous and satisfy condition* W_+ (see § 17) *with respect to* $Y = (y_0, \dots, y_{n-1})$ *in an open region* D. *Then through every* $(t_0, Y_0) \in D$ *there is the right-hand maximum (minimum) solution of* (17.1), *reaching the boundary of* D *by its right-hand extremity* (see § 17).

Now, from Theorem 18.1 we deduce, by the definition of the left-hand maximum (minimum) solution and by the definition of condition W_- (see § 17), the next theorem.

THEOREM 18.2. *If the right-hand side of equation* (17.1) *is continuous and satisfies condition* W_- (see § 17) *with respect to* Y *in an open region* D, *then through every* $(t_0, Y_0) \in D$ *there is the left-hand maximum (minimum) solution of* (17.1), *reaching the boundary of* D *by its left-hand extremity* (see § 17).

§ 19. Basic theorems on nth order ordinary differential inequalities. We start with the following general remark. Consider an nth order differential inequality of the form

$$D_-\varphi^{(n-1)}(t) \leqslant \sigma\big(t, \varphi(t), \varphi'(t), \dots, \varphi^{(n-1)}(t)\big) \tag{19.1}$$

with initial inequalities

$$\varphi^{(j)}(t_0) \leqslant \mathring{y}_j \quad (j = 0, 1, \dots, n-1), \tag{19.2}$$

where $\varphi(t)$ is of class C^{n-1}. It is clear that if $\varphi(t)$ is a solution of (19.1) and (19.2), then $\big(\varphi_0(t), \dots, \varphi_{n-1}(t)\big)$, defined by the formulas

$$\varphi_j(t) = \varphi^{(j)}(t) \quad (j = 0, 1, \dots, n-1), \tag{19.3}$$

is a solution of the system

$$(19.4)\qquad \frac{d\varphi_i(t)}{dt} = \varphi_{i+1}(t) \quad (i = 0, 1, \ldots, n-2),$$
$$D_-\varphi_{n-1}(t) \leqslant \sigma\big(t, \varphi_0(t), \ldots, \varphi_{n-1}(t)\big)$$

with initial inequalities

$$(19.5)\qquad \varphi_j(t_0) \leqslant \mathring{y}_j \quad (j = 0, 1, \ldots, n-1).$$

Following this remark and the definitions and results of §§ 17, 18 we get the next theorem, by Theorems 9.3 and 9.4 applied to the system (19.4).

THEOREM 19.1. *Let the right-hand member* $\sigma(t, y_0, \ldots, y_{n-1})$ *of equation* (17.1) *be continuous and satisfy condition* W_+ *with respect to* $Y = (y_0, \ldots, y_{n-1})$ (see § 17) *in an open region* D. *Let* $(t_0, Y_0) = (t_0, \mathring{y}_0, \ldots, \mathring{y}_{n-1}) \in D$ *and consider the right-hand maximum (minimum) solution* $\omega^+(t; t_0, Y_0)$ $(\omega_+(t; t_0, Y_0))$ (see § 18) *of* (17.1) *through* (t_0, Y_0), *defined in the interval* $\Delta_+ = [t_0, a)$ *and reaching the boundary of* D *by its right-hand extremity* (see § 17). *Suppose that* $\varphi(t)$ *is of class* C^{n-1} *on the interval* $\tilde{\Delta}_+ = [t_0, \tilde{a})$ *and that* $\big(t, \varphi(t), \varphi'(t), \ldots, \varphi^{(n-1)}(t)\big) \in D$.

Under these assumptions, if

$$(19.6)\qquad \varphi^{(j)}(t_0) \leqslant \mathring{y}_j \quad (\varphi^{(j)}(t_0) \geqslant \mathring{y}_j) \quad (j = 0, 1, \ldots, n-1)$$

and

$$(19.7)\qquad D_-\varphi^{(n-1)}(t) \leqslant \sigma\big(t, \varphi(t), \varphi'(t), \ldots, \varphi^{(n-1)}(t)\big)$$
$$\big(D^-\varphi^{(n-1)}(t) \geqslant \sigma\big(t, \varphi(t), \varphi'(t), \ldots, \varphi^{(n-1)}(t)\big)\big) \quad \text{in} \quad \tilde{\Delta}_+,$$

then

$$\varphi^{(j)}(t) \leqslant [\omega^+(t; t_0, Y_0)]^{(j)} \quad (\varphi^{(j)}(t) \geqslant [\omega_+(t; t_0, Y_0)]^{(j)}) \quad (j = 0, 1, \ldots, n-1)$$

for $t \in \Delta_+ \cap \tilde{\Delta}_+$.

The derivative D_- in the differential inequality (19.7) can be substituted by any of the three remaining Dini's derivatives.

Remark 19.1. We want now to explain why in Theorem 19.1 the apparently strong assumption on $\varphi(t)$ to be of class C^{n-1} in Δ_+ is an essential one. To this purpose, let us first introduce the following notation for an arbitrary function $\varphi(t)$ in $\tilde{\Delta}_+$:

$$D_-^{(0)}\varphi(t) = \varphi(t) \quad \text{for} \quad t \in \tilde{\Delta}_+,$$
$$D_-^{(j+1)}\varphi(t) = D_-\big(D_-^{(j)}\varphi(t)\big) \quad \text{for} \quad t \in \tilde{\Delta}_+,$$

whenever $D_-^{(j)}\varphi(t)$ is finite in $\tilde{\Delta}_+$. We might now consider, instead of (19.7), the differential inequality

$$(19.8)\qquad D_-^{(n}\varphi(t) \leqslant \sigma\big(t, \varphi(t), D_-^{(1)}\varphi(t), \ldots, D_-^{(n-1)}\varphi(t)\big) \quad \text{in} \quad \tilde{\Delta}_+$$

with initial inequalities

$$D_-^{(j)}\varphi(t_0) \leqslant \mathring{y}_j \quad (j = 0, 1, \ldots, n-1) \tag{19.9}$$

for a function having all derivatives $D_-^{(j)}$ $(j = 0, 1, \ldots, n-1)$ finite in $\tilde{\Delta}_+$. It is evident that if $\varphi(t)$ is a solution of (19.8) and (19.9), having the above regularity, then $(\varphi_0(t), \ldots, \varphi_{n-1}(t))$, defined by the formulas

$$\varphi_j(t) = D_-^{(j)}\varphi(t) \quad (j = 0, 1, \ldots, n-1),$$

is a solution of the system

$$\begin{aligned} D_-\varphi_i(t) &= \varphi_{i+1}(t) \quad (i = 0, 1, \ldots, n-2), \\ D_-\varphi_{n-1}(t) &\leqslant \sigma(t, \varphi_0(t), \ldots, \varphi_{n-1}(t)) \end{aligned} \tag{19.10}$$

with the initial inequalities (19.5). Hence it follows that the apparently stronger variant of Theorem 19.1 with (19.7) replaced by (19.8) is equivalent with Theorem 9.3 for system (19.9). But for the validity of Theorem 9.3 it is essential to assume $\varphi_j(t)$ $(j = 0, 1, \ldots, n-1)$ to be continuous in $\tilde{\Delta}_+$. Thus, the continuity of the derivatives $D_-^{(j)}\varphi(t)$ $(j = 0, 1, \ldots, n-1)$ in $\tilde{\Delta}_+$ is essential for the above variant of Theorem 19.1; but, by Corollary 2.2, continuity of $D_-^{(j)}\varphi(t)$ implies that of $\varphi^{(j)}(t)$. In this way we are led to that regularity of $\varphi(t)$ which was required in Theorem 19.1.

Now, notice that if we apply the mapping (17.6) and put $\psi(\tau) = \varphi(-\tau)$, then the initial inequalities (19.6) are transformed into

$$(-1)^j\psi^{(j)}(-t_0) \leqslant (-1)^j[(-1)^j\mathring{y}_j] \quad (j = 0, 1, \ldots, n-1)$$

and the differential inequality (19.7) into

$$(-1)^n D^+\psi^{(n-1)}(\tau) \leqslant (-1)^n\left[(-1)^n\sigma\left(-\tau, \psi(\tau), -\psi'(\tau), \ldots, (-1)^{n-1}\psi^{(n-1)}(\tau)\right)\right].$$

Hence, applying the mapping (17.6) we get, by the definitions and results of §§ 17, 18 the next theorem from Theorem 19.1.

THEOREM 19.2. *Let the right-hand member of equation* (17.1) *be continuous and satisfy condition* W$_-$ *with respect to* Y (see § 17) *in an open region* D. *Let* $(t_0, Y_0) \in D$ *and consider the left-hand maximum* (*minimum*) *solution* $\omega^-(t; t_0, Y_0)$ $(\omega_-(t; t_0, Y_0))$ (see § 18) *of* (17.1) *through* (t_0, Y_0), *defined in the interval* $\Delta_- = (\beta, t_0]$ *and reaching the boundary of* D *by its left-hand extremity. Suppose that* $\varphi(t)$ *is of class* C^{n-1} *in the interval* $\tilde{\Delta}_- = (\tilde{\beta}, t_0]$ *and that* $(t, \varphi(t), \varphi'(t), \ldots, \varphi^{(n-1)}(t)) \in D$.

Under these assumptions, if

$$(-1)^j\varphi^{(j)}(t_0) \leqslant (-1)^j\mathring{y}_j \quad ((-1)^j\varphi^{(j)}(t_0) \geqslant (-1)^j\mathring{y}_j) \quad (j = 0, 1, \ldots, n-1)$$

and

$$(-1)^n D^+\varphi^{(n-1)}(t) \leqslant (-1)^n \sigma\big(t, \varphi(t), \varphi'(t), \dots, \varphi^{(n-1)}(t)\big)$$

$$\big((-1)^n D_+\varphi^{(n-1)}(t) \geqslant (-1)^n \sigma\big(t, \varphi(t), \varphi'(t), \dots, \varphi^{(n-1)}(t)\big)\big),$$

then

$$(-1)^j\varphi^{(j)}(t) \leqslant (-1)^j[\omega^-(t;\, t_0,\, Y_0)]^{(j)} \qquad \big((-1)^j\varphi^{(j)}(t) \geqslant (-1)^j[\omega_-(t;\, t_0,\, Y_0)]^{(j)}\big)$$
$$(j = 0, 1, \dots, n-1)$$

for $\Delta_- \cap \tilde{\Delta}_-$.

Theorem 19.2 is true with any of the remaining Dini's derivatives instead of D^+ (D_+).

§ 20. Comparison equation of order n. Equation (17.1) will be called *comparison equation of order n* if the corresponding system (17.3) is a comparison system of type I (see § 14), i.e. if its right-hand side $\sigma(t, y_0, y_1, \dots, y_{n-1})$ is non-negative and continuous and satisfies condition W_+ (see § 17) with respect to Y in

$$\bar{Q}\colon\ t \geqslant 0\,,\ y_j \geqslant 0 \qquad (j = 0, 1, \dots, n-1).$$

Proposition 14.1 implies the following result:

Through every point $(0, H) = (0, \eta_0, \eta_1, \dots, \eta_{n-1})$ *there is the right-hand maximum solution of a comparison equation of order n which we denote by* $\omega(t;\, H)$ *and its existence interval by* $\Delta(H) = [0, a_0(H))$.

Moreover, we have either $a_0(H) = +\infty$, *or* $a_0(H)$ *is finite and*

$$\lim_{t\to a_0} \sqrt{\sum_{j=0}^{n-1} [\omega^{(j)}(t;\, H)]^2} = +\infty\,.$$

Comparison Theorem. *A comparison equation* (17.1) *being given, let $\varphi(t)$ be of class C^{n-1} in an interval $\Delta = [0, \gamma)$ and suppose that* $\varphi^{(j)}(t) \geqslant 0$ $(j = 0, 1, \dots, n-1)$. *Under these assumptions, if*

$$\varphi^{(j)}(0) \leqslant \eta_j \qquad (j = 0, 1, \dots, n-1)$$

and

$$D_-\varphi^{(n-1)}(t) \leqslant \sigma\big(t, \varphi(t), \dots, \varphi^{(n-1)}(t)\big) \qquad \text{in} \quad \Delta\,,$$

then

$$\varphi^{(j)}(t) \leqslant \omega^{(j)}(t;\, H) \qquad (j = 0, 1, \dots, n-1)$$

for $t \in \Delta(H) \cap \Delta$, *where $\omega(t;\, H)$ is the right-hand maximum solution of* (17.1) *through the point* $(0, H) = (0, \eta_0, \eta_1, \dots, \eta_{n-1})$.

This theorem is an immediate consequence of Theorem 19.1.

§ 21. Absolute value estimates. Let a comparison equation (17.1) of order n (see § 20) be given and consider for a function $\varphi(x)$ of class C^{n-1} the differential inequality

$$|D_-\varphi^{(n-1)}(x)| \leqslant \sigma\big(|x - x_0|, |\varphi(x)|, \dots, |\varphi^{(n-1)}(x)|\big)\,. \tag{21.1}$$

It is clear that if $\varphi(x)$ is a solution of (21.1), then $\big(\varphi_0(x), \ldots, \varphi_{n-1}(x)\big)$ defined by the formulas $\varphi_j(x) = \varphi^{(j)}(x)$ $(j = 0, 1, \ldots, n-1)$ is a solution of the system

$$\left|\frac{d\varphi_i}{dx}\right| = |\varphi_{i+1}| \quad (i = 0, 1, \ldots, n-2), \tag{21.2}$$
$$|D_-\varphi_{n-1}| \leqslant \sigma(|x-x_0|, |\varphi_0|, \ldots, |\varphi_{n-1}|).$$

By this remark, the next theorem follows from Theorem 15.1.

THEOREM 21.1. *Let a comparison equation* (17.1) (see § 20) *be given and assume $\varphi(x)$ to be of class C^{n-1} in the interval $|x-x_0| < \gamma$. Suppose that*

$$|\varphi^{(j)}(x_0)| \leqslant \eta_j \quad (j = 0, 1, \ldots, n-1)$$

and

$$|D_-\varphi^{(n-1)}(x)| \leqslant \sigma\big(|x-x_0|, |\varphi(x)|, |\varphi'(x)|, \ldots, |\varphi^{(n-1)}(x)|\big) \quad \textit{for} \quad |x-x_0| < \gamma.$$

Under these assumptions we have the inequalities

$$|\varphi^{(j)}(x)| \leqslant \omega^{(j)}(|x-x_0|; H) \quad (j = 0, 1, \ldots, n-1)$$

for $|x-x_0| < \min\big(\gamma, a_0(H)\big)$, where $\omega(t; H)$ is the right-hand maximum solution of (17.1) *through $(0, H) = (0, \eta_0, \ldots, \eta_{n-1})$, defined in the interval $[0, a_0(H))$.*

Next, from Theorem 15.2 we derive the following

THEOREM 21.2. *Under the assumptions of Theorem* 21.1 *suppose additionally that the right-hand member $\sigma(t, y_0, y_1, \ldots, y_{n-1})$ of the comparison equation* (17.1) *satisfies condition $\overline{\mathrm{W}}_+$ (i.e. increases with respect to all variables y_j) and that*

$$\varphi^{(j)}(x_0) = \eta_j > 0 \quad (\varphi^{(j)}(x_0) = -\eta_j < 0) \quad (j = 0, 1, \ldots, n-1).$$

This being assumed we have

$$\varphi^{(j)}(x) \geqslant 2\eta_j - \omega^{(j)}(|x-x_0|; H) \quad (\varphi^{(j)}(x) \leqslant -2\eta_j + \omega^{(j)}(|x-x_0|; H))$$
$$(j = 0, 1, \ldots, n-1)$$

in the interval $|x-x_0| < \min\big(\gamma, a_0(H)\big)$.

As an immediate corollary of Theorem 21.2 we obtain the next theorem.

THEOREM 21.3. *Under the assumptions of Theorem* 21.2 *suppose that*

$$\eta_j > \tilde{\eta}_j \geqslant 0 \quad (-\eta_j < -\tilde{\eta}_j \leqslant 0) \quad (j = 0, 1, \ldots, n-1)$$

Denote by t_j the least root of the equation in t

$$2\eta_j - \omega^{(j)}(t; H) = \tilde{\eta}_j \quad (-2\eta_j + \omega^{(j)}(t; H) = -\tilde{\eta}_j) \tag{21.3}$$

if such a root exists in the interval $0 < t < a_0$; if it does not exist, put $t_j = +\infty$.

Under these hypotheses we have

$$\varphi^{(j)}(x) > \tilde{\eta}_j \qquad (\varphi^{(j)}(x) < -\tilde{\eta}_j) \qquad (j = 0, 1, \ldots, n-1)$$

in the interval $|x - x_0| < \min(\gamma, a_0, t_0, t_1, \ldots, t_{n-1})$.

EXAMPLE 21.1. Let $\varphi(x)$ be of class C^1 in the interval

$$|x - x_0| < \gamma\,. \tag{21.4}$$

Suppose that $\varphi(x)$ satisfies the initial inequalities

$$|\varphi(x_0)| \leqslant \eta_0\,, \qquad |\varphi'(x_0)| \leqslant \eta_1$$

and the differential inequality

$$|D_-\varphi'(x)| \leqslant a\,|\varphi'(x)| \qquad (a > 0)\,.$$

The comparison equation of second order is here

$$y''(t) = ay'(t)$$

and its unique solution, satisfying the initial conditions

$$y(0) = \eta_0\,, \qquad y'(x) = \eta_1\,,$$

is

$$\omega(t) = \frac{\eta_1}{a}(e^{at} - 1) + \eta_0\,.$$

By Theorem 21.1, we have in the interval (21.4)

$$|\varphi(x)| \leqslant \frac{\eta_1}{a}(e^{a|x-x_0|} - 1) + \eta_0\,, \qquad |\varphi'(x)| \leqslant \eta_1 e^{a|x-x_0|}\,.$$

If, moreover, we assume that

$$\varphi(x_0) = \eta_0 > 0\,, \qquad \varphi'(x_0) = \eta_1 > 0\,,$$

then, by Theorem 21.2,

$$\varphi(x) \geqslant \eta_0 - \frac{\eta_1}{a}(e^{a|x-x_0|} - 1)\,, \qquad \varphi'(x) \geqslant 2\eta_1 - \eta_1 e^{a|x-x_0|}$$

in the interval (21.4). Suppose finally that $\eta_0 > \tilde{\eta}_0 \geqslant 0$, $\eta_1 > \tilde{\eta}_1 \geqslant 0$. Equations (21.3) have now the form

$$\eta_0 - \frac{\eta_1}{a}(e^{at} - 1) = \tilde{\eta}_0\,, \qquad 2\eta_1 - \eta_1 e^{at} = \tilde{\eta}_1\,.$$

Their only solutions are respectively

$$t_0 = \frac{1}{a}\ln\left(1 + \frac{a(\eta_0 - \tilde{\eta}_0)}{\eta_1}\right), \qquad t_1 = \frac{1}{a}\ln\left(1 + \frac{\eta_1 - \tilde{\eta}_1}{\eta_1}\right).$$

Hence, by Theorem 21.3, we have

$$\varphi(x) > \tilde{\eta}_0\,, \qquad \varphi'(x) > \tilde{\eta}_1$$

in the interval $|x - x_0| < \min(\gamma, t_0, t_1)$.

§ 22. Some integral inequalities. Integral inequalities we are going to deal with in this section are closely related with first order ordinary differential inequalities. This will be made clear by the proposition we prove first.

PROPOSITION 22.1. *Let* $\sigma_i(t, y_1, \ldots, y_n)$ $(i = 1, 2, \ldots, n)$ *be continuous in an open region* D *and let* $(t_0, Y_0) = (t_0, \mathring{y}_1, \ldots, \mathring{y}_n) \in D$. *Suppose that* $\Phi(t) = \big(\varphi_1(t), \ldots, \varphi_n(t)\big)$ *is continuous in an interval* $[t_0, \gamma)$ *and that* $\big(t, \Phi(t)\big) \in D$. *Under these assumptions, if*

$$\Phi(t_0) \leqslant Y_0\,, \tag{22.1}$$

and

$$D_-\varphi_i(t) \leqslant \sigma_i\big(t, \varphi_1(t), \ldots, \varphi_n(t)\big) \quad \textit{for} \quad t_0 \leqslant t < \gamma \quad (i = 1, 2, \ldots, n)\,, \tag{22.2}$$

then

$$\varphi_i(t) \leqslant \mathring{y}_i + \int_{t_0}^{t} \sigma_i\big(\tau, \varphi_1(\tau), \ldots, \varphi_n(\tau)\big)\,d\tau \quad \textit{for} \quad t_0 \leqslant t < \gamma \quad (i = 1, 2, \ldots, n)\,. \tag{22.3}$$

Proof. Consider the Picard's transform of $\Phi(t)$

$$\psi_i(t) = \varphi_i(t) - \int_{t_0}^{t} \sigma_i\big(\tau, \varphi_1(\tau), \ldots, \varphi_n(\tau)\big)\,d\tau \qquad (i = 1, 2, \ldots, n)\,.$$

The function $\psi_i(t)$ is continuous in $[t_0, \gamma)$ and, by (22.2), we have

$$D_-\psi_i(t) = D_-\varphi_i(t) - \sigma_i\big(t, \varphi_1(t), \ldots, \varphi_n(t)\big) \leqslant 0\,.$$

Hence, by Remark 2.1, $\psi_i(t)$ is decreasing and since, by (22.1), there is $\psi_i(t_0) \leqslant \mathring{y}_i$, we obtain

$$\psi_i(t) \leqslant \psi_i(t_0) \leqslant \mathring{y}_i \quad \text{on} \quad [t_0, \gamma)\,,$$

which is equivalent to (22.3).

By Proposition 22.1, inequalities (22.1) and (22.2) imply integral inequalities (22.3); but, obviously, (22.3) does not imply (22.2).

Now we know, by Theorem 9.3, that under the assumptions of Proposition 22.1, provided that $\sigma_i(t, Y)$ satisfy condition W_+ (see § 4), from the inequalities (22.1) and (22.2) result the inequalities

$$\Phi(t) \leqslant \Omega(t; t_0, Y_0) \quad \text{for} \quad t_0 \leqslant t < \min(\gamma, a_0)\,, \tag{22.4}$$

where $\Omega(t; t_0, Y_0)$ is the right-hand maximum solution of (5.1) through (t_0, Y_0), defined in $[t_0, a_0)$.

Next, we will prove that (22.4) is also a consequence of the essentially weaker (than (22.1) and (22.2)) inequalities (22.3), provided that the condition W_+ be substituted by the stronger condition $\overline{\mathrm{W}}_+$ (see § 4). In fact, we have the following theorem (see [39] and [65]):

THEOREM 22.1. *Let* $\sigma_i(t, y_1, \ldots, y_n)$ $(i = 1, 2, \ldots, n)$ *be continuous in the open region* $D = \{(t, Y): a < t < b, Y \text{ arbitrary}\}$ *and satisfy condition* $\overline{\mathrm{W}}_+$ (see § 4). *Let* $(t_0, Y_0) = (t_0, \mathring{y}_1, \ldots, \mathring{y}_n) \in D$. *Suppose that* $\Phi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$ *is continuous in an interval* $[t_0, \gamma)$ *and that* $(t, \Phi(t)) \in D$. *Under these assumptions, if*

$$(22.5)\qquad \varphi_i(t) \leqslant \mathring{y}_i + \int_{t_0}^{t} \sigma_i(\tau, \varphi_1(\tau), \ldots, \varphi_n(\tau))\, d\tau \quad \textit{for} \quad t_0 \leqslant t < \gamma \quad (i = 1, 2, \ldots, n),$$

then

$$(22.6)\qquad \Phi(t) \leqslant \Omega(t; t_0, Y_0) \quad \textit{for} \quad t_0 \leqslant t < \min(\gamma, a_0),$$

where $\Omega(t; t_0, Y_0)$ *is the right-hand maximum solution of* (5.1) *through* (t_0, Y_0), *defined on* $[t_0, a_0)$.

Proof. Put

$$\beta_i(t) = \mathring{y}_i + \int_{t_0}^{t} \sigma_i(\tau, \varphi_1(\tau), \ldots, \varphi_n(\tau))\, d\tau \quad (i = 1, 2, \ldots, n).$$

Then, by (22.5) and by condition $\overline{\mathrm{W}}_+$, we have

$$\beta_i'(t) = \sigma_i(t, \varphi_1(t), \ldots, \varphi_n(t)) \leqslant \sigma_i(t, \beta_1(t), \ldots, \beta_n(t)) \quad \text{for} \quad t_0 \leqslant t < \gamma \quad (i = 1, 2, \ldots, n).$$

Moreover, $\beta_i(t_0) = \mathring{y}_i$; therefore, by Theorem 9.3, we get

$$\beta_i(t) \leqslant \omega_i(t; t_0, Y_0) \quad \text{for} \quad t_0 \leqslant t < \min(\gamma, a_0) \quad (i = 1, 2, \ldots, n),$$

whence follows (22.6), since $\varphi_i(t) \leqslant \beta_i(t)$ $(i = 1, 2, \ldots, n)$.

As a corollary of Theorem 22.1 we obtain immediately the following known result (see [10]).

Assume $\varphi(t)$ to be continuous on an interval $[t_0, \gamma)$ and to satisfy the integral inequality

$$\varphi(t) \leqslant y_0 + \int_{t_0}^{t} a(\tau)\varphi(\tau)\, d\tau,$$

where $a(t)$ is continuous and non-negative for $t_0 \leqslant t < \gamma$. Then

$$\varphi(t) \leqslant y_0 \exp\Big(\int_{t_0}^{t} a(\tau)\, d\tau\Big) \quad \text{for} \quad t_0 \leqslant t < \gamma.$$

Remark. One can show (see [39]) that in Theorem 22.1 condition $\overline{\mathrm{W}}_+$ is essential.

From Theorem 22.1 we derive the following corollary:

COROLLARY 22.1. *Under the assumptions of Theorem* 22.1 *suppose that*

$$(22.7)\qquad \varphi_i(t) \leqslant \psi_i(t) + \int_{t_0}^{t} \sigma_i(\tau, \varphi_1(\tau), \ldots, \varphi_n(\tau))\, d\tau \quad (i = 1, 2, \ldots, n)$$

for $t_0 \leqslant t < \gamma$, *where* $\Psi(t) = (\psi_1(t), \ldots, \psi_n(t))$ *is continuous on* $[t_0, \gamma)$. *This being assumed, we have*

$$(22.8) \qquad \Phi(t) \leqslant \Psi(t) + \Omega_\psi(t) \quad \textit{for} \quad t_0 \leqslant t < \min(\gamma, a_0),$$

where $\Omega_\psi(t)$ *is the right-hand maximum solution through* $(t_0, 0, \ldots, 0)$ *of the system*

$$\frac{dy_i}{dt} = \sigma_i(t, \psi_1(t) + y_1, \ldots, \psi_n(t) + y_n) \quad (i = 1, 2, \ldots, n),$$

defined on $[t_0, a_0)$.

Proof. Put

$$\tilde{\sigma}_i(t, Y) = \sigma_i(t, \Psi(t) + Y) \quad (i = 1, 2, \ldots, n).$$

The functions $\tilde{\sigma}_i(t, Y)$ are continuous and satisfy condition $\overline{\mathrm{W}}_+$ in the region D.

If we write

$$\tilde{\varphi}_i(t) = \varphi_i(t) - \psi_i(t) \quad (i = 1, 2, \ldots, n),$$

then, by (22.7), we have

$$\tilde{\varphi}_i(t) \leqslant \int_{t_0}^{t} \tilde{\sigma}_i(\tau, \tilde{\varphi}_1(\tau), \ldots, \tilde{\varphi}_n(\tau))\, d\tau \quad (i = 1, 2, \ldots, n).$$

Therefore, we see that $\tilde{\Phi}(t), \tilde{\sigma}_i(t, Y)$ $(i = 1, 2, \ldots, n)$ satisfy all the assumptions of Theorem 22.1 in the region D with $(t_0, Y_0) = (t_0, 0, \ldots, 0)$. Hence we have

$$\tilde{\Phi}(t) \leqslant \Omega_\psi(t) \quad \text{for} \quad t_0 \leqslant t < \min(\gamma, a_0),$$

which is equivalent with (22.8).

CHAPTER V

CAUCHY PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS

In the present chapter we give a number of applications of results obtained in Chapters III and IV to different questions concerning the Cauchy problem for ordinary differential equations. In particular, we find: estimates of the solution and of its existence interval, estimates of the difference between two solutions, estimates of the error for an approximate solution and uniqueness criteria. Moreover, we discuss continuous dependence of the solution on initial data and on the right-hand sides of the equations, Chaplygin method and approximation of solutions of ordinary differential equations in a normed linear space.

§ 23. Estimates of the solution and of its existence interval. We prove

THEOREM 23.1. *Consider a system of ordinary differential equations*

$$\frac{dy_i}{dx} = f_i(x, y_1, \dots, y_n) \quad (i = 1, 2, \dots, n). \tag{23.1}$$

Suppose the right-hand members $f_i(x, Y)$ *to be defined in the region*

$$|x-x_0| < h, \quad |y_i-\mathring{y}_i| < h_i \quad (i = 1, 2, \dots, n) \tag{23.2}$$

and to satisfy the inequalities

$$|f_i(x, Y)| \leqslant \sigma_i(|x-x_0|, |Y-Y_0|) \quad (i = 1, 2, \dots, n), \tag{23.3}$$

where $Y_0 = (\mathring{y}_1, \dots, \mathring{y}_n)$, *and* $\sigma_i(t, y_1, \dots, y_n)$ *are the right-hand members of a comparison system of type* I (see § 14)

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, \dots, y_n) \quad (i = 1, 2, \dots, n). \tag{23.4}$$

Denote by $\Omega(t; H) = (\omega_1(t; H), \dots, \omega_n(t; H))$ *the right-hand maximum solution of* (23.4) *through* $(0, H) = (0, \eta_1, \dots, \eta_n)$, *defined in the interval* $[0, a_0)$. *Suppose* $Y(x) = (y_1(x), \dots, y_n(x))$ *is a solution of system* (23.1) *satisfying initial inequalities*

$$|y_i(x_0)-\mathring{y}_i| \leqslant \eta_i < h_i \quad (i = 1, 2, \dots, n) \tag{23.5}$$

and reaching the boundary of (23.2) *by both extremities* (see § 7). *Denote by t_i the least root of the equation in t*

$$\omega_i(t; H) = h_i$$

if such a root exists in the interval $(0, a_0)$; *if it does not exist, put $t_i = +\infty$.*

Under these assumptions the solution $Y(x)$ exists in the interval

$$|x - x_0| < h_0 = \min(h, a_0, t_1, \ldots, t_n) \tag{23.6}$$

and satisfies there the inequalities

$$|Y(x) - Y_0| \leqslant \Omega(|x - x_0|; H)\,. \tag{23.7}$$

Proof. Let $(x_0 - \alpha, x_0 + \beta)$ be the maximal existence interval of $Y(x)$ and put

$$\Phi(x) = \big(\varphi_1(x), \ldots, \varphi_n(x)\big) = \big(y_1(x) - \mathring{y}_1, \ldots, y_n(x) - \mathring{y}_n\big);$$

then we have, by (23.3) and (23.5),

$$|\varphi_i'(x)| = |y_i'(x)| = \big|f_i\big(x, Y(x)\big)\big| \leqslant \sigma_i\big(|x - x_0|, |\Phi(x)|\big) \qquad (i = 1, 2, \ldots, n)$$

in the interval $(x_0 - \alpha, x_0 + \beta)$ and

$$|\Phi(x_0)| \leqslant H\,.$$

Hence, by Theorem 15.1, inequality (23.7) is satisfied in the interval

$$|x - x_0| < \min(a_0, \alpha, \beta)\,. \tag{23.8}$$

Therefore, to complete the proof of our theorem it is enough to show that the interval (23.6) is contained in (23.8). We may suppose that, for instance, $\beta \leqslant \alpha$; then we have to show that $h_0 \leqslant \beta$. Suppose the contrary, i.e. $h_0 > \beta$; then the point β would belong to the interval $(0, h_0)$ and since $\omega_i(0; H) = \eta_i < h_i$, we would have, by the definition of t_i,

$$\omega_i(\beta; H) < h_i \qquad (i = 1, 2, \ldots, n)\,. \tag{23.9}$$

Consider now the following compact set:

$$|x - x_0| \leqslant \beta\,, \qquad |y_i - \mathring{y}_i| \leqslant \omega_i(\beta; H) \qquad (i = 1, 2, \ldots, n)\,. \tag{23.10}$$

By (23.9) and by the inequality $\beta < h_0 \leqslant h$, this compact set is contained in (23.2). On the other hand, in view of the inequalities $\beta \leqslant \alpha$, $\beta < h_0 \leqslant a_0$, the interval (23.8) is identical with $|x - x_0| < \beta$, and since inequalities (23.7) are satisfied in (23.8), we would have in particular

$$|Y(x) - Y_0| \leqslant \Omega(|x - x_0|; H) \leqslant \Omega(\beta; H)$$

in the interval

$$0 \leqslant x - x_0 < \beta\,. \tag{23.11}$$

This means that the solution-path $Y = Y(x)$ would be contained, for x belonging to (23.11), in the compact set (23.10) which—as we saw—is contained in (23.2). But this is impossible because the solution $Y(x)$, considered in (23.11), reaches the boundary of (23.2) by its right-hand extremity (see § 7).

By an analogous argument, using Theorem 21.1 we obtain

THEOREM 23.2. *Consider a differential equation of n-th order*

$$y^{(n)}(x) = f\big(x, y(x), y'(x), \ldots, y^{(n-1)}(x)\big) . \tag{23.12}$$

Suppose its right-hand member $f(x, y_0, y_1, \ldots, y_{n-1})$ *to be defined in the region*

$$|x-x_0| < h , \qquad |y_j-\mathring{y}_j| < h_j \qquad (j = 0, 1, \ldots, n-1) \tag{23.13}$$

and to satisfy the inequality

$$|f(t, Y)| \leqslant \sigma(|x-x_0|, |Y-Y_0|) ,$$

where $Y_0 = (\mathring{y}_0, \mathring{y}_1, \ldots, \mathring{y}_{n-1})$, *and* $\sigma(t, y_0, y_1, \ldots, y_{n-1})$ *is the right-hand side of a comparison equation* (see § 20)

$$y^{(n)}(t) = \sigma\big(t, y(t), y'(t), \ldots, y^{(n-1)}(t)\big) . \tag{23.14}$$

Denote by $\omega(t; H)$ *the right-hand maximum solution of* (23.14) *through* $(0, H) = (0, \eta_0, \eta_1, \ldots, \eta_{n-1})$, *defined in the interval* $[0, a_0)$. *Suppose that* $y(x)$ *is a solution of equation* (23.12) *satisfying the initial inequalities*

$$|y^{(j)}(x_0)-\mathring{y}_j| \leqslant \eta_j < h_j \qquad (j = 0, 1, \ldots, n-1)$$

and reaching the boundary of (23.13) *by both extremities* (see § 17). *Denote by* t_j *the least root of the equation in* t

$$\omega^{(j)}(t; H) = h_j$$

if such a root exists in the interval $(0, a_0)$; *if it does not exist, put* $t_j = +\infty$.

Under these assumptions the solution $y(x)$ *exists in the interval*

$$|x-x_0| < \min(h, a_0, t_0, \ldots, t_{n-1})$$

and satisfies the inequalities

$$|y^{(j)}(x)-\mathring{y}_j| \leqslant \omega^{(j)}(|x-x_0|; H) \qquad (j = 0, 1, \ldots, n-1) .$$

§ 24. Estimates of the difference between two solutions. We prove

THEOREM 24.1. *Let the right-hand members of system* (23.1) *and of the system*

$$\frac{dy_i}{dx} = \tilde{f}_i(x, y_1, \ldots, y_n) \qquad (i = 1, 2, \ldots, n) \tag{24.1}$$

be defined in an open region D and satisfy the inequalities

$$(24.2)\qquad |f_i(x, Y)-\tilde{f}_i(x, \tilde{Y})| \leqslant \sigma_i(|x-x_0|, |Y-\tilde{Y}|) \qquad (i = 1, 2, \dots, n),$$

where $\sigma_i(t, y_1, \dots, y_n)$ are the right-hand sides of a comparison system (23.4) *of type* I (see § 14). *Suppose that $Y(x) = (y_1(x), \dots, y_n(x))$ and $\tilde{Y}(x) = (\tilde{y}_1(x), \dots, \tilde{y}_n(x))$ are two solutions of systems* (23.1) *and* (24.1) *respectively, defined in an interval $|x-x_0| < \gamma$ and satisfying the initial inequalities*

$$(24.3)\qquad |Y(x_0)-\tilde{Y}(x_0)| \leqslant H,$$

where $H = (\eta_1, \dots, \eta_n)$. Denote by $\Omega(t; H) = (\omega_1(t; H), \dots, \omega_n(t; H))$ the right-hand maximum solution of the comparison system (23.4) *through* $(0, H)$ *and let it be defined in* $[0, a_0)$.

Under these hypotheses we have the inequalities

$$(24.4)\qquad |Y(x)-\tilde{Y}(x)| \leqslant \Omega(|x-x_0|; H)$$

in the interval

$$(24.5)\qquad |x-x_0| < \min(\gamma, a_0).$$

Proof. In the interval $|x-x_0| < \gamma$ put

$$\Phi(x) = (\varphi_1(x), \dots, \varphi_n(x)) = (y_1(x)-\tilde{y}_1(x), \dots, y_n(x)-\tilde{y}_n(x)).$$

Then, by (24.2) and (24.3), we have

$$\begin{aligned}|\varphi_i'(x)| = |y_i'(x)-\tilde{y}_i'(x)| &= |f_i(x, Y(x))-f_i(x, \tilde{Y}(x))| \\ &\leqslant \sigma_i(|x-x_0|, |\Phi(x)|) \qquad (i = 1, 2, \dots, n),\end{aligned}$$

and $|\Phi(x_0)| \leqslant H$.

Hence, by Theorem 15.1, inequalities (24.4) hold true in the interval (24.5).

In a similar way, using Theorem 21.1 we get

THEOREM 24.2. *Let the right-hand member of equation* (23.12) *and of equation*

$$(24.6)\qquad y^{(n)}(x) = \tilde{f}(x, y(x), y'(x), \dots, y^{(n-1)}(x))$$

be defined in an open region D and satisfy the inequality

$$|f(x, Y)-\tilde{f}(x, \tilde{Y})| \leqslant \sigma(|x-x_0|, |Y-\tilde{Y}|),$$

where $\sigma(t, y_0, y_1, \dots, y_{n-1})$ is the right-hand side of the comparison equation (23.14). *Suppose $y(x)$ and $\tilde{y}(x)$ are two solutions of equation* (23.12) *and* (24.6) *respectively, defined in an interval $|x-x_0| < \gamma$ and satisfying the initial inequalities*

$$|y^{(j)}(x_0)-\tilde{y}^{(j)}(x_0)| \leqslant \eta_j \qquad (j = 0, 1, .., n-1).$$

Denote by $\omega(t; H)$ *the right-hand maximum solution of the comparison equation* (23.14) *through* $(0, H) = (0, \eta_0, \eta_1, \ldots, \eta_{n-1})$ *and let it be defined in* $[0, a_0)$.

Under these assumptions we have the inequalities

$$|y^{(j)}(x) - \tilde{y}^{(j)}(x)| \leqslant \omega^{(j)}(|x - x_0|; H) \qquad (j = 0, 1, \ldots, n-1)$$

in the interval

$$|x - x_0| < \min(\gamma, a_0)\,.$$

§ 25. Uniqueness criteria. Continuous dependence of the solution of Cauchy problem on the initial values and on the right-hand sides. As an immediate consequence of Theorem 24.1 we obtain the following uniqueness criterion:

THEOREM 25.1. *Let the right-hand members of system* (23.1) *be defined in an open region* D, *containing the point* (x_0, Y_0), *and satisfy the inequalities*

$$(25.1) \qquad |f_i(x, Y) - f_i(x, \tilde{Y})| \leqslant \sigma_i(|x - x_0|, |Y - \tilde{Y}|) \qquad (i = 1, 2, \ldots, n)\,,$$

where $\sigma_i(t, y_1, \ldots, y_n)$ *are the right-hand sides of a comparison system of type* I (see § 14). *Suppose that*

$$\sigma_i(t, 0, \ldots, 0) \equiv 0 \qquad (i = 1, 2, \ldots, n)$$

and that

$$\Omega(t) \equiv 0 \qquad \textit{for} \qquad 0 \leqslant t < +\infty\,,$$

where $\Omega(t) = \big(\omega_1(t), \ldots, \omega_n(t)\big)$ *is the right-hand maximum solution of the comparison system through the origin.*

Under these assumptions system (23.1) *admits at most one solution through* (x_0, Y_0) *in* D.

Proof. Let $Y(x) = \big(y_1(x), \ldots, y_n(x)\big)$ and $\tilde{Y}(x) = \big(\tilde{y}_1(x), \ldots, \tilde{y}_n(x)\big)$ be two solutions of system (23.1), defined in some interval $|x - x_0| < \gamma$ and such that

$$Y(x_0) = \tilde{Y}(x_0) = Y_0\,.$$

Then, by Theorem 24.1 (systems (23.1) and (24.1) are now identical) and by our assumptions, we have

$$|Y(x) - \tilde{Y}(x)| \leqslant \Omega(|x - x_0|) = 0$$

and consequently

$$Y(x) \equiv \tilde{Y}(x) \qquad \text{for} \qquad |x - x_0| < \gamma\,.$$

Remark 25.1. In particular, the comparison system with

$$\sigma_i(t, y_1, \ldots, y_n) = K \sum_{j=1}^{n} y_j \qquad (K \geqslant 0)$$

satisfies all the assumptions of Theorem 25.1 and in this case inequalities (25.1) mean that the right-hand members of system (23.1) satisfy a Lipschitz condition with respect to Y.

Remark 25.2. What concerns uniqueness of the solution for $x \geqslant x_0$, i.e. to right from the initial point, condition (25.1) can be substituted by an essentially weaker one, viz.

$$\text{(i)} \quad [f_i(x, Y)-f_i(x, \tilde{Y})]\operatorname{sgn}(y_i-\tilde{y}_i) \leqslant \sigma_i(x-x_0, |Y-\tilde{Y}|) \qquad (i = 1, 2, \ldots, n).$$

In this case the proof of uniqueness is achieved in the following way. Let $y_i(x)$ and $\tilde{y}_i(x)$ $(i = 1, 2, \ldots, n)$ be two solutions issued from (x_0, Y_0) and defined in some interval $0 \leqslant x-x_0 < \gamma$. Put for $0 \leqslant t < \gamma$

$$\varphi_i(t) = |y_i(x_0+t)-\tilde{y}_i(x_0+t)| \qquad (i = 1, 2, \ldots, n).$$

Since $\varphi_i(0) = |y_i(x_0)-\tilde{y}_i(x_0)| = 0$ $(i = 1, 2, \ldots, n)$, it suffices, by Theorem 11.1, to show that

$$\text{(ii)} \qquad \varphi_i'(t) \leqslant \sigma_i\big(t, \varphi_1(t), \ldots, \varphi_n(t)\big)$$

for t in the set

$$E_i = \{t \in (0, \gamma): \varphi_i(t) > 0\}.$$

Now, if $\tilde{t} \in E_i$, then we have

$$\varphi_i(t) = [y_i(x_0+t)-\tilde{y}_i(x_0+t)]\operatorname{sgn}\big(y_i(x_0+\tilde{t})-\tilde{y}_i(x_0+\tilde{t})\big)$$

in some neighborhood of $\tilde{t}$ and consequently we get

$$\varphi_i'(\tilde{t}) = [y_i'(x_0+\tilde{t})-\tilde{y}_i'(x_0+\tilde{t})]\operatorname{sgn}\big(y_i(x_0+\tilde{t})-\tilde{y}_i(x_0+\tilde{t})\big).$$

From the last relation and from (i) we obtain (ii) for $t = \tilde{t}$.

From this remark it follows, in particular, that for one equation

$$\frac{dy}{dx} = f(x, y),$$

with $f(x, y)$ decreasing with respect to y, we have uniqueness to right from the initial point. Indeed, under this assumption equation

$$\frac{dy}{dt} = 0$$

can be taken for a comparison one.

By Theorem 24.2, we get the next theorem.

THEOREM 25.2. *Let the right-hand member of equation* (23.12) *be defined in an open region D, containing the point $(x_0, \mathring{y}_0, \mathring{y}_1, \ldots, \mathring{y}_{n-1})$, and satisfy the inequality*

$$|f(x, Y)-f(x, \tilde{Y})| \leqslant \sigma(|x-x_0|, |Y-\tilde{Y}|),$$

where $\sigma(t, y_0, y_1, \ldots, y_{n-1})$ *is the right-hand side of a comparison equation* (see § 20). *Suppose that*

$$\sigma(t, 0, \ldots, 0) \equiv 0$$

and that

$$\omega(t) \equiv 0 \quad \text{for} \quad 0 \leqslant t < +\infty,$$

where $\omega(t)$ *is the right-hand maximum solution of the comparison equation, satisfying the initial conditions*

$$\omega^{(j)}(0) = 0 \quad (j = 0, 1, \ldots, n-1).$$

Under these hypotheses equation (23.12) *admits at most one solution satisfying the initial conditions*

$$y^{(j)}(x_0) = \mathring{y}_j \quad (j = 0, 1, \ldots, n-1).$$

Next we will show that under the hypotheses of Theorem 25.1 the solution of system (23.1) depends continuously on the initial point and on the right-hand sides.

THEOREM 25.3. *Let the right-hand sides* $f_i(x, Y)$ $(i = 1, 2, \ldots, n)$ *of system* (23.1) *be continuous in an open region* D *and satisfy the assumptions of Theorem* 25.1. *Let* $Y(x) = \big(y_1(x), \ldots, y_n(x)\big)$ *be the solution of system* (23.1) *through* $(x_0, Y_0) \in D$ *and assume it to be defined in an interval* $|x - x_0| < a$. *Suppose that the right-hand members* $\tilde{f}_i(x, Y)$ $(i = 1, 2, \ldots, n)$ *of system* (24.1) *are continuous in* D *and let* $\tilde{Y}(x; \tilde{Y}) = \big(\tilde{y}_1(x; \tilde{Y}), \ldots, \tilde{y}_n(x; \tilde{Y})\big)$ *be any solution of system* (24.1) *through* $(x_0, \tilde{Y}) \in D$, *continued to the boundary of* D *in both directions* (see § 7).

Under these assumptions we have the following propositions:

1. *To every positive* $\gamma < a$ *there is a positive* δ *such that if* $|\tilde{Y} - Y_0| < \delta$ *and*

$$|f_i(x, Y) - \tilde{f}_i(x, Y)| < \delta \quad (i = 1, 2, \ldots, n), \tag{25.2}$$

then the solution $\tilde{Y}(x; \tilde{Y})$ *of system* (24.1) *is defined in the interval*

$$|x - x_0| < \gamma. \tag{25.3}$$

2. *To every* $\varepsilon > 0$ *there is a positive* $\delta_1 < \delta$ *such that inequalities*

$$|\tilde{y}_i(x; \tilde{Y}) - y_i(x)| < \varepsilon \quad (i = 1, 2, \ldots, n)$$

are satisfied in the interval (25.3) *whenever*

$$|\tilde{Y} - Y_0| < \delta_1, \quad |f_i(x, Y) - \tilde{f}_i(x, Y)| < \delta_1 \quad (i = 1, 2. \ldots, n).$$

Proof. For $\mu \geqslant 0$ consider the comparison system

$$\frac{dy_i}{dt} = \sigma_i(t, Y) + \mu \quad (i = 1, 2, \ldots, n) \tag{25.4}$$

and let $\Omega(t; H, \mu) = \big(\omega_1(t; H, \mu), \ldots, \omega_n(t; H, \mu)\big)$ be its right-hand maximum solution through $(0, H) = (0, \eta_1, \ldots, \eta_n)$. Since–in view of our assumptions–there is $\Omega(t; 0, 0) \equiv \Omega(t) \equiv 0$ for $0 \leqslant t < +\infty$, we conclude, by Theorem 10.1, that for any positive $\gamma < a$

1° there is a positive δ such that $\Omega(t; H, \mu)$ is defined in the interval $[0, \gamma]$ whenever $\mu \leqslant \delta$, $0 \leqslant \eta_i < \delta$ $(i = 1, 2, \ldots, n)$,

2° $\lim\limits_{\substack{H\to 0,\mu\to 0\\ H\geqslant 0,\mu\geqslant 0}} \Omega(t; H, \mu) = 0$ uniformly in $[0, \gamma]$.

Suppose (25.2) holds true with the above δ. By (25.1) and (25.2), we have for any two points $(x, Y), (x, \tilde{Y}) \in D$

$$(25.5)\quad |f_i(x, Y) - \tilde{f}_i(x, \tilde{Y})| \leqslant |f_i(x, Y) - f_i(x, \tilde{Y})| + |f_i(x, \tilde{Y}) - \tilde{f}_i(x, \tilde{Y})| \leqslant \sigma_i(|x - x_0|, |Y - \tilde{Y}|) + \delta \quad (i = 1, 2, \ldots, n).$$

Suppose that

$$(25.6)\qquad |\tilde{Y} - Y_0| < \delta;$$

then, putting $\eta_i = |\tilde{y}_i - \mathring{y}_i|$ we have

$$(25.7)\qquad 0 \leqslant \eta_i = |\tilde{y}_i - \mathring{y}_i| < \delta \quad (i = 1, 2, \ldots, n).$$

Denote by $(x_0 - \tilde{\alpha}, x_0 + \tilde{\beta})$ the maximal existence interval of $\tilde{Y}(x; \tilde{Y})$. We may assume that, for instance, $0 < \tilde{\beta} \leqslant \tilde{\alpha}$. Let $\tilde{f}_i$ and $\tilde{Y}$ satisfy (25.2) and (25.6). By (25.7), we have

$$|\tilde{y}_i(x_0; \tilde{Y}) - y_i(x_0)| = |\tilde{y}_i - \mathring{y}_i| = \eta_i < \delta \quad (i = 1, 2, \ldots, n).$$

Hence, by (25.5) and by Theorem 24.1, we get

$$(25.8)\qquad |\tilde{Y}(x; \tilde{Y}) - Y(x)| \leqslant \Omega(|x - x_0|; H, \delta)$$

in the interval

$$(25.9)\qquad |x - x_0| < \min(\gamma, \tilde{\beta}).$$

By 2°, we may assume that δ was chosen small enough, so that the compact set

$$(25.10)\qquad |x - x_0| \leqslant \gamma, \quad |Y - Y(x)| \leqslant \Omega(|x - x_0|; H, \delta)$$

be contained in the region D. In order to prove assertion 1 of our theorem, it is sufficient to show that $\tilde{f}_i$ and $\tilde{Y}$ satisfying (25.2) and (25.6) we have $\tilde{\beta} \geqslant \gamma$. Suppose the contrary, i.e. $\tilde{\beta} < \gamma$; then, by (25.8), the solution path $Y = \tilde{Y}(x; \tilde{Y})$ would be contained in the compact set (25.10) for $0 \leqslant x - x_0 < \tilde{\beta}$, which is impossible since $\tilde{Y}(x; \tilde{Y})$ reaches the boundary of D by its right-hand extremity. Thus, assertion 1 is proved.

Now, take an arbitrary $\varepsilon > 0$. By 2°, there is a positive $\delta_1 < \delta$ such that for $0 \leqslant \eta_i < \delta_1$ $(i = 1, 2, \ldots, n)$ we have

$$(25.11)\qquad \omega_i(t; H, \delta_1) < \varepsilon \quad \text{in} \quad 0 \leqslant t < \gamma \quad (i = 1, 2, \ldots, n).$$

Suppose that

$$|\tilde{Y}-Y_0|<\delta_1\,,\qquad |f_i(x,\,Y)-\tilde{f}_i(x,\,Y)|<\delta_1\qquad (i=1,2,\ldots,n);$$

then by an argument similar to that used in the proof of assertion 1 we conclude, by (25.11), that

$$|\tilde{y}_i(x;\;\tilde{Y})-y_i(x)|\leqslant\omega_i(|x-x_0|;\,H,\,\delta_1)<\varepsilon\qquad (i=1,2,\ldots,n)$$

in the interval (25.3). This completes the proof of assertion 2.

What concerns an nth order ordinary differential equation we have the following

THEOREM 25.4. *Let the right-hand member $f(x,y_0,y_1,\ldots,y_{n-1})$ of equation* (23.12) *be continuous in an open region D and satisfy the assumptions of Theorem* 25.2. *Let $y(x)$ be the solution of equation* (23.12) *satisfying initial conditions*

$$y^{(j)}(x_0)=\mathring{y}_j\qquad (j=0,1,\ldots,n-1)$$

and assume it to be defined in an interval $|x-x_0|<a$. Suppose that the right-hand side $\tilde{f}(x,y_0,y_1,\ldots,y_{n-1})$ of equation (24.6) *is continuous in D and let $\tilde{y}(x;\;\tilde{Y})$ be any solution of equation* (24.6), *satisfying initial conditions*

$$\tilde{y}^{(j)}(x_0;\;\tilde{Y})=\tilde{y}_j\qquad (j=0,1,\ldots,n-1)$$

and continued to the boundary of D in both directions (see § 17).

Under these assumptions the following propositions hold true:

1. *To every positive $\gamma<a$ there is a positive δ such that if*

$$|\tilde{y}_j-\mathring{y}_j|<\delta\qquad (j=0,1,\ldots,n-1)\,,\qquad |f(x,\,Y)-\tilde{f}(x,\,Y)|<\delta\,,$$

then the solution $\tilde{y}(x;\;\tilde{Y})$ of equation (24.6) *is defined in the interval $|x-x_0|<\gamma$.*

2. *To every $\varepsilon>0$ there is a positive $\delta_1<\delta$ such that the inequalities*

$$|\tilde{y}^{(j)}(x;\;\tilde{Y})-y^{(j)}(x)|<\varepsilon\qquad (j=0,1,\ldots,n-1)$$

are satisfied in the interval $|x-x_0|<\gamma$ whenever

$$|\tilde{y}_j-\mathring{y}_j|<\delta_1\qquad (j=0,1,\ldots,n-1)\,,\qquad |f(x,\,Y)-\tilde{f}(x,\,Y)|<\delta_1\,.$$

Now, we are going to prove Kamke's (see [14], p. 139) uniqueness criterion which is more general than the one contained in Theorem 25.1. This time the much weaker assumptions will not assure, in general, the continuous dependence of the solution on the initial point.

THEOREM 25.5. *Let the right-hand members $f_i(x,\,Y)$ $(i=1,2,\ldots,n)$ of system* (23.1) *be defined in an open region D, containing the point $(x_0,\,Y_0)$, and satisfy the inequality*

$$(25.12)\qquad \sum_{i=1}^{n}|f_i(x,\,Y)-f_i(x,\,\tilde{Y})|\leqslant\sigma\Big(|x-x_0|,\,\sum_{j=1}^{n}|y_j-\tilde{y}_j|\Big)\qquad \text{for}\qquad x\neq x_0\,,$$

where $\sigma(t, y)$ *is the right-hand side of a comparison equation of type* III (see § 14). *Then system* (23.1) *admits at most one solution through* (x_0, Y_0) *in* D.

Proof. Suppose $Y(x) = (y_1(x), \ldots, y_n(x))$ and $\tilde{Y}(x) = (\tilde{y}_1(x), \ldots, \tilde{y}_n(x))$ are two solutions of system (23.1), defined in an interval $|x - x_0| < \gamma$ and satisfying initial conditions

$$Y(x_0) = \tilde{Y}(x_0) = Y_0 . \tag{25.13}$$

Since the assumptions of our theorem are invariant under the mapping $\xi = -x + 2x_0$, it is sufficient to prove that

$$\sum_{i=1}^{n} |y_i(x) - \tilde{y}_i(x)| = 0 \tag{25.14}$$

in the interval

$$0 \leqslant x - x_0 < \gamma . \tag{25.15}$$

Put

$$\varphi(t) = \sum_{i=1}^{n} |y_i(x_0 + t) - \tilde{y}_i(x_0 + t)|$$

for

$$0 \leqslant t < \gamma . \tag{25.16}$$

The function $\varphi(t)$ is continuous in the interval (25.16) and, by (25.13), there is

$$\varphi(0) = 0 . \tag{25.17}$$

Further we have

$$D^+\varphi(0) \leqslant \sum_{i=1}^{n} |y_i'(x_0) - \tilde{y}_i'(x_0)| = \sum_{i=1}^{n} |f_i(x_0, Y_0) - \tilde{f}_i(x_0, Y_0)| = 0 . \tag{25.18}$$

Finally, by (25.12), we get for $0 < t < \gamma$

$$\begin{aligned} D_-\varphi(t) &\leqslant \sum_{i=1}^{n} |y_i'(x_0 + t) - \tilde{y}_i'(x_0 + t)| \\ &= \sum_{i=1}^{n} \left| f_i(x_0 + t, Y(x_0 + t)) - \tilde{f}_i(x_0 + t, \tilde{Y}(x_0 + t)) \right| \leqslant \sigma(t, \varphi(t)) . \end{aligned} \tag{25.19}$$

From (25.17), (25.18) and (25.19) it follows, by the third comparison theorem (see § 14), that

$$\varphi(t) \leqslant 0$$

in the interval (25.16). But, since $\varphi(t) \geqslant 0$, we conclude that $\varphi(t) \equiv 0$ in (25.16) and consequently (25.14) is satisfied in the interval (25.15).

Remark 25.3. If the comparison equation of type III is, in particular, equation (β) from Example 14.2, then Theorem 25.5 gives Osgood's uniqueness criterion. Similarly, Theorem 25.5 contains, as a particular case, Nagumo's criterion if the comparison equation is that of the Example 14.3.

Remark 25.4. In view of the Remark 14.3, Theorem 25.5 would be false if property (α_1) of the comparison equation of type III were replaced by the essentially weaker property (α_2). Indeed, if we put

$$f(x, y) = \begin{cases} \dfrac{\varphi'(x)}{\varphi(x)} y & \text{for} \quad x > 0\,,\ y \geqslant 0\,, \\ 0 & \text{elsewhere}\,, \end{cases}$$

then for the equation

$$\frac{dy}{dx} = f(x, y) \tag{25.20}$$

and for the comparison equation (14.13) the assumption (25.12) of Theorem 25.5 is satisfied at the point $(0, 0)$. However, there are two different solutions of (25.20) through the origin, viz. $y(x) = \varphi(x)$ and $\tilde{y}(x) = 0$. In the above counter-example the right-hand member $f(x, y)$ of (25.20) was discontinuous for $x = 0$. It is possible to construct a similar example with $f(x, y)$ continuous in the whole plane [56].

Remark 25.5. In the case of one equation with a continuous right-hand side Kamke's uniqueness criterion is only apparently more general than the criterion of Theorem 25.1. Indeed, the following result, due to C. Olech [37], is true.

Let the function $f(x, y)$ be continuous in the neighborhood of the point (x_0, y_0) and satisfy there the inequality

$$|f(x, y) - f(x, \tilde{y})| \leqslant \sigma(|x - x_0|, |y - \tilde{y}|) \quad \text{for} \quad x \neq x_0\,,$$

where $\sigma(t, y)$ is the right-hand side of a comparison equation of type III; then $f(x, y)$ also satisfies an inequality

$$|f(x, y) - f(x, \tilde{y})| \leqslant \tilde{\sigma}(|x - x_0|, |y - \tilde{y}|)\,,$$

where $\tilde{\sigma}(t, y)$ is the right-hand side of a comparison equation of type I (see § 14) satisfying assumptions of Theorem 25.1.

Remark 25.6. Due to Theorem 15.4 it is easy to check that Theorems 24.1, 25.1 and 25.5 are true for a system (23.1) with x being a real variable, y_i $(i = 1, 2, \ldots, n)$ being vectors in a linear normed space $\mathfrak{L}$, $f_i(x, Y)$ being vector-valued functions with values in $\mathfrak{L}$ and the absolute value being substituted by the norm in $\mathfrak{L}$.

§ 26. Estimates of the error of an approximate solution. In this section we describe a general method by which we can evaluate the error when, instead of the solution of a given ("difficult to solve") system, the solution of an approximate ("easy to solve") one is taken (see [60]).

Let the right-hand members of the ("difficult to solve") system

$$\frac{dy_i}{dx} = f_i(x, y_1, \ldots, y_n) \quad (i = 1, 2, \ldots, n) \tag{26.1}$$

be continuous in an open region D containing the point $(x_0, Y_0) = (x_0, \mathring{y}_1, \ldots, \mathring{y}_n)$. Denote by $Y(x) = \big(y_1(x), \ldots, y_n(x)\big)$ a solution of system (26.1) through (x_0, Y_0). Suppose that the inequalities

$$|f_i(x, Y)| \leqslant \tilde{\sigma}_i(|x-x_0|, |Y-Y_0|) \quad (i = 1, 2, \ldots, n) \tag{26.2}$$

hold true, $\tilde{\sigma}_i(t, y_1, \ldots, y_n)$ being the right-hand sides of a comparison system of type I (see § 14). Let $\tilde{\Omega}(t) = \big(\tilde{\omega}_1(t), \ldots, \tilde{\omega}_n(t)\big)$ be its right-hand maximum solution through the origin. Consider the approximate ("easy to solve") system

$$\frac{dy_i}{dx} = g_i(x, y_1, \ldots, y_n) \quad (i = 1, 2, \ldots, n) \tag{26.3}$$

with right-hand sides continuous in D and let $\tilde{Y}(x) = \big(\tilde{y}_1(x), \ldots, \tilde{y}_n(x)\big)$ be its solution through (x_0, Y_0) in the interval $|x-x_0| < \gamma$. Assume that

$$|g_i(x, Y) - g_i(x, \tilde{Y})| \leqslant \hat{\sigma}_i(|x-x_0|, |Y-\tilde{Y}|) \quad (i = 1, 2, \ldots, n), \tag{26.4}$$

where $\hat{\sigma}_i(t, y_1, \ldots, y_n)$ are the right-hand members of a comparison system of type I (see § 14). Suppose finally that the following limitation of the difference between the right-hand sides of the given system (26.1) and of the approximate one (26.3) is known

$$|f_i(x, Y) - g_i(x, Y)| \leqslant h_i(|x-x_0|, |Y-Y_0|) \quad (i = 1, 2, \ldots, n), \tag{26.5}$$

where the functions $h_i(t, y_1, \ldots, y_n)$ satisfy condition $\overline{\mathrm{W}}_+$ with respect to Y (see § 4).

Under all these assumptions we are able to evaluate the difference between the solution $Y(x)$, which is sought for, and the approximate one $\tilde{Y}(x)$. We do it in two steps.

I step. *Estimate of the solution and of its existence interval.* In view of (26.2) we evaluate, by Theorem 23.1, the existence interval

$$|x-x_0| < h_0 \tag{26.6}$$

of $Y(x)$ and $Y(x)$ itself

$$|Y(x) - Y_0| \leqslant \tilde{\Omega}(|x-x_0|) \tag{26.7}$$

in the interval (26.6).

II step. *Evaluation of the error.* Solution $Y(x)$ of system (26.1) satisfies obviously the system

$$\frac{dy_i}{dx} = \tilde{g}_i(x, y_1, \ldots, y_n) \quad (i = 1, 2, \ldots, n), \tag{26.8}$$

where

$$\tilde{g}_i(x, Y) = g_i(x, Y) + [f_i(x, Y(x)) - g_i(x, Y(x))] \quad (i = 1, 2, \ldots, n).$$

By (26.4), (26.5), (26.7) and by the condition $\overline{\mathrm{W}}_+$ (satisfied by h_i), we get

(26.9) $$|g_i(x, Y) - \tilde{g}_i(x, \tilde{Y})| \leqslant \sigma_i(|x - x_0|, |Y - \tilde{Y}|) \quad (i = 1, 2, \ldots, n),$$

where for $\sigma_i(t, y_1, \ldots, y_n)$ we can take any functions satisfying inequalities

(26.10) $$\sigma_i(t, y_1, \ldots, y_n) \geqslant \hat{\sigma}_i(t, y_1, \ldots, y_n) + h_i(t, \tilde{\Omega}(t)) \quad (i = 1, 2, \ldots, n)$$

and being right-hand sides of a comparison system of type I. Denoting by $\Omega(t) = (\omega_1(t), \ldots, \omega_n(t))$ its right-hand maximum solution through the origin, defined in an interval $[0, \alpha_0)$, we conclude, by (26.9) and by Theorem 24.1 applied to system (26.3) and (26.8), that

(26.11) $$|Y(x) - \tilde{Y}(x)| \leqslant \Omega(|x - x_0|)$$

in the interval

$$|x - x_0| < \min(h_0, \gamma, \alpha_0).$$

Inequalities (26.11) give the evaluation of the error that was sought for.

EXAMPLE 26.1. To illustrate the procedure described above, let us consider the case when the approximate system (26.3) is linear, its right-hand sides being Taylor's expansions up to order one of the right-hand members of the given system (26.1).

Assume then that the right-hand sides $f^i(x, Y)$ of system (26.1) are of class C^2 in the cube

(26.12) $$|x| < h, \quad |y_i| < h \quad (i = 1, 2, \ldots, n),$$

and let $(x_0, Y_0) = (0, 0, \ldots, 0)$. Suppose that we have

(26.13) $$f^i(0, 0, \ldots, 0) = 0, \quad |f^i_x|, |f^i_{y_j}| \leqslant A, \quad |f^i_{xx}|, |f^i_{xy_j}|, |f^i_{y_jy_k}| \leqslant B$$

in the cube (26.12); then we get in (26.12)

$$|f^i(x, Y)| = |f^i(x, Y) - f^i(0, 0)| = \left|f^i_x(\xi, \Xi)x + \sum_{j=1}^{n} f^i_{y_j}(\xi, \Xi)y_j\right|$$
$$\leqslant A\left(|x| + \sum_{j=1}^{n} |y_j|\right).$$

Hence, for $\tilde{\sigma}_i(t, Y)$ in (26.2) we can take

$$\tilde{\sigma}_i(t, Y) = A\left(t + \sum_{j=1}^{n} y_j\right) \quad (i = 1, 2, \ldots, n).$$

The unique solution through the origin of the comparison system with the above right-hand sides is

$$\tilde{\omega}_i(t) = \frac{1}{n^2A}(e^{nAt}-1-nAt) \qquad (i = 1, 2, \ldots, n).$$

Since

$$\tilde{\omega}_i(t) = \frac{1}{n^2A}\left[\frac{(nAt)^2}{2!}+\frac{(nAt)^3}{3!}+\ldots\right] \leqslant \frac{1}{2}At^2e^{nAt},$$

the unique root t_i of the equation in t, $\tilde{\omega}_i(t) = h$, is not less than that of the equation $\frac{1}{2}At^2e^{nAt} = h$. The root of the last equation is, by its turn, not less than

$$\delta = \min\left(h, \sqrt{\frac{2h}{A}e^{-nAh/2}}\right).$$

Hence we have $t_i \geqslant \delta$ $(i = 1, 2, \ldots, n)$ and, by Theorem 23.1, the solution $Y(x) = (y_1(x), \ldots, y_n(x))$ of system (26.1) through the origin exists in the interval

$$|x| < \delta \tag{26.14}$$

and satisfies there the inequalities

$$|y_i(x)| \leqslant \tilde{\omega}_i(|x|) \leqslant \tfrac{1}{2}A|x|^2e^{nAh} \qquad (i = 1, 2, \ldots, n). \tag{26.15}$$

Write

$$\mathring{f}^i_x = f^i_x(0, 0, \ldots, 0), \qquad \mathring{f}^i_{y_j} = f^i_{y_j}(0, 0, \ldots, 0)$$

and take for the right-hand sides of the approximate system (26.3)

$$g^i(x, Y) = x\mathring{f}^i_x + \sum_{j=1}^{n} y_j\mathring{f}^i_{y_j} \qquad (i = 1, 2, \ldots, n).$$

By (26.13), we have then

$$|g^i(x, Y)-g^i(x, \tilde{Y})| \leqslant A\sum_{j=1}^{n}|y_j-\tilde{y}_j| \qquad (i = 1, 2, \ldots, n),$$

and consequently for $\hat{\sigma}_i$ in (26.4) we can choose

$$\hat{\sigma}_i(t, Y) = A\sum_{j=1}^{n} y_j \qquad (i = 1, 2, \ldots, n). \tag{26.16}$$

By Taylor's formula and by (26.13), we get

$$|f^i(x, Y)-g^i(x, Y)| = \left|\frac{1}{2}\left(x\frac{\partial}{\partial x}+\sum_{j=1}^{n} y_j\frac{\partial}{\partial y_j}\right)^{(2)}f^i(\xi, \Xi)\right| \leqslant \frac{1}{2}B\left(|x|+\sum_{j=1}^{n}|y_j|\right)^2.$$

Hence, for $h^i(x, Y)$ in (26.5) we can put

$$h^i(x, Y) = \tfrac{1}{2}B\Big(t+\sum_{j=1}^{n} y_j\Big)^2 \quad (i = 1, 2, \ldots, n).$$

Since in the interval (26.14) we have

$$h^i\big(t, \tilde{\Omega}(t)\big) \leqslant \frac{1}{2}B\Big(t+\frac{n}{2}At^2e^{nAt}\Big)^2 \leqslant \frac{1}{2}B\Big(1+\frac{n}{2}Ahe^{nAh}\Big)^2 t^2,$$

we can choose for $\sigma_i(t, Y)$ in (26.10) (see (26.16))

$$\sigma_i(t, Y) = Ct^2+A\sum_{j=1}^{n} y_j \quad (i = 1, 2, \ldots, n),$$

where

$$C = \tfrac{1}{2}B(1+\tfrac{1}{2}nAhe^{nAh})^2.$$

Now, the only solution through the origin of the comparison system with the right-hand members $\sigma_i(t, Y)$, defined above, is

$$\begin{aligned}\omega_i(t) &= \frac{2C}{(nA)^3}\left[e^{nAt}-1-nAt-\frac{(nAt)^2}{2!}\right] \\ &= \frac{2C}{(nA)^3}\left[\frac{(nAt)^3}{3!}+\frac{(nAt)^4}{4!}+\ldots\right] \leqslant \frac{C}{3}t^3e^{nAt} \quad (i = 1, 2, \ldots, n).\end{aligned}$$

Therefore, we get finally

$$|y_i(x)-\tilde{y}_i(x)| \leqslant \frac{C}{3}|x|^3e^{nA|x|} \quad (i = 1, 2, \ldots, n)$$

in the interval (26.14), where $\tilde{y}_i(x)$ $(i = 1, 2, \ldots, n)$ is the solution through the origin of the approximate (in our case linear) system (26.3).

§ 27. Stability of the solution. We give here a stability criterion which is an immediate consequence of Theorem 23.1.

THEOREM 27.1. *Let the right-hand sides of system* (26.1) *be continuous in the region*

$$x_0 \leqslant x < +\infty, \quad |y_i| < h \quad (i = 1, 2, \ldots, n).$$

Suppose that $f_i(x, 0, \ldots, 0) = 0$ $(i = 1, 2, \ldots, n)$ *and*

$$(27.1) \qquad |f_i(x, Y)| \leqslant \sigma_i(x-x_0, |Y|) \quad (i = 1, 2, \ldots, n),$$

where $\sigma_i(t, Y)$ *are the right-hand members of a comparison system of type* I (see § 14). *Assume that* $\sigma_i(t, 0, \ldots, 0) = 0$ $(i = 1, 2, \ldots, n)$ *and that the null solution of the comparison system is stable* (see [7], p. 314).

Under these assumptions the null solution of system (26.1) *is stable.*

6*

Proof. In view of the stability of the null solution of the comparison system, there is an $h_0 < h$ such that whenever

$$0 \leqslant \eta_i < h_0 \quad (i = 1, 2, \ldots, n),$$

then any solution $\omega_i(t)$ of the comparison system, starting from the point $(0, H) = (0, \eta_1, \ldots, \eta_n)$, is defined in the interval $[0, +\infty)$ and satisfies the inequalities $\omega_i(t) < h$ $(i = 1, 2, \ldots, n)$. Hence, by (27.1) and by Theorem 23.1, any solution of system (26.1) through a point $(x_0, \widetilde{Y})$ exists in the interval $[x_0, +\infty)$, whenever $\widetilde{Y} = (\widetilde{y}_1, \ldots, \widetilde{y}_n)$ satisfies the inequalities

$$|\widetilde{y}_i| < h_0 \quad (i = 1, 2, \ldots, n).$$

Moreover, for any such solution $Y(x; \widetilde{Y}) = \big(y_1(x; \widetilde{Y}), \ldots, y_n(x; \widetilde{Y})\big)$ inequalities

$$|Y(x; \widetilde{Y})| \leqslant \Omega(x - x_0; |\widetilde{Y}|) \tag{27.2}$$

hold true, where $\Omega(t; H)$ is the right-hand maximum solution of the comparison system through $(0, H)$. From (27.2) and from the assumptions on the comparison system follows the conclusion of our theorem.

By the same argument we prove the next theorem.

THEOREM 27.2. *If, under the hypotheses of Theorem 27.1, we additionally assume that the right-hand sides $\sigma_i(t, Y)$ of the comparison system do not depend on t, then the null solution of system (26.1) is uniformly stable.*

§ 28. Differential inequalities in the complex domain. In this section we will obtain an analogue of Theorem 15.1 in the case when $\varphi_k(z)$ $(k = 1, 2, \ldots, n)$ are holomorphic functions of the complex variable z in a disk $|z - z_0| < \gamma$.

In order to apply here the theory of differential inequalities in the real domain, we will have to consider real functions

$$M_k(t) = \max_{|z - z_0| = t} |\varphi_k(z)| \quad \text{for} \quad 0 \leqslant t < \gamma.$$

Therefore, we first prove a lemma on Dini's derivatives $D_- M_k(t)$.

LEMMA 28.1. *Let $\varphi(z)$ be holomorphic in the disk*

$$|z - z_0| < \gamma \tag{28.1}$$

and put

$$M(t) = \max_{|z - z_0| = t} |\varphi(z)| \quad \textit{for} \quad 0 \leqslant t < \gamma.$$

Then, to every $t \in (0, \gamma)$ there is a $\mathfrak{z}$ such that

$$|\mathfrak{z} - z_0| = t, \tag{28.2}$$

$$M(t) = |\varphi(\mathfrak{z})|, \tag{28.3}$$

$$D_- M(t) \leqslant |\varphi'(\mathfrak{z})|. \tag{28.4}$$

Proof. There exists, obviously, a $\mathfrak{z}$ satisfying (28.2) and (28.3). Let

$$\mathfrak{z} = z_0 + te^{\xi i},$$

where i is the imaginary unit, and take a sequence t_ν, $0 < t_\nu < \gamma$, so that $t_\nu < t$, $t_\nu \to t$ and

$$\lim_{\nu\to\infty} \frac{M(t_\nu) - M(t)}{t_\nu - t} = D_- M(t). \tag{28.5}$$

Put

$$\mathfrak{z}_\nu = z_0 + t_\nu e^{\xi i} \quad (\nu = 1, 2, \ldots).$$

Since, by the definition of $M(t)$, there is

$$M(t_\nu) \geqslant |\varphi(\mathfrak{z}_\nu)|,$$

we get, by (28.3),

$$\begin{aligned} \frac{M(t_\nu) - M(t)}{t_\nu - t} &= \frac{M(t_\nu) - |\varphi(\mathfrak{z})|}{t_\nu - t} \leqslant \frac{|\varphi(\mathfrak{z}_\nu)| - |\varphi(\mathfrak{z})|}{t_\nu - t} \\ &\leqslant \frac{\big||\varphi(\mathfrak{z}_\nu)| - |\varphi(\mathfrak{z})|\big|}{|t_\nu - t|} \leqslant \frac{|\varphi(\mathfrak{z}_\nu) - \varphi(\mathfrak{z})|}{|t_\nu - t|} = \left|\frac{\varphi(\mathfrak{z}_\nu) - \varphi(\mathfrak{z})}{\mathfrak{z}_\nu - \mathfrak{z}}\right|. \end{aligned} \tag{28.6}$$

Because of $\mathfrak{z}_\nu \to \mathfrak{z}$, relations (28.5) and (28.6) imply (28.4).

THEOREM 28.1. *Suppose that $\Phi(z) = (\varphi_1(z), \ldots, \varphi_n(z))$ is holomorphic in the disk* (28.1) *and satisfies initial inequality*

$$|\Phi(z_0)| \leqslant H, \tag{28.7}$$

where $H = (\eta_1, \ldots, \eta_n)$, as well as differential inequalities

$$|\varphi_k'(z)| \leqslant \sigma_k(|z - z_0|, |\Phi(z)|) \quad (k = 1, 2, \ldots, n) \tag{28.8}$$

in (28.1), *where $\sigma_k(t, y_1, \ldots, y_n)$ are the right-hand sides of a comparison system of type* I (see § 14).

Under these hypotheses we have

$$|\Phi(z)| \leqslant \Omega(|z - z_0|; H) \tag{28.9}$$

in the disk

$$|z - z_0| < \min(\gamma, a_0(H)), \tag{28.10}$$

where $\Omega(t; H) = (\omega_1(t; H), \ldots, \omega_n(t; H))$ is the right-hand maximum solution through $(0, H)$ of the comparison system in the interval $[0, a_0(H))$.

Proof. Put

$$M_k(t) = \max_{|z - z_0| = t} |\varphi_k(z)| \quad (k = 1, 2, \ldots, n), \qquad M(t) = (M_1(t), \ldots, M_n(t))$$

for $0 \leqslant t < \gamma$. The functions $M_k(t)$ are continuous and satisfy, by (28.7), initial inequalities

$$M(0) \leqslant H. \tag{28.11}$$

By Lemma 28.1, for any $t \in (0, \gamma)$ there is a $\mathfrak{z}_k$ such that

$$(28.12)\qquad |\mathfrak{z}_k - z_0| = t\,,\qquad M_k(t) = |\varphi_k(\mathfrak{z}_k)|\,,\qquad D_- M_k(t) \leqslant |\varphi_k'(\mathfrak{z}_k)| \qquad (k = 1, 2, \ldots, n)\,.$$

Hence, by (28.8), we have

$$(28.13)\qquad D_- M_k(t) \leqslant |\varphi_k'(\mathfrak{z}_k)| \leqslant \sigma_k\big(|\mathfrak{z}_k - z_0|, |\Phi(\mathfrak{z}_k)|\big) \qquad (k = 1, 2, \ldots, n)\,.$$

Further, by the definition of $M_k(t)$ and by (28.12), the following inequalities hold true (see § 4):

$$|\Phi(\mathfrak{z}_k)| \overset{k}{\leqslant} M(t) \qquad (k = 1, 2, \ldots, n)\,.$$

Therefore, in view of condition W_+ (see § 4), we have

$$(28.14)\qquad \sigma_k\big(|\mathfrak{z}_k - z_0|, |\Phi(\mathfrak{z}_k)|\big) \leqslant \sigma_k\big(t, M(t)\big) \qquad (k = 1, 2, \ldots, n)\,.$$

Inequalities (28.13) and (28.14) imply

$$(28.15)\qquad D_- M_k(t) \leqslant \sigma_k\big(t, M(t)\big) \qquad (k = 1, 2, \ldots, n)$$

in the interval $(0, \gamma)$. From (28.11) and (28.15) it follows, by Theorem 9.3, that

$$(28.16)\qquad M(t) \leqslant \Omega(t; H)$$

in the interval $0 \leqslant t < \min\big(\gamma, \alpha_0(H)\big)$; but inequalities (28.16) are equivalent with (28.9) in the disk (28.10), which completes the proof.

§ 29. Estimates of the solution and of its radius of convergence for differential equations in the complex domain. This paragraph deals with an analogue of Theorem 23.1 in the complex domain (see [58]). To start with, we state an analogue of Theorem 7.3, which is easily proved by the method of successive approximations.

THEOREM 29.1. *Let the right-hand sides of the system*

$$(29.1)\qquad \frac{dz_k}{dz} = f_k(z, z_1, \ldots, z_n) \qquad (k = 1, 2, \ldots, n)$$

be analytic functions of $n+1$ complex variables $(z, z_1, \ldots, z_n)$ in the domain

$$(29.2)\qquad |z - \mathfrak{z}_0| < h\,,\qquad |z_k - \mathring{\mathfrak{z}}_k| < h' \qquad (k = 1, 2, \ldots, n)$$

and suppose that in (29.2)

$$(29.3)\qquad |f_k(z, Z)| \leqslant M \qquad (k = 1, 2, \ldots, n)\,.$$

Under these assumptions the unique solution $Z(z) = \big(z_1(z), \ldots, z_n(z)\big)$ of system (29.1), *satisfying initial conditions*

$$(29.4)\qquad z_k(\mathfrak{z}_0) = \mathring{\mathfrak{z}}_k \qquad (k = 1, 2, \ldots, n)\,,$$

is holomorphic in the disk

$$|z-\mathfrak{z}_0| < \min\left(h, \frac{h'}{M}\right). \tag{29.5}$$

THEOREM 29.2. *Suppose that the right-hand members* $f_k(z, z_1, \ldots, z_n)$ *of system* (29.1) *are analytic functions in the complex domain*

$$D\colon |z-z_0| < r, \ |z_k - \mathring{z}_k| < r_k \quad (k = 1, 2, \ldots, n)$$

and satisfy the inequalities

$$|f_k(z, Z)| \leqslant \sigma_k(|z-z_0|, |Z-Z_0|) \quad (k = 1, 2, \ldots, n), \tag{29.6}$$

where $Z_0 = (\mathring{z}_1, \ldots, \mathring{z}_n)$ *and* $\sigma_k(t, y_1, \ldots, y_n)$ *are the right-hand sides of a comparison system of type* I (see § 14). *Denote by* $\Omega(t; H) = (\omega_1(t; H), \ldots, \omega_n(t; H))$ *its right-hand maximum solution through* $(0, H) = (0, \eta_1, \ldots, \eta_n)$, *defined in the interval* $[0, a_0(H))$. *Suppose that* $Z(z) = (z_1(z), \ldots, z_n(z))$ *is a solution of system* (29.1) *satisfying initial inequality*

$$|Z(z_0) - Z_0| \leqslant H < R, \tag{29.7}$$

where $R = (r_1, \ldots, r_n)$. *Denote by* t_k *the least root of the equation in* t

$$\omega_k(t; H) = r_k$$

if such a root exists in the interval $(0, a_0)$; *if it does not exist, put* $t_k = +\infty$.

Under these hypotheses the solution $Z(z)$ *is holomorphic in the disk*

$$|z-z_0| < r_0 = \min(r, a_0, t_1, \ldots, t_n) \tag{29.8}$$

and satisfies there the inequalities

$$|Z(z) - Z_0| \leqslant \Omega(|z-z_0|; H). \tag{29.9}$$

Proof. Let

$$|z-z_0| < \gamma \leqslant r \tag{29.10}$$

be the largest disk in which the solution $Z(z)$ is holomorphic and put

$$\Phi(z) = (\varphi_1(z), \ldots, \varphi_n(z)) = (z_1(z) - \mathring{z}_1, \ldots, z_n(z) - \mathring{z}_n).$$

The function $\Phi(z)$ is holomorphic in the disk (29.10) and, by (29.7), satisfies initial inequality (28.7). By (29.6), we have in (29.10)

$$\begin{aligned} |\varphi_k'(z)| = |z_k'(z)| = |f_k(z, Z(x))| &\leqslant \sigma_k(|z-z_0|, |Z(z) - Z_0|) \\ &= \sigma_k(|z-z_0|, |\Phi(z)|) \quad (k = 1, 2, \ldots, n). \end{aligned}$$

Hence, by Theorem 28.1, inequalities (28.9) are satisfied in the disk (28.10) and consequently inequalities (29.9) hold true in the disk (28.10). Therefore, to complete the proof it remains to show that $r_0 \leqslant \gamma$. Suppose the contrary is true, i.e. $r_0 > \gamma$; then $\gamma \in (0, a_0)$ and, by the definition of t_k, we have $\omega_k(\gamma; H) < r_k$ $(k = 1, 2, \ldots, n)$. Choose γ', b and b' so that

$$\gamma < \gamma' < r_0\,, \tag{29.11}$$

$$\omega_k(\gamma; H) \leqslant b < b' < r_k \quad (k = 1, 2, \ldots, n) \tag{29.12}$$

and consider the compact domain

$$D_1\colon\ |z - z_0| \leqslant \gamma'\,,\ |z_k - \mathring{z}_k| \leqslant b' \quad (k = 1, 2, \ldots, n)\,.$$

Obviously $D_1 \subset D$ and there is an M such that

$$|f_k(z, Z)| \leqslant M \text{ in } D_1 \quad (k = 1, 2, \ldots, n)\,. \tag{29.13}$$

Put

$$h = \frac{\gamma' - \gamma}{2}\,, \quad h' = \frac{b' - b}{2} \tag{29.14}$$

and choose $\varrho > 0$ such that

$$\varrho < \gamma\,, \quad \gamma - \varrho < \min\left(h, \frac{h'}{M}\right)\,. \tag{29.15}$$

Let $\mathfrak{z} = z_0 + \gamma e^{i\xi}$ be an arbitrary point of the circle $|z - z_0| = \gamma$ and put $\mathfrak{z}_0 = z_0 + \varrho e^{i\xi}$, $\mathring{\mathfrak{z}}_k = z_k(\mathfrak{z}_0)$ $(k = 1, 2, \ldots, n)$. Since inequalities (29.9) hold true in the disk (28.10) and since $\varrho < \gamma < \alpha_0$, we have, by (29.12),

$$\begin{aligned} |\mathring{\mathfrak{z}}_k - \mathring{z}_k| &= |z_k(\mathfrak{z}_0) - \mathring{z}_k| \leqslant \omega_k(|\mathfrak{z}_0 - z_0|; H) \\ &= \omega_k(\varrho; H) \leqslant \omega_k(\gamma; H) \leqslant b \quad (k = 1, 2, \ldots, n)\,. \end{aligned} \tag{29.16}$$

Consider the domain

$$D_2\colon\ |z - \mathfrak{z}_0| < h\,,\ |z_k - \mathring{\mathfrak{z}}_k| < h' \quad (k = 1, 2, \ldots, n)\,,$$

with h and h' defined by formulas (29.14). We claim that $D_2 \subset D_1$. Indeed, by (29.11), (29.12), (29.15) and (29.16), we have

$$\begin{aligned} |z - \mathfrak{z}_0| < h \Rightarrow |z - z_0| &\leqslant |z - \mathfrak{z}_0| + |\mathfrak{z}_0 - z_0| < h + \varrho \\ &= \frac{\gamma' - \gamma}{2} + \varrho < \frac{\gamma' - \gamma}{2} + \gamma = \frac{\gamma' + \gamma}{2} < \gamma'\,, \end{aligned}$$

$$\begin{aligned} |z_k - \mathring{\mathfrak{z}}_k| < h' \Rightarrow |z_k - \mathring{z}_k| &\leqslant |z_k - \mathring{\mathfrak{z}}_k| + |\mathring{\mathfrak{z}}_k - \mathring{z}_k| < h' + b \\ &= \frac{b' - b}{2} + b = \frac{b' + b}{2} < b'\,. \end{aligned}$$

Therefore, by (29.13),

$$|f_k(z, Z)| \leqslant M \quad \text{in} \quad D_2$$

and, by Theorem 29.1, the unique solution $\psi_k(z)$ $(k = 1, 2, \ldots, n)$ of system (29.1), satisfying initial conditions

$$\psi_k(\mathfrak{z}_0) = \mathring{\mathfrak{z}}_k = z_k(\mathfrak{z}_0) \quad (k = 1, 2, \ldots, n)\,, \tag{29.17}$$

is holomorphic in the disk (29.5). We claim that the last disk contains the point $\mathfrak{z} = z_0 + \gamma e^{i\xi}$. Indeed, by (29.15),

$$|\mathfrak{z} - \mathfrak{z}_0| = \gamma - \varrho < \min\left(h, \frac{h'}{M}\right);$$

but, in view of (29.17) and of the uniqueness of the solution of the Cauchy problem, we have

$$\psi_k(z) = z_k(z) \qquad (k = 1, 2, \ldots, n)$$

in the intersection of the disk (29.5) and (29.10). That means that $\psi_k(z)$ is the analytic continuation of $z_k(z)$ in the neighborhood of the point $\mathfrak{z} = z_0 + \gamma e^{i\xi}$. Hence, $\mathfrak{z}$ being an arbitrary point of the circle $|z - z_0| = \gamma$, it follows that $Z(z)$ is holomorphic in a larger disk than (29.10), contrary to the definition of the disk (29.10). This contradiction completes the proof of the inequality $r_0 \leqslant \gamma$.

§ 30. Estimates of the difference between two solutions in the complex domain. Here we prove an analogue of Theorem 24.1.

THEOREM 30.1. *Let the right-hand sides of system* (29.1) *and of system*

$$(30.1) \qquad \frac{dz_k}{dz} = \tilde{f}_k(z, z_1, \ldots, z_n) \qquad (k = 1, 2, \ldots, n)$$

be analytic in an open region D and satisfy the inequalities

$$(30.2) \qquad |f_k(z, Z) - \tilde{f}_k(z, \tilde{Z})| \leqslant \sigma_k(|z - z_0|, |Z - \tilde{Z}|) \qquad (k = 1, 2, \ldots, n),$$

where $\sigma_k(t, Y)$ are the right-hand members of a comparison system of type I (see § 14). *Suppose $Z(z) = (z_1(z), \ldots, z_n(z))$ and $\tilde{Z}(z) = (\tilde{z}_1(z), \ldots, \tilde{z}_n(z))$ are two solutions of system* (29.1) *and* (30.1) *respectively, holomorphic in a disk $|z - z_0| < \gamma$ and satisfying the initial inequality*

$$(30.3) \qquad |Z(z_0) - \tilde{Z}(z_0)| \leqslant H,$$

where $H = (\eta_1, \ldots, \eta_n)$. Let $\Omega(t; H) = (\omega_1(t; H), \ldots, \omega_n(t; H))$ be the right-hand maximum solution of the comparison system through $(0, H)$, defined in the interval $[0, a_0)$.

Under these assumptions we have

$$(30.4) \qquad |Z(z) - \tilde{Z}(z)| \leqslant \Omega(|z - z_0|; H)$$

in the disk

$$(30.5) \qquad |z - z_0| < \min(\gamma, a_0).$$

Proof. Put in the disk $|z - z_0| < \gamma$

$$\Phi(z) = (\varphi_1(z), \ldots, \varphi_n(z)) = (z_1(z) - \tilde{z}_1(z), \ldots, z_n(z) - \tilde{z}_n(z));$$

then, by (30.2) and (30.3), we have

$$|\varphi'_k(z)| = |z'_k(z)-\tilde{z}'_k(z)| = \left|f_k(z, Z(z))-f_k(z, \tilde{Z}(z))\right| \leqslant \sigma_k(|z-z_0|, |\Phi(z)|)$$
$$(k = 1, 2, \dots, n)$$

and

$$|\Phi(z_0)| \leqslant H .$$

Hence, by Theorem 28.1, inequalities (30.4) hold true in the disk (30.5).

To close this section we make the following remark. All the results of § 26 are valid for systems (29.1) of ordinary differential equations in the complex domain. Indeed, in our considerations in § 26 we used only Theorems 23.1 and 24.1, while their analogues in the complex domain, viz. Theorems 29.2 and 30.1, have just been proved in § 29 and § 30.

§ 31. Chaplygin method for ordinary differential equations. We consider the differential equation

$$u' = f(t, u) \tag{31.1}$$

with the initial condition

$$u(0) = u_0 , \tag{31.2}$$

where $f(t, u)$ is continuous for $0 \leqslant t \leqslant a$ and arbitrary u. Suppose that $f_u(t, u)$ is continuous in (t, u). Given an arbitrary continuous function $\varphi(t)$, $t \in [0, a]$, we write down the equation

$$u' = f(t, \varphi(t)) + f_u(t, \varphi(t))(u-\varphi(t)) \equiv \underline{\delta}(t, u; \varphi) . \tag{31.3}$$

The right-hand side of this equation is a linear approximation of that of (31.1). This is nothing else but the analogue of Newton's method known for numerical equations. Like in this classical case, we need some a priori bounds for solutions. To begin with we introduce the following definition:

DEFINITION. Let the function $\varphi(t)$ $(\psi(t))$ be differentiable in the interval $[0, a]$. We say that $\varphi(t)$ $(\psi(t))$ is a *lower* (*upper*) *function* if $\varphi'(t) \leqslant f(t, \varphi(t))$, $t \in [0, a]$ $(\psi'(t) \geqslant f(t, \psi(t)))$, $\varphi(0) = u_0$ $(\psi(0) = u_0)$.

Notice now that if $f_u(t, u)$ is continuous, then the Cauchy problem (31.1), (31.2) has the uniqueness property. Denote its unique solution by $u(t)$. It follows then from Theorem 9.5 and from the classical continuation procedure (see Theorem 7.1) that the following proposition holds true:

PROPOSITION 31.1. *Let $f(t, u)$, $f_u(t, u)$ be continuous and suppose that there exist an upper function $\psi(t)$ and a lower one $\varphi(t)$. Then the unique solution $u(t)$ of* (31.1), (31.2) *exists all over the interval $[0, a]$ and $\varphi(t) \leqslant u(t) \leqslant \psi(t)$ for $0 \leqslant t \leqslant a$.*

EXAMPLE. Suppose that

$$-A|u|-B < f(t, u) < A|u|+B .$$

We can take $\varphi(t)$ as the solution of

$$u' = -A|u|-B , \quad \varphi(0) = u_0$$

and $\psi(t)$ as the solution of

$$u' = A|u|+B , \quad \psi(0) = u_0 .$$

Besides the linear approximation of type (31.3) we can approximate equation (31.1) by the equation

$$u' = \bar{\delta}(t, u; \varphi, \psi) \equiv f\big(t, \varphi(t)\big) + \frac{f\big(t, \varphi(t)\big) - f\big(t, \psi(t)\big)}{\varphi(t) - \psi(t)}\big(u - \varphi(t)\big)$$

provided that $\varphi(t) < \psi(t)$. If $\varphi(t) = \psi(t)$, then we put

$$\bar{\delta}(t, u; \varphi, \psi) = f\big(t, \varphi(t)\big) + f_u\big(t, \varphi(t)\big)\big(u - \varphi(t)\big) .$$

We say that the couple (φ, ψ) is *admissible* if $\varphi(t)$ is a lower function and $\psi(t)$ is an upper function.

In what follows we deal with the method originated by Chaplygin in [6] and developed by Lusin [20]. The first theorem is the following one:

THEOREM 31.1. *Suppose that the couple (φ, ψ) is admissible. Let $f(t, u)$ and $f_u(t, u)$ be continuous and suppose that $f_u(t, u)$ increases in u.*

Define now: $\bar{\varphi}(t)$ = the solution of $u' = \underline{\delta}(t, u; \varphi)$ such that $\bar{\varphi}(0) = u_0$, $\bar{\psi}(t)$ = the solution of $u' = \bar{\delta}(t, u; \varphi, \psi)$ such that $\bar{\psi}(0) = u_0$.

Then $(\bar{\varphi}, \bar{\psi})$ is an admissible couple and

$$\varphi(t) \leqslant \bar{\varphi}(t) \leqslant u(t) \leqslant \bar{\psi}(t) \leqslant \psi(t) \quad \textit{for} \quad 0 \leqslant t \leqslant a .$$

Proof. The functions $\bar{\varphi}, \bar{\psi}$ are the solutions of linear equations. Hence they are defined all over the interval $[0, a]$. We have $\varphi'(t) \leqslant f\big(t, \varphi(t)\big) = \underline{\delta}\big(t, \varphi(t); \varphi\big)$, $\varphi(0) = u_0 = \bar{\varphi}(0)$ and $\bar{\varphi}'(t) = \underline{\delta}\big(t, \bar{\varphi}(t); \varphi\big)$. It follows then from Theorem 9.5 that

$$\varphi(t) \leqslant \bar{\varphi}(t) . \tag{31.4}$$

On the other hand, the function $f(t, u)$ is convex in u. Hence $\bar{\varphi}'(t) = \underline{\delta}\big(t, \bar{\varphi}(t); \varphi\big) = f\big(t, \varphi(t)\big) + f_u\big(t, \varphi(t)\big)\big(\bar{\varphi}(t) - \varphi(t)\big) \leqslant f\big(t, \bar{\varphi}(t)\big)$. We see that $\bar{\varphi}(t)$ is a lower function and consequently, by Proposition 31.1, $\bar{\varphi}(t) \leqslant u(t)$.

Notice now that

$$\bar{\delta}\big(t, \psi(t); \varphi, \psi\big) = f\big(t, \psi(t)\big) .$$

But $\psi(t)$ is an upper function. Hence $\psi'(t) \geqslant f(t, \psi(t))$ and consequently

$$\psi'(t) \geqslant \bar{\delta}(t, \psi(t); \varphi, \psi) .$$

Since $\psi(0) = \bar{\psi}(0) = u_0$, Theorem 9.5 applies and we get $\bar{\psi}(t) \leqslant \psi(t)$. Observe that $\bar{\delta}(t, \varphi(t); \varphi, \psi) = f(t, \varphi(t)) \geqslant \varphi'(t)$ and $\varphi(0) = \bar{\psi}(0)$. By Theorem 9.5, we get therefore $\varphi(t) \leqslant \bar{\psi}(t)$. This last inequality together with the convexity of $f(t, u)$ in u proves that $\bar{\psi}'(t) = \bar{\delta}(t, \bar{\psi}(t); \varphi, \psi) \geqslant f(t, \bar{\psi}(t))$, i.e. $\bar{\psi}(t)$ is an upper function. It follows then that $u(t) \leqslant \bar{\psi}(t)$ which completes the proof.

The above theorem defines the transformation $(\varphi, \psi) \to (\bar{\varphi}, \bar{\psi})$. We denote this transformation by C and thus get $(\bar{\varphi}, \bar{\psi}) = C(\varphi, \psi)$. Moreover, Theorem 31.1 shows that C maps admissible couples on admissible ones. If we start with an admissible couple (φ_0, ψ_0), then the sequence $(\varphi_{n+1}, \psi_{n+1}) = C(\varphi_n, \psi_n)$ is well defined. It consists of admissible couples or more precisely the following conditions hold true:

$$\varphi_n(0) = u(0) = \psi_n(0) = u_0 , \tag{31.5}$$

$$\varphi_n'(t) \leqslant f(t, \varphi_n(t)) , \tag{31.6}$$

$$\psi_n'(t) \geqslant f(t, \psi_n(t)) , \tag{31.7}$$

$$\varphi_n(t) \leqslant \varphi_{n+1}(t) \leqslant u(t) \leqslant \psi_{n+1}(t) \leqslant \psi_n(t) , \tag{31.8}$$

$$\varphi_n'(t) = \underline{\delta}(t, \varphi_n(t); \varphi_{n-1}) , \tag{31.9}$$

$$\psi_n'(t) = \bar{\delta}(t, \psi_n(t); \varphi_{n-1}, \psi_{n-1}) . \tag{31.10}$$

The sequence (φ_n, ψ_n) is called the *Chaplygin sequence.*

Next we prove

THEOREM 31.2. *Under the assumptions of Theorem* 31.1, *if* (φ, ψ) *is an admissible couple, the Chaplygin sequence*

$$(\varphi_0, \psi_0) = (\varphi, \psi) , \quad (\varphi_{n+1}, \psi_{n+1}) = C(\varphi_n, \psi_n)$$

is uniformly convergent to $u(t)$ *on* $[0, a]$.

Proof. It follows from (31.8) that the sequences $\{\varphi_n(t)\}$ and $\{\psi_n(t)\}$ are uniformly bounded on $[0, a]$. Let

$$\max\{|\varphi_n(t)|, |\psi_n(t)|\} \leqslant K < +\infty$$

for $n = 0, 1, 2, \ldots,\ 0 \leqslant t \leqslant a$; then

$$|\varphi_n'(t)| \leqslant \left|f(t, \varphi_{n-1}(t))\right| + \left|f_u(t, \varphi_{n-1}(t))\right| \left(|\varphi_n(t)| + |\varphi_{n-1}(t)|\right) ,$$

$$|\psi_n'(t)| \leqslant \left|f(t, \varphi_{n-1}(t))\right| + \left|f_u(t, \theta_n(t))\right| \left(|\psi_n(t)| + |\varphi_{n-1}(t)|\right) ,$$

where

$$\varphi_{n-1}(t) \leqslant \theta_n(t) \leqslant \psi_{n-1}(t) .$$

Write

$$R = \max\{\sup_{0\leqslant t\leqslant a,\,|u|\leqslant K} |f(t,u)|\,, \qquad \sup_{0\leqslant t\leqslant a,\,|u|\leqslant K} |f_u(t,u)|\};$$

then

$$|\varphi_n'(t)| \leqslant R+2RK\,, \qquad |\psi_n'(t)| \leqslant R+2RK$$

and consequently $\varphi_n(t)$ and $\psi_n(t)$ are equicontinuous on $[0,a]$. But, these sequences are equibounded on $[0,a]$. By Arzela's theorem both of them have uniformly convergent subsequences. Since both are monotonic, they must be uniformly convergent. We will show that both limit functions are equal to the unique solution $u(t)$ of the problem (31.1), (31.2). Indeed, we have

$$\left|\varphi_n'(t)-f\big(t,\varphi_{n-1}(t)\big)\right| \leqslant R\,|\varphi_n(t)-\varphi_{n-1}(t)|\,.$$

Hence

$$\left|\varphi_n(t)-u_0-\int_0^t f\big(\tau,\varphi_{n-1}(\tau)\big)\,d\tau\right| \leqslant R\int_0^a |\varphi_n(\tau)-\varphi_{n-1}(\tau)|\,d\tau\,.$$

The right-hand side of the last inequality tends to zero. It follows that $\lim\limits_{n\to\infty}\varphi_n$ is the (unique) solution of problem (31.1), (31.2) and consequently $u(t)=\lim\limits_{n\to\infty}\varphi_n(t)$.

Write $v(t)=\lim\limits_{n\to\infty}\psi_n(t)$. It follows from (31.10) and from the definition of $\bar{\delta}$ that

$$\left|\psi_n'(t)-f\big(t,\varphi_{n-1}(t)\big)\right| \leqslant R\,|\psi_n(t)-\varphi_{n-1}(t)|\,.$$

The integration and the equalities $\psi_n(0)=u_0=\varphi_n(0)$ give us

$$\left|\psi_n(t)-u_0-\int_0^t f\big(\tau,\varphi_{n-1}(\tau)\big)\,d\tau\right| \leqslant R\int_0^t |\psi_n(\tau)-\varphi_{n-1}(\tau)|\,d\tau\,.$$

The limit passage in this inequality and the fact that $u(t)=\lim\limits_{n\to\infty}\varphi_n(t)$ satisfies

$$u(t)=u_0+\int_0^t f\big(\tau,u(\tau)\big)\,d\tau$$

imply that

$$|v(t)-u(t)| \leqslant R\int_0^t |v(\tau)-u(\tau)|\,d\tau\,.$$

By theorem on integral inequalities (see § 22), we get $|v(t)-u(t)|=0$, i.e. $v(t)=u(t)$, as was to be proved.

Following Lusin we will prove

THEOREM 31.3. *Suppose that $f_{uu}(t,u)$ exists, is bounded and $f_{uu}(t,u)\geqslant 0$ in $D=\{(t,u)\colon 0\leqslant t\leqslant a,\ \varphi_0(t)\leqslant u\leqslant\psi_0(t)\}$.*

Let (φ_0, ψ_0) *be an admissible couple and write*

$$C = \frac{1}{2Hae^{Ka}},$$

where $K = \sup_D |f_u(t, u)|$, $H = \sup_D |f_{uu}(t, u)|$.

Assume that $0 \leqslant \psi_0(t) - \varphi_0(t) \leqslant C$. *Then, for the Chaplygin sequence*

$$|\varphi_n(t) - \psi_n(t)| \leqslant \frac{2C}{2^{2^n}} \tag{31.11}$$

and consequently, by (31.8),

$$|u(t) - \varphi_n(t)| \leqslant \frac{2C}{2^{2^n}}, \qquad |u(t) - \psi_n(t)| \leqslant \frac{2C}{2^{2^n}}.$$

Proof. (31.11) holds for $n = 0$. Let it hold for some n. It follows from the definition of $\varphi_{n+1}, \psi_{n+1}$ that

$$\psi'_{n+1}(t) - \varphi'_{n+1}(t) = f_u(t, p)\big(\psi_{n+1}(t) - \varphi_{n+1}(t)\big) + \\ + f_u(t, p)\big(\varphi_{n+1}(t) - \varphi_n(t)\big) - f_u\big(t, \varphi_n(t)\big)\big(\varphi_{n+1}(t) - \varphi_n(t)\big),$$

where

$$\varphi_n(t) \leqslant p \leqslant \psi_n(t). \tag{31.13}$$

On the other hand,

$$f_u(t, p) - f_u\big(t, \varphi_n(t)\big) = f_{uu}(t, q)\big(p - \varphi_n(t)\big), \tag{31.14}$$

where $\varphi_n(t) \leqslant q \leqslant p$. But

$$|f_u(t, p)| \leqslant K, \qquad |f_{uu}(t, q)| \leqslant H.$$

It follows from (31.13) and from (31.14) that

$$|\psi'_{n+1}(t) - \varphi'_{n+1}(t)| \\ \leqslant K|\psi_{n+1}(t) - \varphi_{n+1}(t)| + H|p - \varphi_n(t)||\varphi_{n+1}(t) - \varphi_n(t)|. \tag{31.15}$$

But $|p - \varphi_n(t)| \leqslant |\psi_n(t) - \varphi_n(t)|$ by (31.13). Notice that

$$\varphi_n(t) \leqslant \varphi_{n+1}(t) \leqslant u(t) \leqslant \psi_n(t);$$

hence

$$|\varphi_{n+1}(t) - \varphi_n(t)| \leqslant |\psi_n(t) - \varphi_n(t)|.$$

It follows from (31.15) and from the above inequalities that

$$|\psi'_{n+1}(t) - \varphi'_{n+1}(t)| \leqslant K|\psi_{n+1}(t) - \varphi_{n+1}(t)| + H|\psi_n(t) - \varphi_n(t)|^2.$$

We have assumed that

$$|\psi_n(t) - \varphi_n(t)| \leqslant \frac{2C}{2^{2^n}}.$$

We obtain, therefore,

$$|\psi'_{n+1}(t)-\varphi'_{n+1}(t)| \leqslant K|\psi_{n+1}(t)-\varphi_{n+1}(t)|+H\frac{2^2C^2}{2^{2^{n+1}}}$$

and consequently, by Theorem 15.1 when applied to $\psi_{n+1}(t)-\varphi_{n+1}(t)$,

$$|\psi_{n+1}(t)-\varphi_{n+1}(t)| \leqslant \int_0^t e^{K(t-s)}H\frac{2^2C^2}{2^{2^{n+1}}}ds .$$

Now

$$\frac{2^2C^2}{2^{2^{n+1}}} = \frac{2^2}{2^2H^2a^2e^{2Ka}2^{2^{n+1}}}$$

and

$$\int_0^t e^{K(t-s)}ds \leqslant ae^{Ka} .$$

We get, therefore,

$$|\psi_{n+1}(t)-\varphi_{n+1}(t)| \leqslant \frac{Hae^{Ka}2^2}{2^2H^2a^2e^{2Ka}2^{2^{n+1}}} = \frac{2C}{2^{2^{n+1}}} , \qquad \text{q.e.d.}$$

Let us consider now the system

$$y'_i = f_i(t, y_1, \dots, y_n) \quad (i = 1, 2, \dots, n) \tag{31.16}$$

together with initial conditions

$$y_i(0) = \mathring{y}_i . \tag{31.17}$$

We assume that $f_i(t, y_1, \dots, y_n)$ are defined on $[0, a]\times R^n$. In the vector form (31.16) and (31.17) may be written as

$$Y' = F(t, Y) , \quad Y(0) = \mathring{Y} . \tag{31.18}$$

The vector-valued function $\Phi(t)$ is called *lower* if

$$\Phi(0) = \mathring{Y}, \quad \Phi'(t) \leqslant F\big(t, \Phi(t)\big)\ (^1) \quad \text{on} \quad [0, a] .$$

The definition of an *upper function* is obvious.

Suppose now that f_i have continuous derivatives $\partial f_i/\partial y_j$ $(i, j = 1, 2, \dots, n)$. We write down a linear system in the vector form

$$Y' = F\big(t, \Psi(t)\big)+F_y\big(t, \Psi(t)\big)\big(Y-\Psi(t)\big) \overset{\text{df}}{\equiv} G(t, Y; \Psi) , \tag{31.19}$$

where F_y stands for the matrix $\{\partial f_i/\partial y_j\}$ and $\Psi(t)$ is continuous vector-valued function.

Let us introduce the following condition:

$$F(t, U)+F_y(t, U)(V-U) \leqslant F(t, V) \quad \text{for} \quad V \geqslant U . \tag{31.20}$$

(1) For the meaning of the inequality sign, see § 4.

Suppose now that f_i satisfy condition W_+ (see § 4) and let the solution $Y(t)$ of (31.18) exist all over $[0, a]$. It follows from Theorem 9.3 that if $\Phi(t)$ is lower, then $\Phi(t) \leqslant Y(t)$ on $[0, a]$. On the other hand, given a vector function $\Psi(t)$, continuous on $[0, a]$, we can find a unique solution $\overline{\Psi}(t)$ of (31.19) such that $\overline{\Psi}(0) = \overset{\circ}{Y}$. The system (31.19) is linear. Hence $\overline{\Psi}(t)$ exists on the whole interval $[0, a]$. We have thus the transformation $\Psi \to \overline{\Psi}$; formally $\overline{\Psi} = C(\Psi)$. The question is whether $\overline{\Psi}$ is lower function if Ψ is a lower one. We will prove

THEOREM 31.4. *Suppose that f_i are of class C^1 and satisfy condition* W_+ *and let $Y(t)$ be the solution of* (31.18) *existing on* $[0, a]$. *Let $F(t, Y)$ satisfy* (31.20) *and let $\Phi(t)$ be lower. Then $\Psi = C(\Phi)$ is lower and*

$$\Phi(t) \leqslant \Psi(t) \leqslant Y(t) \quad on \quad [0, a].$$

Proof. Notice that since f_i satisfy condition W_+, then $\partial f_i/\partial y_j \geqslant 0$ for $i \neq j$. It follows then that the right-hand sides of system (31.19) satisfy condition W_+.

We have:

$$\Psi'(t) = G\big(t, \Psi(t); \Phi\big), \quad \Phi'(t) \leqslant F\big(t, \Phi(t)\big) = G\big(t, \Phi(t); \Phi\big), \quad \Phi(0) = \Psi(0).$$

But $G(t, Y; \Phi)$ satisfies condition W_+. By Theorem 9.3 we get, therefore, $\Phi(t) \leqslant \Psi(t)$ and consequently, by (31.20),

$$\Psi'(t) = G\big(t, \Psi(t); \Phi\big) \leqslant F\big(t, \Psi(t)\big).$$

Hence, $\Psi(t)$ is lower what implies $\Psi(t) \leqslant Y(t)$.

The above theorem shows that the sequence

$$\Phi_0 = \Phi, \quad \Phi_{n+1} = C(\Phi_n)$$

is well defined on $[0, a]$ and $\Phi_n(t) \leqslant \Phi_{n+1}(t) \leqslant Y(t)$. This is the Chaplygin sequence for a system of ordinary differential equations. It is easy to check that $\Phi_n(t)$ tends uniformly to $Y(t)$ on $[0, a]$.

§ 32. Approximation of solutions of an ordinary differential equation in a Banach space. Preceding sections concerned scalar differential equations. We could get some estimates for absolute values by using differential inequalities. It is of some interest to consider equation of form (31.1) from the purely metric point of view. What we have in mind is the discussion of problem (31.1), (31.3) in Banach space, without any relation of semi-order, which is the case of scalar equations.

To be more precise, we consider the equation

$$x' = f(t, x), \tag{32.1}$$

where x and $f(t, x)$ take on the values in a Banach space E, the derivative x' being taken in the strong sense.

We add the initial condition

(32.2) $$x(0) = x_0 .$$

The elements of E will be denoted by $x, y, \ldots$ The functions of the real variable t with values in E are denoted by $x(t), y(t), \ldots$; $\|x\|$ stands for the norm of x. We will work under the assumption that $f(t, x)$ is defined for $0 \leqslant t \leqslant a$ and arbitrary x. In what follows we suppose that for every fixed t the function $f(t, x)$ is Fréchet differentiable in x to $f_x(t, x)$ (see [21], p. 300). $f_x(t, x)$ is a linear, bounded operator mapping E into E. We assume that $f_x(t, x)$ is strongly continuous in (t, x), i.e. if $t_\nu \to t$, $x_\nu \to x$ (strongly), then

$$f_x(t_\nu, x_\nu)z \to f_x(t, x)z$$

strongly for every $z \in E$. Next we introduce the assumption:

(32.3) There is a function $\omega(t, u) \geqslant 0$, continuous for $0 \leqslant t \leqslant a$, $u \geqslant 0$, increasing in u and such that $\|f_x(t, x) - f_x(t, y)\| \leqslant \omega(t, \|x-y\|)$.

Suppose now that the function $x_0(t)$ is continous on $[0, a]$ and write the equation

(32.4) $$x' = f\big(t, x_0(t)\big) + f_x\big(t, x_0(t)\big)\big(x - x_0(t)\big)$$

and

(32.5) $$x(0) = x_0 .$$

Notice that $f_x(t, x)$ being continuous, the condition (32.3) implies that $f(t, x)$ satisfies locally the Lipschitz condition in x. Moreover, we assume that $f(t, x)$ is continuous in (t, x). It follows then that (32.1), (32.2) is locally solvable (see [21], p. 291). By the same token (32.4), (32.5) has a unique solution $x(t)$, which by the linearity of (32.4) exists all over the interval $[0, a]$. Hence to every $x_0(\cdot) \in C_E[0, a]$ (1) there corresponds an $x(\cdot) \in C_E[0, a]$ via the equation (32.4). Like in § 31 we have the transformation C defined by $x = Cx_0$ and the sequence

$$x_{n+1} = Cx_n$$

is well defined. It consists of functions $x_n(\cdot) \in C_E[0, a]$ and satisfying the relations

(32.6) $$x_n(0) = x_0 ,$$

(32.7) $$x'_{n+1}(t) = f\big(t, x_n(t)\big) + f_x\big(t, x_n(t)\big)\big(x_{n+1}(t) - x_n(t)\big) .$$

We first prove

THEOREM 32.1. *Let $f(t, x)$ satisfy* (32.3) *and suppose that* $\|x_n(t)\| \leqslant M < +\infty$ *for* $0 \leqslant t \leqslant a$ $(n = 0, 1, 2, \ldots)$. *Then* $\{x_n(t)\}$ *is uniformly convergent on* $[0, a]$ *to the solution* $x(t)$ *of* (32.1), (32.2).

(1) $C_E[0, a]$ denotes here the space of E-valued functions strongly continuous on $[0, a]$.

Proof. It follows from the continuity of $f_x(t, \theta)$ and from the Banach-Steinhaus principle that

$$\sup_{[0,a]} \|f_x(t, \theta)\| = N < +\infty .$$

The difference $z_n(t) = x_{n+1}(t) - x_n(t)$ satisfies the equation

$$z_n'(t) = f_x\big(t, x_n(t)\big) z_n(t) + f\big(t, x_n(t)\big) - f_x\big(t, x_{n-1}(t)\big) z_{n-1}(t) - f\big(t, x_{n-1}(t)\big) \tag{32.8}$$

and

$$z_n(0) = \theta . \tag{32.9}$$

We need the estimate of

$$\Big\| f\big(t, x_n(t)\big) - f_x\big(t, x_{n-1}(t)\big) z_{n-1}(t) - f\big(t, x_{n-1}(t)\big) \Big\| .$$

To do this, notice that by the classical results of the theory of Banach spaces there exists a linear, continuous functional ξ with norm $\|\xi\| \leqslant 1$ such that

$$\begin{aligned} L &= \xi\big[f\big(t, x_n(t)\big) - f\big(t, x_{n-1}(t)\big) - f_x\big(t, x_{n-1}(t)\big) z_{n-1}(t)\big] \\ &= \Big\| f\big(t, x_n(t)\big) - f\big(t, x_{n-1}(t)\big) - f_x\big(t, x_{n-1}(t)\big) z_{n-1}(t) \Big\| . \end{aligned}$$

Consider the real function

$$\varphi(\tau) = \xi f\Big(t, x_{n-1}(t) + \tau\big(x_n(t) - x_{n-1}(t)\big)\Big) .$$

By mean value theorem, there is $\eta \in (0,1)$ such that

$$\varphi(1) - \varphi(0) = \xi f_x\big(t, x_{n-1}(t) + \eta z_{n-1}(t)\big) z_{n-1}(t) .$$

We apply now (32.3) and thus get

$$\begin{aligned} L &= \xi\big[f_x\big(t, x_{n-1}(t) + \eta z_{n-1}(t)\big) z_{n-1}(t) - f_x\big(t, x_{n-1}(t)\big) z_{n-1}(t)\big] \\ &\leqslant \Big\| f_x\big(t, x_{n-1}(t) + \eta z_{n-1}(t)\big) - f_x\big(t, x_{n-1}(t)\big) \Big\| \, \|z_{n-1}(t)\| \\ &\leqslant \omega\big(t, \eta \|z_{n-1}(t)\|\big) \|z_{n-1}(t)\| . \end{aligned}$$

But $\omega(t, u)$ increases in u. Hence

$$\omega\big(t, \eta\|z_{n-1}(t)\|\big) \leqslant \omega\big(t, \|z_{n-1}(t)\|\big)$$

and consequently

$$L \leqslant \omega\big(t, \|z_{n-1}(t)\|\big) \|z_{n-1}(t)\| .$$

The above estimates show that (32.3) implies

$$\begin{aligned} \Big\| f\big(t, x_n(t)\big) - f_x\big(t, x_{n-1}(t)\big) z_{n-1}(t) - f\big(t, x_{n-1}(t)\big) \Big\| \\ \leqslant \omega\big(t, \|z_{n-1}(t)\|\big) \|z_{n-1}(t)\| . \end{aligned} \tag{32.10}$$

Moreover,

(32.11) $$K = N + \max_{[0,a]} \omega(t, M) < +\infty .$$

It follows from (32.8) and (32.10) that

$$\|z_n'(t)\| \leqslant K\|z_n(t)\| + \omega\big(t, \|z_{n-1}(t)\|\big)\|z_{n-1}(t)\|$$

and consequently, by (32.9) and by Theorem 15.4,

$$\|z_n(t)\| \leqslant \int_0^t e^{K(t-s)}\omega\big(s, \|z_{n-1}(s)\|\big)\|z_{n-1}(s)\|ds .$$

But $\|z_n(t)\| \leqslant 2M$; hence,

$$\|z_n(t)\| \leqslant 2M\frac{(Ft)^{n-1}}{(n-1)!} \qquad (n = 1, 2, \ldots),$$

where

$$F = R\exp(Ra), \qquad R = \max\big(K, \max_{[0,a]} \omega(t, 2M)\big) .$$

We infer, by completeness of E, that $\{x_n(t)\}$ is uniformly convergent on $[0, a]$ to a certain limit $y(t)$. By (32.6), (32.7),

$$x_{n+1}(t) = x_0 + \int_0^t \big[f\big(s, x_n(s)\big) + f_x\big(s, x_n(s)\big)\big(x_{n+1}(s) - x_n(s)\big)\big]ds .$$

The limit passage gives us

$$y(t) = x_0 + \int_0^t f\big(s, y(s)\big)ds ,$$

which, by uniqueness of (32.1), (32.2), proves that $y(t) = x(t)$, q.e.d. The Lusin estimates can be generalized as follows:

Theorem 32.2. *Suppose that the assumptions of Theorem* 32.1 *hold true and suppose that*

$$\|x_1(t) - x(t)\| \leqslant w_1(t), \qquad 0 \leqslant t \leqslant a .$$

We define

$$w_{n+1}(t) = \int_0^t e^{K(t-s)}\omega\big(s, w_n(s)\big)w_n(s)ds ,$$

with

$$K = \sup_{[0,a]}\|f_x(t, \theta)\| + \max_{[0,a]} \omega(t, M) .$$

Then $\|x_n(t) - x(t)\| \leqslant w_n(t)$.

Proof. Let $\varphi_n(t) = \|x_n(t) - x(t)\|$. We have

$$x'(t) = f\big(t, x(t)\big)$$

and, by (32.7),

$$[x_n(t)-x(t)]' = f_x(t, x_{n-1}(t))[x_n(t)-x(t)]+ \\ +f_x(t, x_{n-1}(t))[x(t)-x_{n-1}(t)]+[f(t, x_{n-1}(t))-f(t, x(t))] .$$

Condition (32.3) implies that (see the proof of (32.10))

$$D_-\varphi_n(t) \leqslant K\varphi_n(t)+\omega(t, \varphi_{n-1}(t))\varphi_{n-1}(t) .$$

Notice that $\varphi_n(0) = 0$. Hence (see Example 9.1)

$$\varphi_n(t) \leqslant \int_0^t e^{K(t-s)}\omega(s, \varphi_{n-1}(s))\varphi_{n-1}(s)\,ds .$$

Now, an easy induction and the monotonicity of $\omega(t, u)$ in u proves our assertion.

Remark. If $\omega = Qu$ ($Q = \text{const}$), then

$$\|x_n(t)-x(t)\| \leqslant \frac{2C}{2^{2^n}}$$

if

$$w_1(t) \leqslant \frac{1}{2Qa\exp(Qa)} = C .$$

The function w_1 may be chosen in many ways, by using the a priori estimates (see [28]). The most simple choice is $w_1 = 2M$.

The question of boundedness plays an essential role in Theorem 32.1. We will give a certain method of evaluation of the interval of equiboundedness for the sequence $\{x_n(t)\}$. We start with a lemma which is due to T. Ważewski.

LEMMA 32.1. *Suppose that the function $\sigma(t, u, v) \geqslant 0$ is continuous for $0 \leqslant t \leqslant a$; $u, v \geqslant 0$. We assume that $\sigma(t, u, v)$ increases in v. Suppose that for $\eta \geqslant 0$ the right-hand maximum solution $w(t, \eta)$ ($w(0, \eta) = \eta$) of the equation*

$$u' = \sigma(t, u, u)$$

exists on $[0, a]$. Under the above assumptions the right-hand maximum solution $\tilde{w}(t, \eta)$ ($\tilde{w}(0, \eta) = \eta$) of the equation

$$u' = \gamma(t, u) \equiv \sigma(t, u, w(t, \eta))$$

exists on $[0, a]$ and

$$\tilde{w}(t, \eta) \equiv w(t, \eta) .$$

Proof. The maximum solution $\tilde{w}(t, \eta)$ exists in a right-hand neighborhood of zero. Suppose that

$$w(t, \eta) < \tilde{w}(t, \eta) \tag{32.12}$$

for some t within the common part of the existence intervals of considered maximum solutions. The monotonicity of $\sigma(t, u, v)$ in v implies then

$$\tilde{w}'(t, \eta) = \sigma(t, \tilde{w}(t, \eta), w(t, \eta)) \leqslant \sigma(t, \tilde{w}(t, \eta), \tilde{w}(t, \eta)) \tag{32.13}$$

for such t. Hence, (32.12) implies (32.13) what, by Theorem 11.1, proves that $\tilde{w}(t, \eta) \leqslant w(t, \eta)$ in the common existence interval. On the other hand, $w(t, \eta)$ exists on $[0, a]$, $\sigma(t, u, v) \geqslant 0$ and $\tilde{w}(t, \eta)$ can be continued to the boundary (see § 9). It follows then that $\tilde{w}(t, \eta)$ exists all over the interval $[0, a]$. Previous arguments apply and we conclude that $\tilde{w}(t, \eta) \leqslant w(t, \eta)$ on $[0, a]$. Notice now that $w'(t, \eta) = \sigma(t, w(t, \eta), w(t, \eta)) = \gamma(t, w(t, \eta))$. Hence $w(t, \eta) \leqslant \tilde{w}(t, \eta)$ and consequently

$$w(t, \eta) = \tilde{w}(t, \eta) \quad \text{on} \quad [0, a],$$

which completes the proof.

Suppose now that the functions $F(t)$ and $G(t)$ are continuous on the interval $[0, a]$ and

$$\|f(t, x_0)\| \leqslant F(t), \quad \|f_x(t, x_0)\| \leqslant G(t) \quad \text{on} \quad [0, a].$$

Let us take the equation

$$u' = 3G(t)u + 3\omega(t, u)u + F(t)$$

and denote by $\varphi(t)$ its right-hand maximum solution such that $\varphi(0) = 0$. Let us assume that $\varphi(t)$ exists on the interval $[0, a]$. Next we prove the following theorem:

THEOREM 32.3. *Let (32.3) be satisfied and suppose that $x(t) \in C_E[0, a]$ and*

$$x(0) = x_0, \quad \|x(t) - x_0\| \leqslant \varphi(t).$$

Suppose that $y(t)$ satisfies

$$y'(t) = f(t, x(t)) + f_x(t, x(t))(y(t) - x(t)),$$
$$y(0) = x_0.$$

Then $\|y(t) - x_0\| \leqslant \varphi(t)$ on $[0, a]$.

Proof. We have

$$[y(t) - x_0]' = f(t, x(t)) + f_x(t, x(t))[y(t) - x_0] + f_x(t, x(t))[x_0 - x(t)]$$

and

$$\|f_x(t, x(t))(y(t) - x_0)\| \leqslant [G(t) + \omega(t, \varphi(t))]\,\|y(t) - x_0\|,$$
$$\|f_x(t, x(t))(x_0 - x(t)) + f(t, x(t))\| \leqslant 2\omega(t, \varphi(t))\varphi(t) + 2G(t)\varphi(t) + F(t).$$

Hence,

$$\|[y(t) - x_0]'\| \leqslant [G(t) + \omega(t, \varphi(t))]\,\|y(t) - x_0\| + 2\omega(t, \varphi(t))\varphi(t) + 2G(t)\varphi(t) + F(t) \quad \text{on} \quad [0, a]$$

and, by Theorem 15.4,

$$\|y(t)-x_0\| \leqslant \psi(t)\,, \tag{32.14}$$

where $\psi(0) = 0$ and $\psi(t)$ is the right-hand maximum solution of

$$u' = \big[G(t)+\omega\big(t,\varphi(t)\big)\big]u+\big[2\omega\big(t,\varphi(t)\big)+2G(t)\big]\varphi(t)+F(t)\,.$$

By Lemma 32.1, applied for

$$\sigma(t,u,v) = 2\omega(t,v)v+2G(t)v+F(t)+[\omega(t,v)+G(t)]u$$

we get $\psi(t) = \varphi(t)$ which, by (32.14), completes the proof.

It follows from the above theorem that if x_0 is given, then $[0, a]$ is determined by x_0, $f(t, x)$ and by $\omega(t, u)$. On the interval $[0, a]$ we get then

$$\|x_n(t)-x_0\| \leqslant \varphi(t)$$

if $x_0(t) \equiv x_0$. Hence $\{x_n(t)\}$ is equibounded on $[0, a]$. We may then evaluate a priori the interval of equiboundedness with a special choice of constant initial function $x_0(t) \equiv x_0$.

CHAPTER VI

SOME AUXILIARY THEOREMS

The theory of ordinary differential inequalities, developed in Chapter IV, enables us to get estimates for functions of one variable. Now, in the subsequent chapters we are going to deal with applications of ordinary differential inequalities to partial differential equations. Since solutions of partial differential equations are functions of several variables, we will have to associate with a given function $\varphi(t, X) = \varphi(t, x_1, \dots, x_n)$ a function $M(t)$ of one variable only, so that $\varphi(t, X) \leqslant M(t)$. In this way, an estimate from above obtained for the function $M(t)$, by means of ordinary differential inequalities, will yield automatically an estimate from above for the function $\varphi(t, X)$.

§ 33. Maximum of a continuous function of $n+1$ variables on n-dimensional planes. To begin with, we introduce the definition of a region of special type.

Region of type C. A region D in the space of points $(t, x_1, \dots, x_n)$ will be called *region of type C* if the following conditions are satisfied:

(a) D is open, contained in the zone $t_0 < t < t_0+T \leqslant +\infty$, and the intersection of the closure of D with any closed zone $t_0 \leqslant t \leqslant t_1 < t_0+T$ is bounded.

(b) The projection S_{t_1} on the space $(x_1, \dots, x_n)$ of the intersection of the closure of D with the plane $t = t_1$ is, for any $t_1 \in [t_0, t_0+T)$, non-empty.

(c) The point (t, X) being arbitrarily fixed in the closure of D, to every sequence t_ν such that $t_\nu \in [t_0, t_0+T)$ and $t_\nu \to t$, there is a sequence X_ν, so that $X_\nu \in S_{t_\nu}$ and $X_\nu \to X$.

EXAMPLES 33.1. (α) Let G be an open, bounded region in the space $(x_1, \dots, x_n)$. Then the topological product $D = (t_0, t_0+T) \times G$ is a region of type C.

(β) Another example of a region of type C is a pyramid defined by the inequalities

$$t_0 < t < t_0+T\,, \qquad |x_i - \mathring{x}_i| \leqslant a_i - L(t-t_0) \qquad (i = 1, 2, \dots, n)\,,$$

where $0 \leqslant L < +\infty$, $0 < a_i < +\infty$ and $T \leqslant \min\limits_i (a_i/L)$.

(γ) Put

$$D_1 = \{(t, X)\colon\ 0 < t < 1\ ,\ 0 < x < 2\}\ ,$$
$$D_2 = \{(t, X)\colon\ 1 \leqslant t < 2\ ,\ 0 < x < 1\}\ ,$$
$$D\ = D_1 \cup D_2\ .$$

Then D is not a region of type C.

In fact, condition (c) is not satisfied, for example, at the point $(1, \frac{3}{2})$.

THEOREM 33.1. *Let* $\varphi(t, X) = \varphi(t, x_1, \ldots, x_n)$ *be continuous in the closure of a region* D *of type* C *and put*

$$M(t) = \max_{X \in S_t} \varphi(t, X) \qquad \textit{for} \qquad t_0 \leqslant t < t_0 + T\ .$$

Then

1° *For every* $t^* \in [t_0, t_0+T)$ *there is a point* $X^* \in S_{t^*}$ *such that*

$$M(t^*) = \varphi(t^*, X^*)\ . \tag{33.1}$$

2° *If* (33.1) *holds true for an interior point* $(t^*, X^*) \in D$ *and if* $\varphi_t(t^*, X^*)$ *exists, then*

$$D^- M(t^*) \leqslant \varphi_t(t^*, X^*)\ . \tag{33.2}$$

3° $M(t)$ *is continuous in the interval* $[t_0, t_0+T)$.

Proof. Because of conditions (a) and (b), satisfied by a region of type C, S_t is a non-empty, compact set for any $t \in [t_0, t_0+T)$; hence, by the continuity of $\varphi(t, X)$, follows 1°.

Now, let (33.1) hold true for an interior point $(t^*, X^*) \in D$ and suppose that $\varphi_t(t^*, X^*)$ exists. Choose a sequence t_ν, so that $t_\nu < t^*$, $t_\nu \to t^*$ and

$$D^- M(t^*) = \lim_{\nu \to \infty} \frac{M(t_\nu) - M(t^*)}{t_\nu - t^*}\ . \tag{33.3}$$

The point (t^*, X^*) being interior we have $(t_\nu, X^*) \in D$ for ν sufficiently large and

$$\lim_{\nu \to \infty} \frac{\varphi(t_\nu, X^*) - \varphi(t^*, X^*)}{t_\nu - t^*} = \varphi_t(t^*, X^*)\ . \tag{33.4}$$

On the other hand, by the definition of $M(t)$ and by (33.1), for ν sufficiently large we have

$$\frac{M(t_\nu) - M(t^*)}{t_\nu - t^*} \leqslant \frac{\varphi(t_\nu, X^*) - \varphi(t^*, X^*)}{t_\nu - t^*}\ . \tag{33.5}$$

From (33.3), (33.4) and (33.5) follows (33.2) and thus 2° is proved. Next, fix $t \in [t_0, t_0+T)$ and take an arbitrary sequence $t_\nu \in [t_0, t_0+T)$ such that $t_\nu \to t$.

To prove 3°, it is sufficient to show that there is a subsequence t_{ν_μ} such that

$$M(t_{\nu_\mu}) \to M(t)\,. \tag{33.6}$$

By 1°, there are $\tilde{X}_\nu \in S_{t_\nu}$ and $X \in S_t$ such that

$$M(t_\nu) = \varphi(t_\nu, \tilde{X}_\nu)\,, \qquad M(t) = \varphi(t, X)\,. \tag{33.7}$$

By condition (a), there exists a subsequence $\tilde{X}_{\nu_\mu}$ such that $\tilde{X}_{\nu_\mu} \to \tilde{X} \in S_t$. Hence, by the continuity of $\varphi(t, X)$, we get

$$\varphi(t_{\nu_\mu}, \tilde{X}_{\nu_\mu}) \to \varphi(t, \tilde{X})\,. \tag{33.8}$$

In view of (33.7) and (33.8), relation (33.6) will be proved if we show that

$$\varphi(t, \tilde{X}) = M(t)\,. \tag{33.9}$$

By condition (c), since $(t, X) \in \bar{D}$ and $t_{\nu_\mu} \to t$, there is a sequence X_{ν_μ} such that $X_{\nu_\mu} \in S_{t_{\nu_\mu}}$ and $X_{\nu_\mu} \to X$. Because of continuity we have, by (33.7),

$$\varphi(t_{\nu_\mu}, X_{\nu_\mu}) \to \varphi(t, X) = M(t)\,. \tag{33.10}$$

Further, by the definition of $M(t)$ and by (33.7), we get

$$\varphi(t_{\nu_\mu}, X_{\nu_\mu}) \leqslant M(t_{\nu_\mu}) = \varphi(t_{\nu_\mu}, \tilde{X}_{\nu_\mu})\,.$$

Hence, from (33.8) and (33.10) it follows that

$$M(t) \leqslant \varphi(t, \tilde{X})\,.$$

The last inequality together with the obvious inequality (by the definition of $M(t)$)

$$M(t) \geqslant \varphi(t, \tilde{X})$$

yields (33.9), which completes the proof.

Remark 33.1. Condition (c) is essential for the continuity of function $M(t)$ in Theorem 33.1. Indeed, take for D the region from the Example 33.1, (γ) and put

$$\varphi(t, x) = \begin{cases} 0 & \text{for} \quad 0 \leqslant t \leqslant 2\,,\ 0 \leqslant x \leqslant 1\,, \\ x-1 & \text{for} \quad 0 \leqslant t \leqslant 1\,,\ 1 \leqslant x \leqslant 2\,. \end{cases}$$

Then $\varphi(t, x)$ is continuous in the closure of D, but $M(t)$ is discontinuous for $t = 1$ since obviously we have $M(t) = 1$ for $0 \leqslant t \leqslant 1$ and $M(t) = 0$ for $1 < t \leqslant 2$.

Remark 33.2. It is easily seen that if in point 2° of Theorem 33.1 the derivative $\varphi_t(t^*, X^*)$ does not exist, then (33.2) holds true with φ_t replaced by Dini's derivative D^- with respect to t.

§ 34. Maximum of the absolute value of functions of $n+1$ variables on n dimensional planes. We prove

THEOREM 34.1. *Let the functions $\varphi_l(t, X)$ $(l = 1, 2, \ldots, k)$ be continuous in the closure of a region D of type C* (see § 33). *Put*

$$W(t) = \max_{l} \{\max_{X \in S_t} |\varphi_l(t, X)|\},$$

$$M_l(t) = \max_{X \in S_t} \varphi_l(t, X) \qquad (l = 1, 2, \ldots, k),$$

$$N_l(t) = \max_{X \in S_t} \big(-\varphi_l(t, X)\big) \qquad (l = 1, 2, \ldots, k).$$

Under these assumptions the function $W(t)$ is continuous on the interval $[t_0, t_0+T)$ and for every $t \in [t_0, t_0+T)$ there is an index j and a point $X \in S_t$ such that either

$$W(t) = M_j(t) = \varphi_j(t, X), \quad D^- W(t) \leqslant D^- M_j(t), \tag{34.1}$$

or

$$W(t) = N_j(t) = -\varphi_j(t, X), \quad D^- W(t) \leqslant D^- N_j(t). \tag{34.2}$$

Relations (34.1) *or* (34.2) *are true with D^- replaced by D^+.*

Proof. Continuity of $W(t)$ follows from Theorem 33.1, 3°. Fix a $t \in [t_0, t_0+T)$ and take a sequence t_ν such that $t_\nu < t$, $t_\nu \to t$ and

$$D^- W(t) = \lim_{\nu\to\infty} \frac{W(t_\nu) - W(t)}{t_\nu - t}. \tag{34.3}$$

Obviously, for every ν there is an index j_ν and a point $X_\nu \in S_{t_\nu}$ such that either

$$W(t_\nu) = M_{j_\nu}(t_\nu) = \varphi_{j_\nu}(t_\nu, X_\nu), \tag{34.4}$$

or

$$W(t_\nu) = N_{j_\nu}(t_\nu) = -\varphi_{j_\nu}(t_\nu, X_\nu). \tag{34.5}$$

It is clear that for infinitely many indices ν we have either (34.4) with the same index, say j, or (34.5). Taking, if necessary, a suitable subsequence we may suppose that, for instance,

$$W(t_\nu) = M_j(t_\nu) = \varphi_j(t_\nu, X_\nu) \quad \text{for} \quad \nu = 1, 2, \ldots \tag{34.6}$$

Further taking, if necessary, another subsequence we may suppose (by condition (a) of a region of type C) that

$$X_\nu \to X \in S_t. \tag{34.7}$$

By (34.6), (34.7) and by the continuity of $W(t)$, $M_j(t)$ and $\varphi_j(t, X)$, we get

$$W(t) = M_j(t) = \varphi_j(t, X). \tag{34.8}$$

On the other hand, from (34.3), (34.6) and (34.8) it follows that

$$D^- W(t) = \lim_{\nu\to\infty} \frac{W(t_\nu) - W(t)}{t_\nu - t} = \lim_{\nu\to\infty} \frac{M_j(t_\nu) - M_j(t)}{t_\nu - t} \leqslant D^- M_j(t) .$$

The last inequality together with (34.8) gives (34.1). For D^+ the proof is quite similar.

§ 35. Maximum of a continuous function of several variables on plane sections of a pyramid. Here we get stronger results than those of Theorem 33.1, taking for the region D a pyramid and imposing stronger regularity requirements on the function $\varphi(t, X)$.

THEOREM 35.1. *Let $\varphi(t, X)$ be continuous in the pyramid*

$$(35.1) \qquad t_0 \leqslant t < t_0+T , \qquad |x_i - \mathring{x}_i| \leqslant a_i - L(t-t_0) \qquad (i = 1, 2. ..., n) ,$$

where $0 \leqslant L < +\infty$, $0 < a_i < +\infty$ and $T \leqslant \min\limits_i (a_i/L)$. Put

$$M(t) = \max_{X \in S_t} \varphi(t, X) \qquad \textit{for} \qquad t_0 \leqslant t < t_0+T ,$$

where S_{t_1} is the projection on $(x_1, ..., x_n)$ of the intersection of the pyramid (35.1) with the plane $t = t_1$.

Under these assumptions,

1° For every $\tilde{t} \in (t_0, t_0+T)$ there is a point $\tilde{X} \in S_{\tilde{t}}$ such that

$$(35.2) \qquad M(\tilde{t}) = \varphi(\tilde{t}, \tilde{X})$$

and the following implication holds true: if either

I. *$(\tilde{t}, \tilde{X})$ is an interior point of the pyramid and the derivatives $\varphi_t(\tilde{t}, \tilde{X})$, $\varphi_{x_i}(\tilde{t}, \tilde{X})$ $(i = 1, 2, ..., n)$ exist,*

or

II. *$(\tilde{t}, \tilde{X})$ is a point on the side surface of the pyramid and $\varphi(t, X)$ possesses Stolz's differential at $(\tilde{t}, \tilde{X})$,*

then

$$(35.3) \qquad D^- M(\tilde{t}) \leqslant \varphi_t(\tilde{t}, \tilde{X}) - L \sum_{i=1}^{n} |\varphi_{x_i}(t, X)| .$$

2° If, moreover, $\varphi_t(t, X)$ exists for $t_0 \leqslant t < t_0+\varepsilon$ and is continuous with respect to (t, X) for $t = t_0$, then there is a point $X_0 \in S_{t_0}$ such that

$$(35.4) \qquad D^+ M(t_0) \leqslant \varphi_t(t_0, X_0) .$$

Proof. By Theorem 33.1, 1°, there is a point $\tilde{X} \in S_{\tilde{t}}$ such that (35.2) holds true. Suppose first that I is true. Then, $\varphi(\tilde{t}, X)$ attaining its maximum at the interior point $\tilde{X}$ and possessing there first order derivatives, we have

$$(35.5) \qquad \varphi_{x_i}(\tilde{t}, \tilde{X}) = 0 \qquad (i = 1, 2, ..., n) .$$

On the other hand, by Theorem 33.1, 2°, we get

$$D^- M(\tilde{t}) \leqslant \varphi_t(\tilde{t}, \tilde{X}) . \tag{35.6}$$

Relations (35.5) and (35.6) imply (35.3).

Suppose now that II holds true; then, changing—if necessary—the numbering of variables, we may assume that

$$\begin{cases} \tilde{x}_p - \mathring{x}_p = a_p - L(\tilde{t}-t_0) & (p = 1, 2, \ldots, k) , \\ \tilde{x}_q - \mathring{x}_q = -a_q + L(\tilde{t}-t_0) & (q = k+1, \ldots, k+l) , \\ |\tilde{x}_r - \mathring{x}_r| < a_r - L(\tilde{t}-t_0) & (r = k+l+1, \ldots, n) . \end{cases} \tag{35.7}$$

Introduce the mapping

$$t = t , \quad \eta_i = \frac{a_i(x_i - \mathring{x}_i)}{a_i - L(t-t_0)} \quad (i = 1, 2, \ldots, n) ,$$

which transforms the pyramid (35.1) into the parallelepipede

$$t_0 \leqslant t < t_0 + T , \quad |\eta_i| \leqslant a_i \quad (i = 1, 2, \ldots, n) . \tag{35.8}$$

Put in (35.8)

$$\begin{aligned} \psi(t, H) &= \psi(t, \eta_1, \ldots, \eta_n) \\ &= \varphi\left(t, \mathring{x}_1 + \frac{\eta_1(a_1 - L(t-t_0))}{a_1}, \ldots, \mathring{x}_n + \frac{\eta_n(a_n - L(t-t_0))}{a_n}\right) . \end{aligned}$$

Then

$$M(t) = \max_{H \in \tilde{S}_t} \psi(t, H) ,$$

where $\tilde{S}_{t_1}$ is the projection on $(\eta_1, \ldots, \eta_n)$ of the intersection of the parallelepipede (35.8) with the plane $t = t_1$. Write

$$\tilde{\eta}_i = \frac{a_i(\tilde{x}_i - \mathring{x}_i)}{a_i - L(\tilde{t}-t_0)} \quad (i = 1, 2, \ldots, n) .$$

Then, by (35.7), we have

$$\tilde{\eta}_p = a_p , \quad \tilde{\eta}_q = -a_q , \quad |\tilde{\eta}_r| < a_r . \tag{35.9}$$

By our assumption that II holds true, $\varphi(t, X)$ possesses Stolz's differential at the point $(\tilde{t}, \tilde{X})$. Therefore, $\psi_t(\tilde{t}, \tilde{H}) = \psi_t(\tilde{t}, \tilde{\eta}_1, \ldots, \tilde{\eta}_n)$ exists and

$$\psi_t(\tilde{t}, \tilde{H}) = \varphi_t(\tilde{t}, \tilde{X}) - L \sum_{i=1}^{n} \frac{\tilde{\eta}_i}{a_i} \varphi_{x_i}(\tilde{t}, \tilde{X}) ,$$

whence, by (35.9),

(35.10)
$$\varphi_t(\tilde{t}, \tilde{H}) = \varphi_t(\tilde{t}, \tilde{X}) - L\sum_p \varphi_{x_p}(\tilde{t}, \tilde{X}) + L\sum_q \varphi_{x_q}(\tilde{t}, \tilde{X}) - L\sum_r \frac{\eta_r}{a_r}\varphi_{x_r}(\tilde{t}, \tilde{X}) .$$

By an argument similar to that used in the proof of Theorem 33.1, 2°, we get

(35.11)
$$D^- M(\tilde{t}) \leqslant \varphi_t(\tilde{t}, \tilde{H}) .$$

Now, consider the function of one variable x_p

$$\varphi(\tilde{t}, \tilde{x}_1, \ldots, \tilde{x}_{p-1}, x_p, \tilde{x}_{p+1}, \ldots, \tilde{x}_n)$$

in the interval

$$[\mathring{x}_p - a_p + L(\tilde{t} - t_0), \mathring{x}_p + a_p - L(\tilde{t} - t_0)] .$$

Since this function attains its maximum at the right-hand extremity $\tilde{x}_p = \mathring{x}_p + a_p - L(\tilde{t} - t_0)$ of the interval, we have

(35.12)
$$\varphi_{x_p}(\tilde{t}, \tilde{X}) \geqslant 0 .$$

In a similar way we obtain

(35.13)
$$\varphi_{x_q}(\tilde{t}, \tilde{X}) \leqslant 0 , \quad \varphi_{x_r}(\tilde{t}, \tilde{X}) = 0 .$$

From (35.10), (35.11), (35.12) and (35.13) follows (35.3). Thus part 1° of our theorem is proved.

Suppose now that φ_t is continuous for $t = t_0$. Take a sequence t_ν, $t_\nu > t_0$, $t_\nu \to t_0$ such that

(35.14)
$$D^+ M(t_0) = \lim_{\nu\to\infty} \frac{M(t_\nu) - M(t_0)}{t_\nu - t_0}$$

and let $M(t_\nu) = \varphi(t_\nu, X_\nu)$, where $X_\nu \in S_{t_\nu}$. Then we have

(35.15)
$$\frac{M(t_\nu) - M(t_0)}{t_\nu - t_0} \leqslant \frac{\varphi(t_\nu, X_\nu) - \varphi(t_0, X_\nu)}{t_\nu - t_0} = \varphi_t(\tilde{t}_\nu, X_\nu) ,$$

where $t_0 < \tilde{t}_\nu < t_\nu$. We may suppose—taking, if necessary, a subsequence—that $(\tilde{t}_\nu, X_\nu) \to (t_0, X_0)$, where $X_0 \in S_{t_0}$. Then, by the continuity of φ_t for $t = t_0$, we get

(35.16)
$$\lim_{\nu\to\infty} \varphi_t(\tilde{t}_\nu, X_\nu) = \varphi_t(t_0, X_0) .$$

Relations (35.14), (35.15) and (35.16) imply (35.4).

Remark 35.1. It is not difficult to construct a counter-example showing that continuity of φ_t at t_0 is essential for part 2° of Theorem 33.1.

Remark 35.2. It is easy to check that if $(\tilde{t}, \tilde{X})$ is an interior point of the pyramid and $\varphi_t(\tilde{t}, \tilde{X})$ does not exist, then (35.3) holds true with $\varphi_t(\tilde{t}, \tilde{X})$ replaced by Dini's derivative D^- of φ with respect to t.

§ 36. Comparison systems with right-hand sides depending on parameters. To close the present chapter we prove rather special theorems which will be needed in Chapter VII.

THEOREM 36.1. *Let the functions* $\sigma_i(t, V) = \sigma_i(t, v_1, \ldots, v_m)$ $(i = 1, 2, \ldots, m)$ *be the right-hand members of a comparison system of type* I (see § 14). *Denote by* $\Omega(t; H) = (\omega_1(t; H), \ldots, \omega_m(t; H))$ *its right-hand maximum solution through* $(0, H) = (0, \eta_1, \ldots, \eta_m)$ *in the interval* $[0, a_0(H))$. *Consider, for an arbitrary* $\lambda \geqslant 0$, *the comparison system of type* I

$$S(\lambda):\quad \frac{dv_i}{dt} = \lambda\sigma_i(\lambda t, v_1, \ldots, v_m) \qquad (i = 1, 2, \ldots, m)\,.$$

Under these hypotheses, $\Omega(\lambda t; H)$ *is the right-hand maximum solution of system* $S(\lambda)$ *through* $(0, H)$ *in the interval*

$$0 \leqslant t < \frac{a_0(H)}{\lambda}\,. \tag{36.1}$$

Proof. Observe that if $V(t) = (v_1(t), \ldots, v_m(t))$ is any solution of system $S(1)$ through $(0, H)$ in an interval $[0, \gamma)$, then $\tilde{V}(t) = (v_1(\lambda t), \ldots, v_m(\lambda t))$ is obviously a solution of system $S(\lambda)$ through $(0, H)$ in the interval $[0, \gamma/\lambda)$. In particular, $\Omega(\lambda t; H)$ is a solution of system $S(\lambda)$ through $(0, H)$ in the interval (36.1). Hence, the theorem will be proved if we show that for any solution $\tilde{V}(t)$ of system $S(\lambda)$ through $(0, H)$, defined in an interval $[0, \tilde{\gamma})$, we have

$$\tilde{V}(t) \leqslant \Omega(\lambda t; H) \quad \text{for} \quad 0 \leqslant t < \min\left(\tilde{\gamma}, a_0(H)/\lambda\right)\,. \tag{36.2}$$

For $\lambda = 0$ it is trivial. Now let $\lambda > 0$ and let $\tilde{V}(t)$ be any such solution; then $V(t) = \tilde{V}(t/\lambda)$ is a solution of system $S(1)$ through $(0, H)$, defined in the interval $[0, \lambda\tilde{\gamma})$. Hence we have

$$V(t) \leqslant \Omega(t; H) \quad \text{for} \quad 0 \leqslant t < \min\left(\lambda\tilde{\gamma}, a_0(H)\right)\,,$$

which is equivalent with (36.2).

THEOREM 36.2. *Let* $\sigma(t, v)$ *be the right-hand side of a comparison equation of type* II (see § 14). *Then, for any* $\lambda > 0$, *the equation*

$$\frac{dv}{dt} = \lambda\sigma(\lambda t, v) \tag{36.3}$$

is a comparison equation of type II.

Proof. Let $\tilde{v}(t)$ be any solution of (36.3) satisfying the condition

$$\lim_{t\to 0}\tilde{v}(t) = 0 .$$

Then, obviously, $v(t) = \tilde{v}(t/\lambda)$ is a solution of the comparison equation of type II

$$\frac{dv}{dt} = \sigma(t, v)$$

and satisfies condition $\lim_{t\to 0} v(t) = 0$. Hence, $v(t) \equiv 0$ and consequently $\tilde{v}(t) \equiv 0$, which completes the proof.

In a similar way we prove

THEOREM 36.3. *Let $\sigma(t, v)$ be the right-hand member of a comparison equation of type* III (see § 14). *Then, for any $\lambda > 0$, equation* (36.3) *is a comparison equation of type* III.

CHAPTER VII

CAUCHY PROBLEM FOR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

In this chapter we discuss a number of questions referring to the Cauchy problem for systems of first order partial differential equations of the form

$$u_x^i = f^i(x, y_1, \dots, y_n, u^1, \dots, u^m, u_{y_1}^i, \dots, u_{y_n}^i) \qquad (i = 1, 2, \dots, m)$$

with initial conditions

$$u^i(x_0, y_1, \dots, y_n) = \mu^i(y_1, \dots, y_n) \qquad (i = 1, 2, \dots, m)$$

and, more generally, for overdetermined systems of the form

$$u_{x_j}^i = f_j^i(x_1, \dots, x_p, y_1, \dots, y_n, u^1, \dots, u^m, u_{y_1}^i, \dots, u_{y_n}^i)$$
$$(i = 1, 2, \dots, m;\ j = 1, 2, \dots, p)$$

with initial data

$$u^i(\mathring{x}_1, \dots, \mathring{x}_p, y_1, \dots, y_n) = \mu^i(y_1, \dots, y_n) \qquad (i = 1, 2, \dots, m)\,.$$

The above systems are of special hyperbolic type since each equation contains first order derivatives of only one unknown function.

In particular, we will give applications of the theory of ordinary differential inequalities to questions like: estimates of the solution and of its domain of existence, estimates of the difference between two solutions, estimates of the error for an approximate solution, uniqueness criteria and continuous dependence of the solution on initial data and on the right-hand sides of the system.

§ 37. Comparison theorems for systems of partial differential inequalities. In order to simplify formulation of subsequent theorems, we first introduce the following definition.

A function $u(X, Y) = u(x_1, \dots, x_p, y_1, \dots, y_n)$ will be called the *function of class $\mathfrak{D}$ in a pyramid*

$$\sum_{r=1}^{p} |x_r - \mathring{x}_r| < \gamma\,, \qquad |y_k - \mathring{y}_k| \leqslant a_k - L\sum_{r=1}^{p} |x_r - \mathring{x}_r| \qquad (k = 1, 2, \dots, n)\,,$$

where $0 \leqslant L < +\infty$, $0 < a_k < +\infty$, $\gamma \leqslant \min_k (a_k/L)$, if $u(X, Y)$ is continuous in the pyramid, possesses Stolz's differential with regard to (X, Y) on its side surface and has first derivatives with respect to Y and Stolz's differential with regard to X in its interior.

If, moreover, the derivatives $u_{x_i}(X, Y)$ $(i = 1, 2, \ldots, n)$ are continuous with respect to (X, Y) for $X = X_0 = (\mathring{x}_1, \ldots, \mathring{x}_p)$, then $u(X, Y)$ will be called the *function of class* $\mathfrak{D}_0$.

THEOREM 37.1. *Let the functions* $U(x, Y) = \big(u^1(x, Y), \ldots, u^m(x, Y)\big)$ *be of class* $\mathfrak{D}$ *in the pyramid*

$$(37.1) \qquad |x-x_0| < \gamma\,, \quad |y_k-\mathring{y}_k| \leqslant a_k - L|x-x_0| \quad (k = 1, 2, \ldots, n)\,,$$

where $0 \leqslant L < +\infty$, $0 < a_k < +\infty$, $\gamma \leqslant \min_k (a_k/L)$. *Suppose the initial inequalities*

$$(37.2) \qquad |U(x_0, Y)| \leqslant H\,,$$

where $H = (\eta_1, \ldots, \eta_m)$, *and the differential inequalities*

$$(37.3) \qquad |u_x^i| \leqslant \sigma_i(|x-x_0|, |U|) + L\sum_{k=1}^{n} |u_{y_k}^i| \quad (i = 1, 2, \ldots, m)$$

are satisfied in the pyramid (37.1), *where* $\sigma_i(t, v_1, \ldots, v_m)$ $(i = 1, 2, \ldots, m)$ *are the right-hand members of a comparison system of type* I (see § 14). *Let* $\Omega(t; H) = \big(\omega_1(t; H), \ldots, \omega_m(t; H)\big)$ *be its right-hand maximum solution through* $(0, H)$ *and assume it to be defined in the interval* $[0, a_0)$.

Under these assumptions,

$$(37.4) \qquad |U(x, Y)| \leqslant \Omega(|x-x_0|; H)$$

in the pyramid (37.1) *for* $|x-x_0| < \min(\gamma, a_0)$.

Proof. Since the assumptions of our theorem are invariant under the mapping $\xi = -x+2x_0$, it is sufficient to prove (37.4) in the right-hand pyramid

$$(37.5) \qquad 0 \leqslant x-x_0 < \delta = \min(\gamma, a_0)\,, \quad |y_k-\mathring{y}_k| \leqslant a_k - L(x-x_0) \quad (k = 1, 2, \ldots, n)\,.$$

Put, for $0 \leqslant t < \delta$,

$$W^i(t) = \max_{Y \in S_t} |u^i(x_0+t, Y)|\,,$$
$$M^i(t) = \max_{Y \in S_t} u^i(x_0+t, Y) \qquad (i = 1, 2, \ldots, m)\,,$$
$$N^i(t) = \max_{Y \in S_t} \big(-u^i(x_0+t, Y)\big)\,,$$

where S_t is the projection on $(y_1, \ldots, y_n)$ of the intersection of the pyramid (37.5) with the plane $x = x_0 + t$. It is obvious that (37.4) in (37.5) is equivalent with

$$W^i(t) \leqslant \omega_i(t; H) \quad \text{for} \quad t \in [0, \delta) \quad (i = 1, 2., \ldots, m). \tag{37.6}$$

Now, we will prove (37.6) using the theory of ordinary differential inequalities. By (37.2), we have

$$W(0) \leqslant H, \tag{37.7}$$

where $W(t) = \big(W^1(t), \ldots, W^m(t)\big)$, and, by Theorem 34.1, $W^j(t)$ are continuous on $[0, \delta)$. By the same theorem, for every fixed j and for every $t \in (0, \delta)$, there is a point $Y \in S_t$ such that either

$$W^j(t) = M^j(t) = u^j(x_0 + t, Y), \quad D_- W^j(t) \leqslant D^- M^j(t), \tag{37.8}$$

or

$$W^j(t) = N^j(t) = -u^j(x_0 + t, Y), \quad D_- W^j(t) \leqslant D^- N^j(t). \tag{37.9}$$

Fix a j and $t \in (0, \delta)$ and suppose that, for instance, relations (37.8) hold true. By Theorem 35.1, 1°, we have

$$D^- M^j(t) \leqslant u^j_x(x_0 + t, Y) - L \sum_{k=1}^{n} |u^j_{y_k}(x_0 + t, Y)|. \tag{37.10}$$

On the other hand, since in view of (37.8) and of the definition of $W^i(t)$ we have (see § 4)

$$|U(x_0 + t, Y)| \overset{j}{\leqslant} W(t),$$

we get, by (37.3) and by condition W_+ (see § 4) imposed on $\sigma_i(t, V)$,

$$\begin{aligned} u^j_x(x_0 + t, Y) &\leqslant \sigma_j\big(t, |U(x_0 + t, Y)|\big) + L \sum_{k=1}^{n} |u^j_{y_k}(x_0 + t, Y)| \\ &\leqslant \sigma_j\big(t, W(t)\big) + L \sum_{k=1}^{n} |u^j_{y_k}(x_0 + t, Y)|. \end{aligned}$$

From (37.8), (37.10) and from the last inequality it follows that the differential inequalities

$$D_- W^j(t) \leqslant \sigma_j\big(t, W(t)\big)$$

are satisfied for every fixed j and $t \in (0, \delta)$. Hence, and by (37.7), we get inequalities (37.6) in virtue of the first comparison theorem (see § 14). This completes the proof.

COROLLARY 37.1. *If under the assumptions of Theorem* 37.1 *inequalities* (37.3) *are, in particular, linear*

$$|u_x^i| \leqslant K \sum_{j=1}^{m} |u^j| + L \sum_{k=1}^{n} |u_{y_k}^i| + C \qquad (K \geqslant 0,\ C \geqslant 0) \qquad (i = 1, 2, \ldots, m)$$

(*Haar's inequalities* [11]) *and if* $\eta_i = \eta$ $(i = 1, 2, \ldots, m)$, *then we get*

$$|u^i(x, Y)| \leqslant \begin{cases} e^{nK|x-x_0|}\left(\eta + \dfrac{C}{nK}\right) - \dfrac{C}{nK} & \text{for} \quad K > 0\,, \\ C|x-x_0| + \eta & \text{for} \quad K = 0\,. \end{cases} \qquad (i = 1, 2, \ldots, m)$$

in the pyramid (37.1).

THEOREM 37.2. *Let the functions* $U(x, Y) = \big(u^1(x, Y), \ldots, u^m(x, Y)\big)$ *be of class* $\mathfrak{D}$ *in the pyramid* (37.1). *Assume that*

$$U(x_0, Y) = 0 \tag{37.11}$$

and that the inequalities

$$|u_x^i| \leqslant \sigma(|x-x_0|, \max_l |u^l|) + L \sum_{k=1}^{n} |u_{y_k}^i| \qquad (i = 1, 2, \ldots, m) \tag{37.12}$$

are satisfied in the pyramid (37.1) *for* $x \neq x_0$, *where* $\sigma(t, v)$ *is the right-hand side of a comparison equation of type* II (see § 14).

Under these hypotheses we have

$$U(x, Y) \equiv 0$$

in the pyramid (37.1).

Proof. Like in Theorem 37.1, it is sufficient to prove our theorem in the right-hand pyramid

$$P_+\colon\ 0 \leqslant x - x_0 < \gamma\,,\ |y_k - \mathring{y}_k| \leqslant a_k - L(x - x_0) \qquad (k = 1, 2, \ldots, n)\,.$$

Put, for $0 \leqslant t < \gamma$,

$$\begin{aligned} W(t) &= \max_l \{\max_{Y \in S_t} |u^l(x_0 + t, Y)|\}\,, \\ M^i(t) &= \max_{Y \in S_t} u^i(x_0 + t, Y)\,, \qquad\qquad (i = 1, 2, \ldots, m) \\ N^i(t) &= \max_{Y \in S_t} \big(-u^i(x_0 + t, Y)\big)\,. \end{aligned}$$

Identities to be proved in the pyramid P_+ are obviously equivalent with

$$W(t) \equiv 0 \quad \text{for} \quad t \in [0, \gamma)\,. \tag{37.13}$$

We will prove (37.13) using the second comparison theorem (see § 14). By (37.11), we have

$$W(0) = 0 \tag{37.14}$$

and, by Theorem 34.1, $W(t)$ is continuous on $[0, \gamma)$. By the same theorem, for every $t \in (0, \gamma)$ there is an index j and a point $Y \in S_t$ such that either

$$W(t) = M^j(t) = u^j(x_0+t, Y), \quad D_-W(t) \leqslant D^-M^j(t), \tag{37.15}$$

or

$$W(t) = N^j(t) = -u^j(x_0+t, Y), \quad D_-W(t) \leqslant D^-N^j(t). \tag{37.16}$$

Suppose, for example, that for a $t \in (0, \gamma)$ relations (37.16) hold true. By Theorem 35.1, 1°, we have

$$D^-N^j(t) \leqslant -u_x^j(x_0+t, Y) - L\sum_{k=1}^{n} |u_{y_k}^j(x_0+t, Y)|. \tag{37.17}$$

Since, by (37.16),

$$-u^j(x_0+t, Y) = W(t) = \max_l |u^l(x_0+t, Y)|,$$

we get from (37.12)

$$-u_x^j(x_0+t, Y) \leqslant \sigma\big(t, W(t)\big) + L\sum_{k=1}^{n} |u_{y_k}^j(x_0+t, Y)|. \tag{37.18}$$

From (37.16), (37.17) and (37.18) it follows that the inequality

$$D_-W(t) \leqslant \sigma\big(t, W(t)\big) \tag{37.19}$$

is satisfied for any $t \in (0, \gamma)$. Hence, by (37.14) and by the second comparison theorem (see § 14), we conclude that $W(t) \leqslant 0$ in $[0, \gamma)$ and, since obviously $W(t) \geqslant 0$, we finally obtain (37.13), which completes the proof.

THEOREM 37.3. *Let the functions* $U(x, Y) = \big(u^1(x, Y), \dots, u^m(x, Y)\big)$ *be of class* $\mathfrak{D}_0$ *in the pyramid* (37.1). *Assume that*

$$U(x_0, Y) = U_x(x_0, Y) = 0, \tag{37.20}$$

where $U_x(x, Y) = \big(u_x^1(x, Y), \dots, u_x^m(x, Y)\big)$, *and that the inequalities* (37.12) *are satisfied in the pyramid* (37.1) *for* $x \neq x_0$, *where* $\sigma(t, v)$ *is the right-hand member of a comparison equation of type* III (see § 14).

Under these assumptions we have

$$U(x, Y) \equiv 0$$

in the pyramid (37.1).

Proof. Again it is sufficient to prove the theorem in the right-hand pyramid P_+. With the notations in the proof of Theorem 37.2, identity

$U(x, Y) \equiv 0$ in P_+ is equivalent with (37.13). This time we will prove (37.13) using the third comparison theorem (see § 14). By (37.20), we have

$$W(0) = 0 \,. \tag{37.21}$$

Next, by Theorem 34.1, there is an index j such that either

$$D^+ W(0) \leqslant D^+ M^j(0) \,, \tag{37.22}$$

or

$$D^+ W(0) \leqslant D^+ N^j(0) \,. \tag{37.23}$$

Suppose, for instance, that (37.22) holds true. Then, by Theorem 35.1, 2°, there is a point $Y_0 \in S_0$, such that

$$D^+ W(0) \leqslant D^+ M^j(0) \leqslant u_x^j(x_0, Y_0) \,.$$

Hence, by (37.20), it follows that

$$D^+ W(0) \leqslant 0 \,. \tag{37.24}$$

Now, like in Theorem 37.2, we prove that (37.19) is satisfied for $t \in (0, \gamma)$. Therefore, due to (37.21) and (37.24) we conclude, by the third comparison theorem (see § 14), that $W(t) \leqslant 0$ for $t \in [0, \gamma)$ and consequently (37.13) holds true, which completes the proof.

Remark 37.1. By Remark 35.2, all theorems of § 37 are true without the requirement that u_x^i exist in the interior of the pyramid, provided that u_x^i be replaced by Dini's derivative D^- of u^i with regard to x.

Remark 37.2. All theorems of § 37 hold true if, instead of the pyramid (37.1), we have the zone

$$|x - x_0| < \gamma \,, \qquad y_1, \ldots, y_n \text{ arbitrary} \,, \tag{37.25}$$

provided that the functions $u^i(x, Y)$ be continuous and possess Stolz's differential in (37.25), and in Theorem 37.3 the derivatives $u_x^i(x, Y)$ be, in addition, continuous for $x = x_0$.

Indeed, under these assumptions, all the hypotheses of theorems in question are satisfied in any pyramid (37.1) with arbitrary finite a_k, and hence follows our remark.

§ 38. Comparison theorems for overdetermined systems of partial differential inequalities. We prove

THEOREM 38.1. *Let the functions* $U(X, Y) = \big(u^1(X, Y), \ldots, u^m(X, Y)\big) = \big(u^1(x_1, \ldots, x_p, y_1, \ldots, y_n), \ldots, u^m(x_1, \ldots, x_p, y_1, \ldots, y_n)\big)$ *be of class* $\mathfrak{D}$ (see § 37) *in the pyramid*

$$\sum_{j=1}^{p} |x_j - \mathring{x}_j| < \gamma \,, \quad |y_k - \mathring{y}_k| \leqslant a_k - L \sum_{j=1}^{p} |x_j - \mathring{x}_j| \quad (k = 1, 2, \ldots, n) \,, \tag{38.1}$$

where $0 \leqslant L < +\infty$, $0 < a_k < +\infty$, $\gamma \leqslant \min_k (a_k/L)$. *Suppose that the initial inequality*

$$|U(X_0, Y)| \leqslant H, \tag{38.2}$$

where $X_0 = (\mathring{x}_1, \ldots, \mathring{x}_p)$, $H = (\eta_1, \ldots, \eta_m)$, *and the differential inequalities*

$$|u^i_{x_j}| \leqslant \sigma_i\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|, |U|\Big) + L \sum_{k=1}^{n} |u^i_{y_k}| \tag{38.3}$$

$$(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p)$$

hold true in the pyramid (38.1), *where the functions* $\sigma_i(t, v_1, \ldots, v_m)$ *are the right-hand sides of a comparison system of type* I (see § 14). *Let its right-hand maximum solution* $\Omega(t; H) = \big(\omega_1(t; H), \ldots, \omega_m(t; H)\big)$ *through* $(0, H)$ *be defined in an interval*

$$0 \leqslant t < a_0(H). \tag{38.4}$$

Under these hypotheses we have

$$|U(X, Y)| \leqslant \Omega\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|; H\Big) \tag{38.5}$$

in the pyramid

$$\sum_{j=1}^{p} |x_j - \mathring{x}_j| < \min\big(\gamma, a_0(H)\big), \quad |y_k - \mathring{y}_k| \leqslant a_k - L \sum_{j=1}^{p} |x_j - \mathring{x}_j| \tag{38.6}$$

$$(k = 1, 2, \ldots, n).$$

Proof. By means of Mayer's transformation

$$X = X_0 + \Lambda x, \tag{38.7}$$

where $\Lambda = (\lambda_1, \ldots, \lambda_p)$, we will reduce our theorem to Theorem 37.1.

For $\Lambda = (\lambda_1, \ldots, \lambda_p)$, consider the comparison system of type I

$$\frac{dv_i}{dt} = \lambda \sigma_i(\lambda t, v_1, \ldots, v_m) \quad (i = 1, 2, \ldots, m),$$

where $\lambda = \sum_{j=1}^{p} |\lambda_j|$. By Theorem 36.1 we know that $\Omega(\lambda t; H)$ is its right-hand maximum solution through $(0, H)$ in the interval $[0, a_0(H)/\lambda)$. In particular, for $\lambda < a_0(H)$, we have

$$\frac{a_0(H)}{\lambda} > 1. \tag{38.8}$$

Suppose that

$$\lambda = \sum_{j=1}^{p} |\lambda_j| < \min\big(\gamma, a_0(H)\big) \tag{38.9}$$

and put

$$\widetilde{U}(x, Y; \Lambda) = U(X_0 + \Lambda x, Y). \tag{38.10}$$

It is clear that, for $\Lambda = (\lambda_1, \ldots, \lambda_p)$ satisfying (38.9), $\widetilde{U}(x, Y; \Lambda) = \left(\widetilde{u}^1(x, Y; \Lambda), \ldots, \widetilde{u}^m(x, Y; \Lambda)\right)$ is of class $\mathfrak{D}$ (see § 37) in the pyramid

$$|x| < \frac{\gamma}{\lambda}, \quad |y_k - \mathring{y}_k| \leqslant a_k - L\lambda|x| \quad (k = 1, 2, \ldots, n), \tag{38.11}$$

where, by (38.9),

$$\frac{\gamma}{\lambda} > 1. \tag{38.12}$$

In virtue of (38.2) and (38.3) we get

$$|\widetilde{U}(0, Y; \Lambda)| \leqslant H$$

and

$$|\widetilde{u}^i_x| \leqslant \lambda\sigma_i(\lambda|x|, |\widetilde{U}|) + L\lambda \sum_{k=1}^{n} |\widetilde{u}^i_{y_k}| \quad (i = 1, 2, \ldots, m)$$

in the pyramid (38.11). Hence, by Theorem 37.1, we have

$$|\widetilde{U}(x, Y; \Lambda)| \leqslant \Omega(\lambda|x|; H)$$

in the pyramid (38.11) for

$$|x| < \min\left(\frac{\gamma}{\lambda}, \frac{a_0(H)}{\lambda}\right).$$

Since, by (38.8) and (38.12),

$$\min\left(\frac{\gamma}{\lambda}, \frac{a_0(H)}{\lambda}\right) > 1,$$

we have, putting $x = 1$,

$$|\widetilde{U}(1, Y; \Lambda)| \leqslant \Omega(\lambda; H) \tag{38.13}$$

for $\Lambda = (\lambda_1, \ldots, \lambda_p)$ satisfying (38.9). Hence, if (X, Y) is any point in the pyramid (38.6) and if we set $\Lambda = X - X_0 = (x_1 - \mathring{x}_1, \ldots, x_p - \mathring{x}_p)$, then

$$|U(X, Y)| = |\widetilde{U}(1, Y; X - X_0)| \leqslant \Omega\left(\sum_{r=1}^{p} |x_r - \mathring{x}_r|; H\right),$$

what was to be proved.

THEOREM 38.2. *Let the functions* $U(X, Y) = \left(u^1(X, Y), \ldots, u^m(X, Y)\right)$ *be of class* $\mathfrak{D}$ (see § 37) *in the pyramid* (38.1). *Suppose that*

$$U(X_0, Y) = 0 \tag{38.14}$$

and

$$(38.15)\qquad |u^i_{x_j}| \leqslant \sigma\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|, \max_l |u^l|\Big) + L\sum_{k=1}^{n} |u^i_{y_k}| \quad \textit{for} \quad X \neq X_0$$

$$(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p)$$

in the pyramid (38.1), *where* $\sigma(t, v)$ *is the right-hand member of a comparison equation of type* II (see § 14).

Under these assumptions we have

$$(38.16)\qquad U(X, Y) \equiv 0$$

in the pyramid (38.1).

Proof. Like in the proof of Theorem 38.1 we introduce Mayer's transformation (38.7) and we define $\widetilde{U}(x, Y; \Lambda)$ by formula (38.10), for an arbitrary vector $\Lambda = (\lambda_1, \ldots, \lambda_p)$ satisfying

$$(38.17)\qquad 0 < \lambda = \sum_{j=1}^{p} |\lambda_j| < \gamma .$$

Then $\widetilde{U}(x, Y; \Lambda) = \big(\widetilde{u}^1(x, Y; \Lambda), \ldots, \widetilde{u}^m(x, Y; \Lambda)\big)$ is of class $\mathfrak{D}$ (see § 37) in the pyramid (38.11) and inequality (38.12) is satisfied. In view of (38.14) and (38.15) we obtain

$$\widetilde{U}(0, Y; \Lambda) = 0$$

and

$$|\widetilde{u}^i_x| \leqslant \lambda\sigma(\lambda|x|, \max_l |\widetilde{u}^l|) + L\lambda\sum_{k=1}^{p} |\widetilde{u}^i_{y_k}| \quad \text{for} \quad x \neq 0 \quad (i = 1, 2, \ldots, m),$$

in the pyramid (38.11). Since, by our assumptions and by Theorem 36.2, $\lambda\sigma(\lambda t, v)$ is—for any $\lambda > 0$—the right-hand member of a comparison equation of type II, we conclude, by Theorem 37.2, that

$$\widetilde{U}(x, Y; \Lambda) \equiv 0 ,$$

for Λ satisfying (38.17), in the pyramid (38.11). Because of (38.12), we have in particular

$$\widetilde{U}(1, Y; \Lambda) = 0 .$$

Hence, if (X, Y) is any point in the pyramid (38.1) such that $X \neq X_0$ and if we set $\Lambda = X - X_0$, then

$$U(X, Y) = \widetilde{U}(1, Y; X - X_0) = 0 ,$$

which completes the proof, since for $X = X_0$ the last identity holds true by (39.14).

In a similar way, using Theorems 36.3 and 37.3 we obtain

THEOREM 38.3. *Let the functions* $U(X, Y) = \big(u^1(X, Y), \ldots, u^m(X, Y)\big)$ *be of class* $\mathfrak{D}_0$ (see § 37) *in the pyramid* (38.1). *Suppose that*

$$U(X_0, Y) = U_{x_j}(X_0, Y) = 0 \qquad (j = 1, 2, \ldots, p),$$

where $U_{x_j} = (u^1_{x_j}, \ldots, u^m_{x_j})$ *and that inequalities* (38.15) *hold true in the pyramid* (38.1) *with* $\sigma(t, v)$ *being the right-hand side of a comparison equation of type* III (see § 14). *Then we have* (38.16) *in the pyramid* (38.1).

Remark 38.1. All theorems of § 38 remain true if, in place of the pyramid (38.1), we have the zone

$$(38.18) \qquad \sum_{r=1}^{p} |x_r - \mathring{x}_r| < \gamma, \qquad y_1, \ldots, y_n \text{ arbitrary},$$

provided that the functions $u^i(X, Y)$ be continuous and possess Stolz's differential in (38.18) and in Theorem 38.3 the derivatives $u^i_{x_j}(X, Y)$ be, in addition, continuous for $X = X_0$. This remark is a consequence of the argument used in Remark 37.2.

§ 39. Estimates of the solution. Since a system

$$u^i_x = f^i(x, y_1, \ldots, y_n, u^1, \ldots, u^m, u^i_{y_1}, \ldots, u^i_{y_n}) \qquad (i = 1, 2, \ldots, m)$$

is a particular case, for $p = 1$, of the overdetermined system

$$(39.1) \qquad u^i_{x_j} = f^i_j(x_1, \ldots, x_p, y_1, \ldots, y_n, u^1, \ldots, u^m, u^i_{y_1}, \ldots, u^i_{y_n})$$
$$(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p),$$

where the ith equation contains derivatives of u^i only, we consider in subsequent sections systems (39.1). We will give first some estimates of solutions of system (39.1).

THEOREM 39.1. *Let the right-hand members*

$$f^i_j(X, Y, U, Q) = f^i_j(x_1, \ldots, x_p, y_1, \ldots, y_n, u^1, \ldots, u^m, q_1, \ldots, q_n)$$
$$(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p)$$

of system (39.1) *be defined in a region whose projection on the space* $(x_1, \ldots, x_p, y_1, \ldots, y_n)$ *contains the pyramid*

$$(39.2) \qquad \sum_{r=1}^{p} |x_r - \mathring{x}_r| < \gamma, \quad |y_k - \mathring{y}_k| \leqslant a_k - L \sum_{r=1}^{p} |x_r - \mathring{x}_r| \quad (k = 1, 2, \ldots, n),$$

where $0 \leqslant L < +\infty$, $0 < a_k < +\infty$, $\gamma \leqslant \min_k (a_k/L)$. *Suppose that*

$$(39.3) \qquad |f^i_j(X, Y, U, Q)| \leqslant \sigma_i\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|, |U|\Big) + L \sum_{k=1}^{n} |q_k|$$
$$(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p),$$

where $\sigma_i(t, v_1, \dots, v_m)$ *are the right-hand sides of a comparison system of type* I (see § 14). *Let* $\Omega(t; H) = (\omega_1(t; H), \dots, \omega_m(t; H))$ *be its right-hand maximum solution through* $(0, H) = (0, \eta_1, \dots, \eta_m)$ *defined in an interval* $[0, a_0)$. *Let* $U(X, Y) = (u^1(X, Y), \dots, u^m(X, Y))$ *be a solution of system* (39.1), *of class* $\mathfrak{D}$ *in the pyramid* (39.2) (see § 37) *and satisfying initial inequality*

$$|U(X_0, Y)| \leqslant H . \tag{39.4}$$

This being assumed, we have

$$|U(X, Y)| \leqslant \Omega\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|, H\Big) \tag{39.5}$$

in the pyramid

$$\sum_{r=1}^{p} |x_r - \mathring{x}_r| < \min(\gamma, a_0), \qquad |y_k - \mathring{y}_k| \leqslant a_k - L\sum_{r=1}^{p} |x_r - \mathring{x}_r| \qquad (k = 1, 2, \dots, n). \tag{39.6}$$

Proof. By (39.3) and (39.4), the solution $U(X, Y)$ satisfies all the assumptions of Theorem 38.1 and, hence, inequalities (39.5) hold true in the pyramid (39.6).

§ 40. Estimate of the existence domain of the solution. In the present section we restrict ourselves to the Cauchy problem for one equation

$$u_x = f(x, y_1, \dots, y_n, u, u_{y_1}, \dots, u_{y_n}) \tag{40.1}$$

with the initial data

$$u(x_0, y_1, \dots, y_n) = \varphi(y_1, \dots, y_n) . \tag{40.2}$$

We will discuss here briefly—without insisting on detailed computations—how the existence domain of the solution of the above problem may be evaluated. As for details omitted here we refer to T. Ważewski's paper [57]. Using the theory of ordinary differential inequalities we will construct the solution by means of the Cauchy characteristics.

Suppose that the right-hand member $f(x, Y, u, Q) = f(x, y_1, \dots, y_n, u, q_1, \dots, q_n)$ and the initial function $\varphi(y_1, \dots, y_n)$ are of class C^2 in the cube

$$|x| < b, \quad |y_k| < b, \quad |u| < b, \quad |q_k| < b \qquad (k = 1, 2, \dots, n) \tag{40.3}$$

and

$$\varphi(0, \dots, 0) = \varphi_{y_k}(0, \dots, 0) = 0 \qquad (k = 1, 2, \dots, n) .$$

Assume further that f and φ together with their first and second derivatives are bounded by a constant M in the cube (40.3).

Under these assumptions, there are two numbers $a(b, n, M)$ and $\delta(b, n, M)$ (which can be effectively evaluated, for instance

$a = b/4n(M+1)$, $\delta = b^2/[(n+1)(M+b+1)]^2$) depending only on b, n, M so that the solution of problem (40.1), (40.2) exists and is of class C^1 in the pyramid

$$|x| < \delta(b, n, M), \quad |y_k| \leqslant a(b, n, M) - M|x| \quad (k = 1, 2, \dots, n). \tag{40.4}$$

We will indicate the way of proving this statement. Consider the characteristic equations

$$\begin{cases} \dfrac{dy_k}{dx} = -f_{q_k}(x, Y, u, Q), \\ \dfrac{dq_k}{dx} = f_{y_k}(x, Y, u, Q) + q_k f_u(x, Y, u, Q) \quad (k = 1, 2, \dots, n), \\ \dfrac{du}{dx} = f(x, Y, u, Q) - \sum\limits_{j=1}^{n} q_j f_{q_j}(x, Y, u, Q), \end{cases} \tag{40.5}$$

and let

$$y_k = \tilde{y}_k(x, \eta_1, \dots, \eta_n), \quad q_k = \tilde{q}_k(x, \eta_1, \dots, \eta_n), \quad u = \tilde{u}(x, \eta_1, \dots, \eta_n) \quad (k = 1, 2, \dots, n) \tag{40.6}$$

be the solution of system (40.5), satisfying the initial conditions

$$\tilde{y}_k(0, H) = \eta_k, \quad \tilde{q}_k(0, H) = \varphi_{y_k}(H), \quad \tilde{u}(0, H) = \varphi(H) \quad (k = 1, 2, \dots, n),$$

where $H = (\eta_1, \dots, \eta_n)$ is any point from the cube

$$|\eta_k| < b \quad (k = 1, 2, \dots, n).$$

Now, Cauchy's method consists in solving, with respect to $\eta_1, \dots, \eta_n$, the system of equations

$$y_k = \tilde{y}_k(x, \eta_1, \dots, \eta_n) \quad (k = 1, 2, \dots, n), \tag{40.7}$$

thus finding the inverse mapping

$$\eta_k = \tilde{\eta}_k(x, y_1, \dots, y_n) \quad (k = 1, 2, \dots, n), \tag{40.8}$$

and in making the substitution

$$u(x, y_1, \dots, y_n) = \tilde{u}(x, \tilde{\eta}_1(x, y_1, \dots, y_n), \dots, \tilde{\eta}_n(x, y_1, \dots, y_n)). \tag{40.9}$$

If the mapping (40.7) is one-to-one and of class C^1 in some domain

$$|x| < \delta, \quad |\eta_k| < c \quad (k = 1, 2, \dots, n) \tag{40.10}$$

with the Jacobian

$$\frac{D(\tilde{y}_1, \dots, \tilde{y}_n)}{D(\eta_1, \dots, \eta_n)} \neq 0 \tag{40.11}$$

and if the domain $D \subset (x, y_1, \dots, y_n)$ is the image of (40.10) by means of the mapping (40.7), then the function $u(x, y_1, \dots, y_n)$, defined by formula (40.9), is the solution of the problem (40.1), (40.2), of class C^1 in D. Therefore, in order to prove our statement concerning the existence of the solution in the pyramid (40.4), it is sufficient to find a cube (40.10) such that

1° The mapping (40.7) is one-to-one and of class C^1 in (40.10) with the Jacobian satisfying (40.11).

2° The domain D contains the pyramid (40.4).

Now, this is achieved in several steps.

I. By Theorem 23.1, we evaluate the interval $|x| < b_0$, in which the functions (40.6) exist for $|\eta_k| < b$ $(k = 1, 2, \ldots, n)$, and the functions themselves, thus obtaining estimates of the form

$$(40.12) \qquad |\tilde{y}_k(x, H)| \leqslant a_k(|x|), \quad |\tilde{q}_k(x, H)| \leqslant \beta_k(|x|), \quad |\tilde{u}(x, H)| \leqslant \gamma(|x|) \quad (k = 1, 2, \ldots, n).$$

Under our assumptions on $f(x, Y, u, Q)$ we may choose for the corresponding comparison system a linear one, whose solution is $a_k(t)$, $\beta_k(t)$, $\gamma(t)$ $(k = 1, 2, \ldots, n)$.

II. The functions (40.6) are of class C^1 and their derivatives with respect to η_k satisfy a linear system of ordinary differential equations. Applying Theorem 23.1 to this system (for the comparison system may be chosen a linear one) and remembering that $\tilde{y}_k(0, H) = \eta_k$ and hence

$$\frac{\partial \tilde{y}_k(0, H)}{\partial \eta_j} = \delta_{kj} \qquad (k, j = 1, 2, \ldots, n),$$

we find $\delta(b, n, M)$ and $c(b, n, M)$, so that inequalities

$$(40.13) \qquad \left| \frac{\partial \tilde{y}_k(x, H)}{\partial \eta_j} - \delta_{kj} \right| < \frac{1}{n} \qquad (k, j = 1, 2, \ldots, n)$$

hold true in cube (40.10). With such choice of δ and c point 1° is achieved.

III. Point 2°, which consists in finding $a(b, n, M)$, is achieved by any method allowing to evaluate the existence domain of the inverse mapping (40.8).

Observe that, since for the function $u(x, Y)$ defined by formula (40.9) we have

$$u_{y_k}(x, Y) = \tilde{q}_k\big(x, \tilde{\eta}_1(x, Y), \ldots, \tilde{\eta}_n(x, Y)\big) \qquad (k = 1, 2, \ldots, n),$$

from (40.12) we get the estimates

$$(40.14) \qquad |u(x, Y)| \leqslant \gamma(|x|), \qquad |u_{y_k}(x, Y)| \leqslant \beta_k(|x|) \qquad (k = 1, 2, \ldots, n).$$

We close this paragraph with the following remark. Using the above results concerning one equation (40.1) with one unknown function it is possible to construct the solution and to evaluate its existence domain for a non-overdetermined system by means of successive approximations (see [52]). The last result enables us to do the same for an overdetermined system (39.1) by means of Mayer's transformation (38.7); this time, we have to require that the right-hand sides of system (39.1) satisfy compatibility conditions (see [52]).

§ 41. Estimates of the difference between two solutions.

THEOREM 41.1. *Let the right-hand members of system* (39.1) *and of system*

(41.1) $$u^i_{x_j} = g^i_j(x_1, \ldots, x_p, y_1, \ldots, y_n, u^1, \ldots, u^m, u^i_{y_1}, \ldots, u^i_{y_n}) \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p)$$

be defined in a region, whose projection on the space of points $(x_1, \ldots, x_p, y_1, \ldots, y_n)$ *contains the pyramid* (39.2), *and satisfy the inequalities*

$$|f^i_j(X, Y, U, Q) - g^i_j(X, Y, \widetilde{U}, \widetilde{Q})| \leqslant \sigma_i\Big(\sum_{r=1}^{p}|x_r - \mathring{x}_r|, |U - \widetilde{U}|\Big) + L\sum_{k=1}^{n}|q_k - \widetilde{q}_k| \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n),$$

where $\sigma_i(t, v_1, \ldots, v_m)$ *are the right-hand sides of a comparison system of type* I (see § 14). *Denote by* $\Omega(t; H) = \big(\omega_1(t; H), \ldots, \omega_m(t; H)\big)$ *its right-hand maximum solution through* $(0, H) = (0, \eta_1, \ldots, \eta_m)$, *defined in the interval* $[0, a_0)$. *Suppose that* $U(X, Y) = \big(u^1(X, Y), \ldots, u^m(X, Y)\big)$ *and* $V(X, Y) = \big(v^1(X, Y), \ldots, v^m(X, Y)\big)$ *are two solutions of system* (39.1) *and* (41.1) *respectively, of class* $\mathfrak{D}$ *in the pyramid* (39.2) (see § 37) *and satisfying initial inequality*

(41.2) $$|U(X_0, Y) - V(X_0, Y)| \leqslant H.$$

Under these assumptions we have

(41.3) $$|U(X, Y) - V(X, Y)| \leqslant \Omega\Big(\sum_{r=1}^{p}|x_r - \mathring{x}_r|; H\Big)$$

in the pyramid (39.6).

Proof. If we put $\widetilde{U}(X, Y) = U(X, Y) - V(X, Y)$, then $\widetilde{U}(X, Y)$ satisfies all the assumptions of Theorem 38.1 and hence (41.3) holds true.

§ 42. Uniqueness criteria. The next theorem is an immediate conclusion from Theorem 41.1.

THEOREM 42.1. *Let the right-hand members of system* (39.1) *be defined in a region, whose projection on the space* $(x_1, \ldots, x_p, y_1, \ldots, y_n)$ *contains the pyramid* (39.2), *and satisfy inequalities*

(42.1) $$|f^i_j(X, Y, U, Q) - f^i_j(X, Y, \widetilde{U}, \widetilde{Q})| \leqslant \sigma_i\Big(\sum_{r=1}^{p}|x_r - \mathring{x}_r|, |U - \widetilde{U}|\Big) + L\sum_{k=1}^{n}|q_k - \widetilde{q}_k| \quad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p),$$

where $\sigma_i(t, v_1, \ldots, v_m)$ *are the right-hand sides of a comparison system of type* I (see § 14). *Suppose that*

$$\sigma_i(t, 0) \equiv 0 \quad (i = 1, 2, \ldots, m) \tag{42.2}$$

and that

$$\Omega(t) \equiv 0 \quad \textit{for} \quad 0 \leqslant t < +\infty, \tag{42.3}$$

where $\Omega(t)$ *is the right-hand maximum solution of the comparison system through the origin.*

Under these assumptions, Cauchy problem for system (39.1) *with initial data*

$$U(X_0, Y) = \Phi(Y) \tag{42.4}$$

admits at most one solution of class $\mathfrak{D}$ (see § 37) *in the pyramid* (39.2).

Proof. For two solutions, satisfying the same initial conditions (42.4), relations (41.2) hold true with $H = 0$; hence, by (41.3) and (42.3), their difference is identically zero.

Remark 42.1. In particular, for $\sigma_i(t, V) = K \sum_{j=1}^{m} v_j$ $(K \geqslant 0)$, inequalities (42.1) mean that the right-hand sides of system (39.1) satisfy a Lipschitz condition with regard to U.

Next we will prove uniqueness criteria of Kamke's type.

THEOREM 42.2. *Let the right-hand members of system* (39.1) *be defined in a region, whose projection on the space* $(x_1, \ldots, x_p, y_1, \ldots, y_n)$ *contains the pyramid* (39.2), *and satisfy inequalities*

$$\begin{gathered} |f_j^i(X, Y, U, Q) - f_j^i(X, Y, \tilde{U}, \tilde{Q})| \\ \leqslant \sigma\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|, \max_l |u^l - \tilde{u}^l|\Big) + L \sum_{k=1}^{n} |q_k - \tilde{q}_k| \\ (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p), \end{gathered} \tag{42.5}$$

where $\sigma(t, v)$ *is the right-hand side of a comparison equation of type* II (*of type* III) (see § 14).

This being assumed, Cauchy problem for system (39.1) *with initial data* (42.4) *admits at most one solution of class* $\mathfrak{D}$ (*of class* $\mathfrak{D}_0$) *in the pyramid* (39.2) (see § 37).

Proof. For two such solutions $U(X, Y) = \big(u^1(X, Y), \ldots, u^m(X, Y)\big)$ and $V(X, Y) = \big(v^1(X, Y), \ldots, v^m(X, Y)\big)$, put $\tilde{U}(X, Y) = U(X, Y) - V(X, Y) = \big(\tilde{u}^1(X, Y), \ldots, \tilde{u}^m(X, Y)\big)$. Then we have

$$\tilde{U}(X_0, Y) = U(X_0, Y) - V(X_0, Y) = 0 \tag{42.6}$$

and, by (42.5),

$$|\tilde{u}^i_{x_j}| \leqslant \sigma\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|, \max_l |\tilde{u}^l|\Big) + L\sum_{k=1}^{n} |\tilde{u}^i_{y_k}|$$
$$(i = 1, 2, \ldots, m;\; j = 1, 2, \ldots, p).$$

Further, by (42.6),

$$U_{y_k}(X_0, Y) \equiv V_{y_k}(X_0, Y) \qquad (k = 1, 2, \ldots, n)$$

and hence, writing $u^i_Y = (u^i_{y_1}, \ldots, u^i_{y_n})$, $v^i_Y = (v^i_{y_1}, \ldots, v^i_{y_n})$, we get $u^i_Y(X_0, Y) = v^i_Y(X_0, Y)$ and consequently

$$\tilde{u}^i_{x_j}(X_0, Y) = u^i_{x_j}(X_0, Y) - v^i_{x_j}(X_0, Y)$$
$$= f^i_j\big(X_0, Y, U(X_0, Y), u^i_Y(X_0, Y)\big) - f^i_j\big(X_0, Y, V(X_0, Y), v^i_Y(X_0, Y)\big) = 0$$
$$(i = 1, 2, \ldots, m;\; j = 1, 2, \ldots, p).$$

Therefore, we see that $\tilde{U}(X, Y)$ satisfies all the assumptions of Theorem 38.2 (of Theorem 38.3) and hence we have

$$\tilde{U}(X, Y) \equiv 0$$

in the pyramid (39.2), what was to be proved.

Remark 42.2. If, in particular, $\sigma(t, v)$ in Theorem 42.2 is the right-hand member of the equation (β) from Example 14.2 or of the equation from Example 14.3, we get uniqueness criteria of Osgood's and Nagumo's type.

§ 43. Continuous dependence of the solution on initial data and on right-hand sides of system. We now prove

THEOREM 43.1. *Let the right-hand members $f^i_j(X, Y, U, Q)$ of system* (39.1) *satisfy assumptions of Theorem* 42.1 *in a region D. Suppose that the right-hand sides $g^i_j(X, Y, V, Q)$ of system* (41.1) *are defined in D. Let $U(X, Y) = \big(u^1(X, Y), \ldots, u^m(X, Y)\big)$ be the solution of system* (39.1), *of class* $\mathfrak{D}$ (see § 37) *and satisfying initial conditions* (42.4) *in the pyramid* (39.2), *and $V(X, Y) = \big(v^1(X, Y), \ldots, v^m(X, Y)\big)$ be a similar solution of system* (41.1) *with initial data*

$$V(X_0, Y) = \Psi(Y). \tag{43.1}$$

Under these assumptions, to every $\varepsilon > 0$, there is a $\delta > 0$ such that if

$$|f^i_j(X, Y, U, Q) - g^i_j(X, Y, U, Q)| < \delta \tag{43.2}$$
$$(i = 1, 2, \ldots, m;\; j = 1, 2, \ldots, p)$$

in D and

$$|\Phi(Y) - \Psi(Y)| < \Delta, \tag{43.3}$$

where $\Delta = (\delta, \ldots, \delta)$, *then we have*

$$|U(X, Y) - V(X, Y)| < E, \tag{43.4}$$

where $E = (\varepsilon, \ldots, \varepsilon)$, *in the pyramid* (39.2).

Proof. Due to Theorem 10.1, to $\varepsilon > 0$ we can choose $\delta > 0$, so that the right-hand maximum solution $\Omega(t; H, \delta) = \big(\omega_1(t; H, \delta), \ldots, \omega_m(t; H, \delta)\big)$ of the comparison system

$$\frac{dv_i}{dt} = \sigma_i(t, v_1, \ldots, v_m) + \delta \qquad (i = 1, 2, \ldots, m),$$

passing through $(0, H) = (0, \eta_1, \ldots, \eta_m)$, be defined in the interval $[0, \gamma)$ and satisfy inequalities

$$\Omega(t; H, \delta) < E \quad \text{for} \quad 0 \leqslant t < \gamma, \tag{43.5}$$

provided that

$$0 \leqslant H < 2\Delta. \tag{43.6}$$

Suppose that (43.2) and (43.3) hold true with the above chosen δ; then, by (43.3), we have

$$|U(X_0, Y) - V(X_0, Y)| \leqslant H$$

with some H satisfying (43.6) and, by (42.1) and (43.2), we get

$$|f_j^i(X, Y, U, Q) - g_j^i(X, Y, \widetilde{U}, \widetilde{Q})| \\ \leqslant \sigma_i\Big(\sum_{r=1}^{p} |x_r - \mathring{x}_r|, |U - \widetilde{U}|\Big) + \delta + L\sum_{k=1}^{n} |q_k - \widetilde{q}_k| \\ (i = 1, \ldots, m;\ j = 1, 2, \ldots, p)$$

in the region D. Hence, by Theorem 41.1, inequality

$$|U(X, Y) - V(X, Y)| \leqslant \Omega\Big(\sum_{r=1}^{n} |x_r - \mathring{x}_r|; H, \delta\Big) \tag{43.7}$$

holds true in the pyramid (39.2). From (43.5) and (43.7) follows (43.4).

Remark 43.1. All theorems of §§ 39-43 are true if, in place of the pyramid (39.2), we have the zone

$$\sum_{r=1}^{p} |x_r - \mathring{x}_r| < \gamma, \quad y_1, \ldots, y_n \text{ arbitrary}, \tag{43.8}$$

provided that the solution be continuous and possess Stolz's differential in (43.8) and in Theorem 42.2 their derivatives with respect to x_j be, in addition, continuous for $X = X_0$. This remark is an immediate consequence of Remark 38.1.

§ 44. Estimate of the error of an approximate solution. In this section, like in § 40, we restrict ourselves to the Cauchy problem for equation (40.1) with initial conditions (40.2). We will indicate a procedure by which we can evaluate the error when, instead of the solution of a given ("difficult to solve") problem (40.1), (40.2), the solution of an approximate ("easy to solve") one is taken.

Let the right-hand member $f(x, Y, u, Q)$ of equation (40.1) and the initial function $\varphi(Y)$ satisfy assumptions introduced in § 40.

Consider the approximate ("easy to solve") equation

$$u_x = g(x, y_1, \dots, y_n, u, u_{y_1}, \dots, u_{y_n}) \tag{44.1}$$

with $g(x, Y, u, Q)$ defined in the cube (40.3) and the approximate initial condition

$$u(0, Y) = \psi(Y) . \tag{44.2}$$

Suppose that

$$|g(x, Y, u, Q)-g(x, Y, \tilde{u}, \tilde{Q})| \leqslant \tilde{\sigma}(|x|, |u-\tilde{u}|)+M\sum_{k=1}^{n}|q_k-\tilde{q}_k| \tag{44.3}$$

where $\tilde{\sigma}(t, v)$ is the right-hand side of a comparison equation of type I (see § 14). Let $v(x, Y)$ be a solution of the approximate problem (44.1), (44.2) in a pyramid

$$|x| < \tilde{\delta}, \quad |y_k| \leqslant \tilde{a}-M|x| \quad (k = 1, 2, \dots, n) .$$

Suppose finally that the limitation

$$|f(x, Y, u, Q)-g(x, Y, u, Q)| \leqslant h(|x|, |u|, |Q|) \tag{44.4}$$

is known, where $h(t, v, q_1, \dots, q_n)$ satisfies condition $\overline{\mathrm{W}}_+$ with respect to $(v, q_1, \dots, q_n)$ (see § 14), and

$$|\varphi(Y)-\psi(Y)| \leqslant \eta . \tag{44.5}$$

Under these hypotheses we can evaluate the difference between the solution $u(x, Y)$ of problem (40.1), (40.2), which is sought for, and the approximate one $v(x, Y)$. We do it in two steps.

I step. *Estimate of the solution and of its existence domain.* Following the results of § 40 we evaluate the pyramid (40.4), in which $u(x, Y)$ is of class C^1, and find the functions $\gamma(t)$ and $\beta_k(t)$ for which inequalities (40.14) hold true. The functions $u(x, Y)$ and $v(x, Y)$ are then both defined in the pyramid

$$|x| < \min(\delta, \tilde{\delta}), \quad |y_k| \leqslant \min(a, \tilde{a})-M|x| \quad (k = 1, 2, \dots, n) . \tag{44.6}$$

II step. *Evaluation of the error.* Solution $u(x, Y)$ satisfies obviously the equation

$$u_x = \tilde{g}(x, Y, u, u_{y_1}, \dots, u_{y_n}) , \tag{44.7}$$

where

$$\tilde{g}(x, Y, u, Q) = g(x, Y, u, Q) + \\ + \big[f\big(x, Y, u(x, Y), u_Y(x, Y)\big) - g\big(x, Y, u(x, Y), u_Y(x, Y)\big)\big].$$

By (44.3), (44.4), (40.14) and by the condition $\overline{\mathrm{W}}_+$, imposed on h, we get

$$(44.8) \qquad |g(x, Y, u, Q) - \tilde{g}(x, Y, \tilde{u}, \tilde{Q})| \leqslant \sigma(|x|, |u-\tilde{u}|) + M \sum_{k=1}^{n} |q_k - \tilde{q}_k|,$$

where

$$\sigma(t, v) = \tilde{\sigma}(t, v) + h\big(t, \gamma(t), \beta_1(t), \ldots, \beta_n(t)\big)$$

is the right-hand member of a comparison equation of type I (see § 14). Denoting by $\omega(t)$ its right-hand maximum solution through $(0, \eta)$, defined in an interval $[0, a_0)$, we conclude, by (44.5), (44.8) and by Theorem 41.1 applied to equations (44.1) and (44.7), that inequality

$$|u(x, Y) - v(x, Y)| \leqslant \omega(|x|)$$

holds true in the pyramid (44.6) for $|x| < \min(\delta, \tilde{\delta}, a_0)$. This is the estimate of the error that was sought for.

§ 45. Systems with total differentials. A system with total differentials

$$(45.1) \qquad u^i_{x_j} = f^i_j(X, u^1, \ldots, u^m) \qquad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p)$$

or shortly

$$du^i = \sum_{j=1}^{m} f^i_j(X, u^1, \ldots, u^m)\, dx_j \qquad (i = 1, 2, \ldots, m)$$

is a particular case of the overdetermined system (39.1) dealt with in the preceding paragraphs. Cauchy initial conditions for system (45.1) have the form

$$(45.2) \qquad u^i(X_0) = \mathring{u}^i \qquad (i = 1, 2, \ldots, m).$$

Now, it is clear that all theorems of §§ 41-43 hold true for the Cauchy problem (45.1), (45.2).

CHAPTER VIII

MIXED PROBLEMS FOR SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS OF PARABOLIC AND HYPERBOLIC TYPE

In the first paragraphs of the present chapter we deal with parabolic solutions (see the subsequent definitions) of nonlinear systems of second order partial differential equations of the form (see [53] and [54])

$$u_t^i = f^i(t, x_1, \ldots, x_n, u^1, \ldots, u^m, u^i_{x_1}, \ldots, u^i_{x_n}, u^i_{x_1x_1}, u^i_{x_1x_2}, \ldots, u^i_{x_nx_n}) \quad (i = 1, 2, \ldots, m),$$

where the ith equation contains derivatives of only one unknown function u^i. We discuss a number of questions concerning mixed problems in a region $D \subset (t, x_1, \ldots, x_n)$ of type C (see § 33). In particular, using the theory of ordinary differential inequalities we treat questions referring to mixed problems like: estimates of the solution, estimates of the difference between two solutions, uniqueness criteria, continuous dependence of the solution on initial and boundary values and on the right-hand sides of system and, finally, stability of the solution.

In the last paragraphs we derive, by means of ordinary differential inequalities, energy estimates of Friedrichs-Levy type for the solution of a system of linear hyperbolic equations (see [51])

$$\sum_{j,k=1}^{n} a^i_{jk}(X) u^i_{x_jx_k} = \sum_{l=1}^{m} \sum_{j=1}^{n} b^{il}_j(X) u^l_{x_j} + \sum_{l=1}^{m} c^{il}(X) u^l + f^i(X) \quad (i = 1, 2, \ldots, m),$$

where the ith equation contains second derivatives of only one unknown function u^i.

§ 46. Ellipticity and parabolicity. To begin with, we recall the definition of a positive (negative) quadratic form and prove, for the convenience of the reader, a lemma.

A real quadratic form in $\lambda_1, \ldots, \lambda_n$, $\sum_{j,k=1}^{n} a_{jk}\lambda_j\lambda_k$ $(a_{jk} = a_{kj})$ is called *positive* (*negative*) if for arbitrary $\lambda_1, \ldots, \lambda_n$ we have

$$\sum_{j,k=1}^{n} a_{jk}\lambda_j\lambda_k \geqslant 0 \quad (\leqslant 0)$$

LEMMA 46.1. *Let the quadratic form* $\Phi(\Lambda) = \Phi(\lambda_1, \ldots, \lambda_n) = \sum_{j,k=1}^{n} a_{jk}\lambda_j\lambda_k$ *be positive and the quadratic form* $\Psi(\Lambda) = \Psi(\lambda_1, \ldots, \lambda_n) = \sum_{j,k=1}^{n} b_{jk}\lambda_j\lambda_k$ *be negative; then we have*

$$\sum_{j,k=1}^{n} a_{jk}b_{jk} \leqslant 0 . \tag{46.1}$$

Proof. The form $\Phi(\Lambda)$ being positive we have, for suitably chosen coefficients α_{pq} $(p, q = 1, 2, \ldots, n)$,

$$\Phi(\Lambda) = \sum_{j,k=1}^{n} a_{jk}\lambda_j\lambda_k = \sum_{p=1}^{n}\Big(\sum_{q=1}^{n} \alpha_{pq}\lambda_q\Big)^2;$$

hence

$$a_{jk} = \sum_{p=1}^{n} \alpha_{pj}\alpha_{pk} \qquad (j, k = 1, 2, \ldots, n)$$

and consequently

$$\sum_{j,k=1}^{n} a_{jk}b_{jk} = \sum_{p=1}^{n}\Big(\sum_{j,k=1}^{n} b_{jk}\alpha_{pj}\alpha_{pk}\Big) = \sum_{p=1}^{n} \Psi(\alpha_{p1}, \ldots, \alpha_{pn}) \leqslant 0 . \tag{46.2}$$

DEFINITION OF ELLIPTICITY. Let the function

$$f^i(t, X, U, Q, R) = f^i(t, x_1, \ldots, x_n, u^1, \ldots, u^m, q_1, \ldots, q_n, r_{11}, r_{12}, \ldots, r_{nn})$$

be defined for (t, X) belonging to a region $D \subset (t, x_1, \ldots, x_n)$ and for arbitrary U, Q, R. Suppose that $U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big)$ is defined and possesses first derivatives with respect to x_j at a point $(\tilde{t}, \tilde{X}) \in D$. Write

$$u^i_X = (u^i_{x_1}, \ldots, u^i_{x_n}) .$$

Under these assumptions, we say that the *function* $f^i(t, X, U, Q, R)$ *is elliptic with respect to* $U(t, X)$ *at the point* $(\tilde{t}, \tilde{X}) \in D$ if for any two sequences of numbers $R = (r_{11}, r_{12}, \ldots, r_{nn})$ and $\tilde{R} = (\tilde{r}_{11}, \tilde{r}_{12}, \ldots, \tilde{r}_{nn})$ $(r_{jk} = r_{kj}, \tilde{r}_{jk} = \tilde{r}_{kj})$ such that the quadratic form in $\lambda_1, \ldots, \lambda_n$

$$\sum_{j,k=1}^{n} (r_{jk} - \tilde{r}_{jk})\lambda_j\lambda_k \text{ is negative ,} \tag{46.3}$$

we have

$$f^i\big(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), u^i_X(\tilde{t}, \tilde{X}), R\big) \leqslant f^i\big(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), u^i_X(\tilde{t}, \tilde{X}), \tilde{R}\big) . \tag{46.4}$$

If the above property holds true for every point $(\tilde{t}, \tilde{X}) \in D$, then we say that $f^i(t, X, U, Q, R)$ *is elliptic with respect to* $U(t, X)$ *in* D.

EXAMPLE 46.1. Consider the second order linear equation

$$u_t = \sum_{j,k=1}^{n} a_{jk}(t, X)u_{x_j x_k} + \sum_{j=1}^{n} b_j(t, X)u_{x_j} + c(t, X)u + d(t, X)\,, \tag{46.5}$$

where $a_{jk}(t, X)$, $b_j(t, X)$, $c(t, X)$ and $d(t, X)$ are defined in a region D. Equation (46.5) is called *parabolic at a point* $(\tilde{t}, \tilde{X}) \in D$ if the quadratic form in $\lambda_1, \ldots, \lambda_n$

$$\sum_{j,k=1}^{n} a_{jk}(\tilde{t}, \tilde{X})\lambda_j\lambda_k \text{ is positive}\,. \tag{46.6}$$

Now, by Lemma 46.1, we conclude that the right-hand member

$$f(t, X, u, Q, R) = d(t, X) + c(t, X)u + \sum_{j=1}^{n} b_j(t, X)q_j + \sum_{j,k=1}^{n} a_{jk}(t, X)r_{jk}$$

of a parabolic equation at a point $(\tilde{t}, \tilde{X})$ is elliptic at $(\tilde{t}, \tilde{X})$ with respect to any function $u(t, X)$ having first derivatives u_{x_j} at $(\tilde{t}, \tilde{X})$.

Remark 46.1. If, in particular, $f^i(t, X, U, Q, R)$ is independent of R, then it is trivially elliptic with regard to any $U(t, X)$.

DEFINITION OF PARABOLIC SOLUTION. Consider a system of second order partial differential equations

$$u_t^i = f^i(t, x_1, \ldots, x_n, u^1, \ldots, u^m, u^i_{x_1}, \ldots, u^i_{x_n}, u^i_{x_1x_1}, u^i_{x_1x_2}, \ldots, u^i_{x_nx_n}) \quad (i = 1, 2, \ldots, m) \tag{46.7}$$

with right-hand sides $f^i(t, X, U, Q, R)$ defined for $(t, X) \in D$ and U, Q, R arbitrary. A solution $U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big)$ of (46.7) in D is called *parabolic at a point* $(\tilde{t}, \tilde{X}) \in D$ if all the functions $f^i(t, X, U, Q, R)$ $(i = 1, 2, \ldots, m)$ are elliptic with respect to $U(t, X)$ at $(\tilde{t}, \tilde{X})$.

If this property holds true for every point in D, then the solution is called *parabolic in D*.

According to Example 46.1 every solution of a parabolic equation (46.5) is a parabolic one.

Remark 46.2. In virtue of Remark 46.1, every solution of a system (46.7) is parabolic if its right-hand sides do not depend on second derivatives, i.e. if it reduces to a system of first order partial differential equations or of ordinary differential equations with parameters.

§ 47. Mixed problems. Before formulating the mixed problems we are going to deal with in the present chapter, we introduce some definitions and assumptions.

DEFINITION OF SETS Σ AND Σ_a. Consider a region $D \subset (t, x_1, \ldots, x_n)$ of type C (see § 33). We denote by Σ the side surface of D, i.e. that part of the boundary of D which is contained in the open zone $t_0 < t < t_0+T$.

A function $a(t, X)$ being given on Σ we denote by Σ_a the subset of Σ on which $a(t, X) \neq 0$.

ASSUMPTIONS A. *A region $D \subset (t, x_1, \ldots, x_n)$ of type C* (see § 33) *being given, let the functions $a^i(t, X)$ $(i = 1, 2, \ldots, m)$ be defined on its side surface Σ. Suppose that*

$$a^i(t, X) \geqslant 0 \qquad (i = 1, 2, \ldots, m). \tag{47.1}$$

For every $(t, X) \in \Sigma_{a^i}$, let a direction $l^i(t, X)$ be given, so that l^i is orthogonal to the t-axis and some segment, with one extremity at (t, X), of the straight half-line from (t, X) in the direction l^i is contained in the closure of D.

Regular solutions and mixed problems. Consider a system (46.7) with right-hand sides $f^i(t, X, U, Q, R)$ $(i = 1, 2, \ldots, m)$ defined for $(t, X) \in D$ of type C (see § 33) and for arbitrary U, Q, R. Let the functions $a^i(t, X)$ and directions $l^i(t, X)$ $(i = 1, 2, \ldots, m)$, satisfying Assumptions A, be given on the side surface Σ of D. A solution $U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big)$ of (46.7) in D will be called *regular solution* if it is continuous in the closure of D, possesses continuous derivatives $\partial/\partial t$, $\partial/\partial x_j$, $\partial^2/\partial x_j \partial x_k$, and satisfies (46.7) in the interior of D. If, in addition, for every i the derivative du^i/dl^i exists at each point $(t, X) \in \Sigma_{a^i}$, then the solution is called *Σ_a-regular solution*. Being given

1. a system (46.7) with right-hand sides $f^i(t, X, U, Q, R)$ $(i = 1, 2, \ldots, m)$ defined for $(t, X) \in D$ of type C (see § 33) and for arbitrary U, Q, R,
2. functions $a^i(t, X)$ and directions $l^i(t, X)$ $(i = 1, 2, \ldots, m)$ on the side surface Σ of D, satisfying Assumptions A,
3. functions $\psi^i(t, X)$ on Σ and $\beta^i(t, X)$ on Σ_{a^i} $(i = 1, 2, \ldots, m)$ where

$$\beta^i(t, X) > 0 \text{ on } \Sigma_{a^i} \qquad (i = 1, 2, \ldots, m), \tag{47.2}$$

4. functions $\varphi^i(X)$ $(i = 1, 2, \ldots, m)$ on S_{t_0} (for the definition of S_t, see § 33, definition of a region of type C),

the *first mixed problem* with initial values $\varphi^i(X)$ and boundary values $\psi^i(t, X)$ consists in finding a Σ_a-regular solution $U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big)$ of (46.7) in D, satisfying the initial conditions

$$U(t_0, X) = \Phi(X) \quad \text{for} \quad X \in S_{t_0}, \tag{47.3}$$

where $\Phi(X) = \big(\varphi^1(X), \ldots, \varphi^m(X)\big)$, and boundary conditions, called of *first type*,

$$\begin{aligned} \beta^i(t, X)u^i(t, X) - a^i(t, X)\frac{du^i}{dl^i} &= \psi^i(t, X) \quad \text{for} \quad (t, X) \in \Sigma_{a^i}, \\ u^i(t, X) &= \psi^i(t, X) \quad \text{for} \quad (t, X) \in \Sigma - \Sigma_{a^i} \\ & \qquad (i = 1, 2, \ldots, m). \end{aligned} \tag{47.4}$$

If, in particular, $a^i(t, X) \equiv 0$ $(i = 1, 2, \ldots, m)$, the boundary conditions (47.4) are of Dirichlet's type and the first mixed problem reduces to the classical first Fourier's problem. If condition (47.2) is not imposed on $\beta^i(t, X)$, the problem described above is called *second mixed problem* and the boundary conditions (47.4) are called *of second type.*

In particular, when $a^i(t, X) \equiv 1$, $\beta^i(t, X) \equiv 0$ $(i = 1, 2, \ldots, m)$, the boundary conditions (47.4) are of Neumann's type and the second mixed problem reduces to the classical second Fourier's problem.

To close this paragraph, we prove a lemma which will be of use in our subsequent considerations.

LEMMA 47.1. *Suppose we are given a region D of type C* (see § 33), *a function $a(t, X)$ and a direction $l(t, X)$ satisfying (for $m = 1$) Assumptions* A *on the side surface Σ of D, and a function $\beta(t, X)$ on Σ_a such that*

$$\beta(t, X) > B \geqslant 0 \quad \text{for} \quad (t, X) \in \Sigma_a. \tag{47.5}$$

Let the function $u(t, X)$ be continuous in the closure of D and possess the derivative du/dl on Σ_a. Suppose that

$$\begin{aligned} \beta(t, X)u(t, X) - a(t, X)\frac{du}{dl} &\leqslant B\eta(t) \;\; (< B\eta(t)) \quad \text{for} \quad (t, X) \in \Sigma_a, \\ u(t, X) &\leqslant \eta(t) \;\; (< \eta(t)) \quad \text{for} \quad (t, X) \in \Sigma - \Sigma_a, \end{aligned} \tag{47.6}$$

where $\eta(t) \geqslant 0$. Denote by $S_{\tilde{t}}$ (see § 33) *the projection on the space $(x_1, \ldots, x_n)$ fo the intersection of the closure of D with the plane $t = \tilde{t}$.*

Under these assumptions, if for a point $(\tilde{t}, \tilde{X}) \in \bar{D}$ $(t_0 < \tilde{t} < t_0 + T)$ we have

$$\max_{X \in S_{\tilde{t}}} u(\tilde{t}, X) = u(\tilde{t}, \tilde{X}) > \eta(\tilde{t}) \quad (\geqslant \eta(\tilde{t})), \tag{47.7}$$

then $(\tilde{t}, \tilde{X})$ is an interior point of D.

Proof. Suppose that the assertion of our lemma is false; then $(\tilde{t}, \tilde{X}) \in \Sigma$ and there are two possible cases to be distinguished: I. $(\tilde{t}, \tilde{X}) \in \Sigma - \Sigma_a$, II. $(\tilde{t}, \tilde{X}) \in \Sigma_a$.

In the case I we have, by (47.6),

$$u(\tilde{t}, \tilde{X}) \leqslant \eta(\tilde{t}) \quad (< \eta(\tilde{t})),$$

contrary to (47.7). Now in the case II we get, by (47.6)

$$(47.8)\qquad \beta(\tilde{t},\tilde{X})u(\tilde{t},\tilde{X})-a(\tilde{t},\tilde{X})\frac{du}{dl}\bigg|_{(\tilde{t},\tilde{X})}\leqslant B\eta(\tilde{t})\qquad (<B\eta(\tilde{t}))\,.$$

The straight half-line from $(\tilde{t},\tilde{X})$ in the direction $l(\tilde{t},\tilde{X})$ has the parametric equation

$$X=\tilde{X}+\tau\operatorname{vers}l(\tilde{t},\tilde{X})\,,\qquad \tau\geqslant 0\,.$$

By Assumptions A, some segment of this half-line, say $0\leqslant\tau<\tau_0$, belongs to $S_{\tilde{t}}$. Hence the function

$$\varphi(\tau)=u\big(\tilde{t},\tilde{X}+\tau\operatorname{vers}l(\tilde{t},\tilde{X})\big)$$

is defined for $0\leqslant\tau<\tau_0$ and attains, by (47.7), its maximum at the left-hand extremity of this interval. Therefore,

$$(47.9)\qquad \varphi'(0)=\frac{du}{dl}\bigg|_{(\tilde{t},\tilde{X})}\leqslant 0\,.$$

Since $a(\tilde{t},\tilde{X})\geqslant 0$ (by Assumptions A), it follows from (47.8) and (47.9) that

$$\beta(\tilde{t},\tilde{X})u(\tilde{t},\tilde{X})\leqslant B\eta(\tilde{t})\qquad (<B\eta(\tilde{t}))$$

and hence, by (47.5),

$$u(\tilde{t},\tilde{X})\leqslant\eta(\tilde{t})\qquad (<\eta(\tilde{t}))\,,$$

what contradicts (47.7). This completes the proof of our lemma.

§ 48. Estimates of the solution of the first mixed problem. We prove

THEOREM 48.1. *Assume the right-hand members* $f^i(t,X,U,Q,R)$ $(i=1,2,\dots,m)$ *of system* (46.7) *to be defined for* $(t,X)\in D$ *of type* C (see §33) *and for arbitrary* U,Q,R. *Suppose that* (¹)

$$(48.1)\qquad f^i(t,X,U,0,0)\operatorname{sgn}u^i\leqslant\sigma_i(t-t_0,|U|)\qquad (i=1,2,\dots,m)\,,$$

where $\sigma_i(t,V)$ *are the right-hand sides of a comparison system of type* I (see § 14). *Denote by* $\Omega(t;H)=\big(\omega_1(t;H),\dots,\omega_m(t;H)\big)$ *its right-hand maximum solution through* $(0,H)=(0,\eta_1,\dots,\eta_m)$, *defined in an interval* $[0,a_0(H))$. *Let the functions* $a^i(t,X)$ *and the directions* $l^i(t,X)$ $(i=1,2,\dots,m)$ *satisfy Assumptions* A (see § 47) *on the side surface* Σ *of* D. *Let* $\beta^i(t,X)$ *be defined on* Σ_{a^i} $(i=1,2,\dots,m)$ *and satisfy inequalities*

$$(48.2)\qquad \beta^i(t,X)>B^i\geqslant 0\ \ on\ \Sigma_{a^i}\qquad (i=1,2,\dots,m)\,.$$

(¹) $\operatorname{sgn}x$ denotes 1 if $x\geqslant 0$, and -1 if $x<0$.

Suppose finally that $U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big)$ *is a parabolic* (see § 46), Σ_a*-regular* (see § 47) *solution of system* (46.7) *in* D, *satisfying initial inequalities*

$$|U(t_0, X)| \leqslant H \quad \textit{for} \quad X \in S_{t_0} \tag{48.3}$$

and boundary inequalities

$$\begin{aligned} &\left|\beta^i(t, X)u^i(t, X) - \alpha^i(t, X)\frac{du^i}{dl^i}\right| \leqslant B^i\omega_i(t-t_0; H) \quad \textit{for} \quad (t, X) \in \Sigma_{a^i}, \\ &\qquad\qquad\qquad\qquad |u^i| \leqslant \omega_i(t-t_0; H) \qquad \textit{for} \quad (t, X) \in \Sigma - \Sigma_{a^i} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 1, 2, \ldots, m)\,. \end{aligned} \tag{48.4}$$

Under these assumptions inequality

$$|U(t, X)| \leqslant \Omega(t-t_0; H) \tag{48.5}$$

holds true in D *for*

$$0 \leqslant t-t_0 < \min\big(T, a_0(H)\big) = \delta\,.$$

Proof. Since the assumptions of our theorem are invariant under the mapping $\tau = t-t_0$, we may assume, without loss of generality, that $t_0 = 0$. Denoting by $S_{\tilde{t}}$ the projection on $(x_1, \ldots, x_n)$ of the intersection of $\bar{D}$ with the plane $t = \tilde{t}$ (see § 33) put, for $0 \leqslant t < T$,

$$\begin{aligned} W^i(t) &= \max_{X \in S_t} |u^i(t, X)|\,, \quad W(t) = \big(W^1(t), \ldots, W^m(t)\big)\,, \\ M^i(t) &= \max_{X \in S_t} u^i(t, X) \quad (i = 1, 2, \ldots, m)\,, \\ N^i(t) &= \max_{X \in S_t} \big(-u^i(t, X)\big)\,. \end{aligned}$$

By Theorem 34.1, the functions $W^i(t)$ are continuous in the interval $[0, T)$ and, by (48.3), we have

$$W(0) \leqslant H\,. \tag{48.6}$$

Inequalities (48.5) are obviously equivalent with

$$W(t) \leqslant \Omega(t; H) \quad \text{for} \quad 0 \leqslant t < \min\big(T, a_0(H)\big) = \delta\,.$$

Now, in view of (48.6) and of the first comparison theorem (see § 14), the last relation will be proved if we show that, for every fixed j, differential inequality

$$D_- W^j(t) \leqslant \sigma_j\big(t, W(t)\big) \tag{48.7}$$

holds true in the set

$$E^j = \{t \in (0, \delta)\colon\ W^j(t) > \omega_j(t; H)\}\,. \tag{48.8}$$

Fix an index j and let $\tilde{t} \in E^j$; then, we have

$$(48.9) \qquad W^j(\tilde{t}) > \omega_j(t; H).$$

By Theorem 34.1, there is a point $\tilde{X} \in S_{\tilde{t}}$, so that either

$$(48.10) \qquad W^j(\tilde{t}) = M^j(\tilde{t}) = u^j(\tilde{t}, \tilde{X}), \quad D_- W^j(\tilde{t}) \leqslant D^- M^j(\tilde{t}),$$

or

$$(48.11) \qquad W^j(\tilde{t}) = N^j(\tilde{t}) = -u^j(\tilde{t}, \tilde{X}), \quad D_- W^j(\tilde{t}) \leqslant D^- N^j(\tilde{t}).$$

Suppose we have, for instance, (48.11). Then, in view of (48.2), (48.4) and (48.9) we conclude, by Lemma 47.1, that $(\tilde{t}, \tilde{X})$ is an interior point of D. The function $-u^j(\tilde{t}, X)$ attains its maximum at the interior point $\tilde{X}$ and is of class C^2 in its neighborhood. Therefore,

$$(48.12) \qquad u^j_X(\tilde{t}, \tilde{X}) = 0$$

and the quadratic form in $\lambda_1, \ldots, \lambda_n$

$$(48.13) \qquad -\sum_{l,k=1}^{n} u^j_{x_l x_k}(\tilde{t}, \tilde{X})\lambda_l \lambda_k \quad \text{is negative}.$$

By Theorem 33.1, 2°, we have

$$D^- N^j(\tilde{t}) \leqslant -u^j_t(\tilde{t}, \tilde{X});$$

hence, by (48.11), we get

$$(48.14) \quad D_- W^j(\tilde{t}) \leqslant -u^j_t(\tilde{t}, \tilde{X}) = -f^j\big(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), u^j_X(\tilde{t}, \tilde{X}), u^j_{XX}(\tilde{t}, \tilde{X})\big),$$

where we have put

$$u^j_{XX}(t, X) = \big(u^j_{x_1x_1}(t, X), u^j_{x_1x_2}(t, X), \ldots, u^j_{x_nx_n}(t, X)\big).$$

Since, by (48.11), we have

$$\operatorname{sgn} u^j(\tilde{t}, \tilde{X}) = -1,$$

it follows from (48.14), by (48.12), that

$$(48.15) \quad D_- W^j(\tilde{t}) \leqslant \big[f^j\big(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), 0, 0\big) - \\ -f^j\big(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), 0, u^j_{XX}(\tilde{t}, \tilde{X})\big)\big] + f^j\big(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), 0, 0\big) \operatorname{sgn} u^j(\tilde{t}, \tilde{X}).$$

The difference in brackets is, by the parabolicity of solution $U(t, X)$ (see § 46) and by (48.13), non-positive. Hence, from (48.1) and (48.15) we obtain

$$(48.16) \qquad D_- W^j(\tilde{t}) \leqslant \sigma_j\big(\tilde{t}, |U(\tilde{t}, \tilde{X})|\big).$$

But, by the definition of $W^i(t)$ and by (48.11), we have (see § 4)

$$|U(\tilde{t}, \tilde{X})| \overset{j}{\leqslant} W(\tilde{t}) .$$

Therefore, in view of the condition W_+ (see § 4) imposed on functions $\sigma_i(t, V)$, inequality (48.16) implies that (48.7) is satisfied for $t = \tilde{t}$, which completes the proof.

Remark 48.1. Under the assumptions of Theorem 48.1 it may happen that the differential inequality (48.7) does not hold for any $t \in (0, \delta)$. In this case Theorem 9.3 does not enable us to conclude on the validity of inequality $W(t) \leqslant \Omega(t; H)$, whereas the first comparison theorem (see § 14)—which is a consequence of Theorem 11.1—does.

The above situation occurs in the following trivial example. Let $n = m = 1$ and put

$$f(t, x, u, q, r) = r , \qquad D = \{(t, x) \colon 0 < t < T , \ 0 < x < 1\} .$$

The system (46.7) reduces now to the heat equation and its right-hand side satisfies inequality (48.1) with $\sigma(t, v) \equiv 0$. Put

$$\alpha(t, x) \equiv 0 , \qquad \beta(t, x) \equiv 1 , \qquad \eta = e^{T+1};$$

then $u(t, x) = e^{t+x}$ is a solution of the heat equation, satisfying assumptions of Theorem 48.1. But, since obviously

$$W(t) = \max_{0 \leqslant x \leqslant 1} |u(t, x)| = e^{t+1} ,$$

we have $W'(t) > 0$ and inequality (48.7) does not hold for any $t \in (0, \delta)$. This remark shows the usefulness of Theorem 11.1.

§ 49. Estimates of the difference between two solutions of the first mixed problem. Now we prove

THEOREM 49.1. *Suppose the right-hand members* $f^i(t, X, U, Q, R)$ $(i = 1, 2, \ldots, m)$ *of system* (46.7) *and of system*

$$(49.1) \qquad u_t^i = g^i(t, x_1, \ldots, x_n, u^1, \ldots, u^m, u_{x_1}^i, \ldots, u_{x_n}^i, u_{x_1x_1}^i, u_{x_1x_2}^i, \ldots, u_{x_nx_n}^i) \quad (i = 1, 2, \ldots, m)$$

are defined for $(t, X) \in D$ *of type* C (see § 33) *and for arbitrary* U, Q, R. *Assume that*

$$(49.2) \qquad [f^i(t, X, U, Q, R) - g^i(t, X, \tilde{U}, Q, R)] \operatorname{sgn}(u^i - \tilde{u}^i) \leqslant \sigma_i(t - t_0, |U - \tilde{U}|) \quad (i = 1, 2, \ldots, m) ,$$

where $\sigma_i(t, V)$ *are the right-hand sides of a comparison system of type* I (see § 14). *Let* $\Omega(t; H) = (\omega_1(t; H), \ldots, \omega_m(t; H))$ *be its right-hand maximum solution through* $(0, H) = (0, \eta_1, \ldots, \eta_m)$, *defined on an interval* $[0, a_0(H))$.

Let $a^i(t, X)$, $l^i(t, X)$ $(i = 1, 2, \dots, m)$ *satisfy Assumptions* A (see § 47) *and* $\beta^i(t, X)$ $(i = 1, 2, \dots, m)$ *inequalities* (48.2). *Suppose, finally, that* $U(t, X) = \big(u^1(t, X), \dots, u^m(t, X)\big)$ *is a parabolic* (see § 46), Σ_a*-regular* (see § 47) *solution of system* (46.7) *in* D *and* $V(t, X) = \big(v^1(t, X), \dots, v^m(t, X)\big)$ *is a* Σ_a*-regular solution of system* (49.1) *in* D, *satisfying initial inequalities*

$$(49.3) \qquad |U(t_0, X) - V(t_0, X)| \leqslant H \quad \text{for} \quad X \in S_{t_0}$$

and boundary inequalities

$$(49.4) \qquad \begin{aligned} &\left|\beta^i(t, X)[u^i(t, X) - v^i(t, X)] - a^i(t, X)\frac{d[u^i - v^i]}{dl^i}\right| \leqslant B^i\omega_i(t - t_0; H) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad (t, X) \in \Sigma_{a^i}, \\ &|u^i(t, X) - v^i(t, X)| \leqslant \omega_i(t - t_0; H) \quad \text{for} \quad (t, X) \in \Sigma - \Sigma_{a^i} \\ &\qquad\qquad\qquad\qquad\qquad\qquad (i = 1, 2, \dots, m). \end{aligned}$$

Under these assumptions we have inequalities

$$(49.5) \qquad |U(t, X) - V(t, X)| \leqslant \Omega(t - t_0; H)$$

in D *for*

$$0 \leqslant t - t_0 < \min\big(T, a_0(H)\big) = \delta .$$

Proof. Like in Theorem 48.1 we assume, without loss of generality, that $t_0 = 0$. Put, for $0 \leqslant t < T$,

$$W^i(t) = \max_{X \in S_t} |u^i(t, X) - v^i(t, X)|, \qquad W(t) = \big(W^1(t), \dots, W^m(t)\big),$$

$$M^i(t) = \max_{X \in S_t} \big(u^i(t, X) - v^i(t, X)\big) \qquad (i = 1, 2, \dots, m),$$

$$N^i(t) = \max_{X \in S_t} \big(v^i(t, X) - u^i(t, X)\big).$$

Just like in the proof of Theorem 48.1, it is sufficient to show that inequality (48.7) holds true in the set E^j defined by (48.8). Fix an index j and let $\tilde{t} \in E^j$; then we have (48.9) and, by Theorem 34.1, there is a point $\tilde{X} \in S_{\tilde{t}}$ such that either

$$(49.6) \qquad W^j(\tilde{t}) = M^j(\tilde{t}) = u^j(\tilde{t}, \tilde{X}) - v^j(\tilde{t}, \tilde{X}), \qquad D_-W^j(\tilde{t}) \leqslant D^-M^j(\tilde{t}),$$

or

$$(49.7) \qquad W^j(\tilde{t}) = N^j(\tilde{t}) = v^j(\tilde{t}, \tilde{X}) - u^j(\tilde{t}, \tilde{X}), \qquad D_-W^j(\tilde{t}) \leqslant D^-N^j(\tilde{t}).$$

Suppose we have, for instance, (49.6); then, like in the proof of Theorem 48.1, we conclude that $(\tilde{t}, \tilde{X})$ is an interior point of D. Hence we have

$$(49.8) \qquad u^j_X(\tilde{t}, \tilde{X}) = v^j_X(\tilde{t}, \tilde{X})$$

and the quadratic form in $\lambda_1, \ldots, \lambda_n$

$$(49.9) \qquad \sum_{l,k=1}^{n} [u^j_{x_l x_k}(\tilde{t}, \tilde{X}) - v^j_{x_l x_k}(\tilde{t}, \tilde{X})]\lambda_l \lambda_k \quad \text{is negative}.$$

By Theorem 33.1, 2°, we have

$$D^- M^j(\tilde{t}) \leqslant u^j_t(\tilde{t}, \tilde{X}) - v^j_t(\tilde{t}, \tilde{X});$$

therefore, by (49.6), we obtain

$$\begin{aligned} D_- W^j(\tilde{t}) &\leqslant u^j_t(\tilde{t}, \tilde{X}) - v^j_t(\tilde{t}, \tilde{X}) \\ &= f^j\big(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), u^j_X(\tilde{t}, \tilde{X}), u^j_{XX}(\tilde{t}, \tilde{X})\big) - \\ &\qquad - g^j\big((\tilde{t}, \tilde{X}, V(\tilde{t}, \tilde{X}), v^j_X(\tilde{t}, \tilde{X}), v^j_{XX}(\tilde{t}, \tilde{X})\big). \end{aligned}$$

From the last inequality it follows, by (49.8), that

$$\begin{aligned} D_- W^j(\tilde{t}) &\leqslant \big[f^j\big(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), u^j_X(\tilde{t}, \tilde{X}), u^j_{XX}(\tilde{t}, \tilde{X})\big) - \\ &\qquad - f^j\big(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), u^j_X(\tilde{t}, \tilde{X}), v^j_{XX}(\tilde{t}, \tilde{X})\big)\big] + \\ &\quad + \big[f^j\big(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), u^j_X(\tilde{t}, \tilde{X}), v^j_{XX}(\tilde{t}, \tilde{X})\big) - \\ &\qquad - g^j\big(\tilde{t}, \tilde{X}, V(\tilde{t}, \tilde{X}), u^j_X(\tilde{t}, \tilde{X}), v^j_{XX}(\tilde{t}, \tilde{X})\big)\big]. \end{aligned}$$

The first difference in brackets is, by the parabolicity of solution $U(t, X)$ (see § 46) and by (49.9), non-positive. Since, by (49.6),

$$u^j(\tilde{t}, \tilde{X}) \geqslant v^j(\tilde{t}, \tilde{X}),$$

we get in virtue of inequality (49.2)

$$D_- W^j(\tilde{t}) \leqslant \sigma_j\big(\tilde{t}, |U(\tilde{t}, \tilde{X}) - V(\tilde{t}, \tilde{X})|\big).$$

From the last inequality if follows, like in the proof of Theorem 48.1, that (48.7) holds true for $t = \tilde{t}$, which completes the proof.

Using the results contained in Example 46.1 we get from Theorem 49.1 the following corollary:

COROLLARY 49.1. *Let the linear equation*

$$u_t = \sum_{j,k=1}^{n} a_{jk}(t, X) u_{x_j x_k} + \sum_{j=1}^{n} b_j(t, X) u_{x_j} + c(t, X) u + d(t, X)$$

be parabolic (see Example 46.1) *in a region D of type C* (see § 33). *Suppose that*

$$c(t, X) \leqslant 0$$

and

$$\beta(t, X) > B \geqslant 0 \quad \textit{for} \quad (t, X) \in \Sigma_a,$$

and that $a(t, X)$, $l(t, X)$ *satisfy Assumptions* A (see § 47). *This being assumed we have, for any two* Σ_a*-regular solutions* (see § 47) $u(t, X)$ *and* $v(t, X)$, *the inequality*

$$|u(t, X)-v(t, X)| \leqslant \eta \quad in \quad D,$$

provided that

$$|u(t_0, X)-v(t_0, X)| \leqslant \eta \quad for \quad X \in S_{t_0},$$

$$\left|\beta(t, X)[u(t, X)-v(t, X)]-a(t, X)\frac{d[u-v]}{dl}\right| \leqslant B\eta \quad for \quad (t, X) \in \Sigma_a,$$

$$|u(t, X)-v(t, X)| \leqslant \eta \quad for \quad (t, X) \in \Sigma-\Sigma_a.$$

Proof. All the assumptions of Theorem 49.1 are satisfied with $m = 1$, system (49.1) being identical to the above equation, and with $\sigma(t, v) \equiv 0$ and $\omega(t; \eta) \equiv \eta$.

EXAMPLE 49.1 (see [33]). Consider a system of almost linear equations

$$(49.10) \qquad u_t^i = \sum_{l,k=1}^{n} a_{lk}^i(X)u_{x_lx_k}^i + h^i(t, X, u^1, \ldots, u^m) \qquad (i = 1, 2, \ldots, m)$$

with $a_{lk}^i(X)$, $h^i(t, X, U)$ defined for $(t, X) \in D$ and U arbitrary, where D is a cylinder

$$D = (0, +\infty)\times G,$$

and G is a bounded region in the space $(x_1, \ldots, x_n)$. Suppose that for every i and $X \in G$ the quadratic form in $\lambda_1, \ldots, \lambda_n$

$$\sum_{l,k=1}^{n} a_{lk}^i(X)\lambda_l\lambda_k$$

is positive. Assume that for any positive h we have

$$(49.11) \qquad |h^i(t+h, X, U)-h^i(t, X, \tilde{U})| \leqslant M\sum_{j=1}^{m}|u^j-\tilde{u}^j| + Rh^a \qquad (i = 1, 2, \ldots, m).$$

where M and R are positive constants and $0 < a \leqslant 1$. Let $U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big)$ be a regular (see § 47) solution of system (49.10) in D, such that for every positive h we have

$$(49.12) \qquad |u^i(0, X)-u^i(h, X)| \leqslant Kh^\beta \quad \text{for} \quad X \in \bar{G} \quad (i = 1, 2, \ldots, m),$$

$$(49.13) \qquad |u^i(t+h, X)-u^i(t, X)| \leqslant Kh^\beta \quad \text{for} \quad (t, X) \in (0, +\infty)\times\partial G,$$

where K is a positive constant and $0 < \beta \leqslant 1$. Under these assumptions, for any positive h, inequalities

$$(49.14) \qquad |u^i(t+h, X) - u^i(t, X)| \leqslant Ke^{Mmt}h^\beta + \frac{Rh^\alpha}{Mm}(e^{Mmt}-1) \qquad (i = 1, 2, \ldots, m)$$

are satisfied in D.

Indeed, fix an $h > 0$ and put

$$(49.15) \qquad \begin{aligned} g^i(t, X, U, Q, R) &= \sum_{l,k=1}^{n} a^i_{lk}(X) r_{lk} + h^i(t+h, X, U) \qquad (i = 1, 2, \ldots, m), \\ v^i(t, X) &= u^i(t+h, X) \qquad (i = 1, 2, \ldots, m). \end{aligned}$$

Then $V(t, X) = (v^1(t, X), \ldots, v^m(t, X))$ is a regular (see § 47) solution of system (49.1) with g^i defined by formula (49.15). If we denote by $f^i(t, X, U, Q, R)$ the right-hand sides of system (49.10), then we can easily check that all the assumptions of Theorem 49.1 are satisfied with

$$\sigma_i(t, V) \equiv M\sum_{j=1}^{m} v_j + Rh^\alpha \qquad (i = 1, 2, \ldots, m),$$

$$\alpha^i(t, X) \equiv 0, \qquad \beta^i(t, X) \equiv 1, \qquad \eta_i = Kh^\beta \qquad (i = 1, 2, \ldots, m),$$

$$\omega_i(t; H) = Ke^{Mmt}h^\beta + \frac{Rh^\alpha}{Mm}(e^{Mmt}-1) \qquad (i = 1, 2, \ldots, m).$$

Therefore Theorem 49.1 yields inequalities (49.14).

The result just obtained may be summarized less precisely in the following form: if the functions $h^i(t, X, U)$ are Hölderian with respect to t and Lipschitzian with respect to U, then any regular solution of system (49.10) in D is Hölderian with respect to t in every bounded subdomain, provided that it be Hölderian with regard to t in the set $(0, +\infty) \times \partial G$ and for $t = 0$.

§ 50. Uniqueness criteria for the solution of the first mixed problem. We prove

THEOREM 50.1. *Let the right-hand members $f^i(t, X, U, Q, R)$ $(i = 1, 2, \ldots, m)$ of system* (46.7) *be defined for $(t, X) \in D$ of type C* (see § 33) *and for arbitrary U, Q, R. Assume that*

$$(50.1) \qquad [f^i(t, X, U, Q, R) - f^i(t, X, \widetilde{U}, Q, R)]\operatorname{sgn}(u^i - \widetilde{u}^i) \leqslant \sigma_i(t - t_0, |U - \widetilde{U}|) \qquad (i = 1, 2, \ldots, m),$$

where $\sigma_i(t, V)$ are the right-hand sides of a comparison system of type I (see § 14). *Suppose that*

$$\sigma_i(t, 0) \equiv 0 \qquad (i = 1, 2, \ldots, m)$$

and that

$$\Omega(t; 0) \equiv 0 \quad in \quad [0, +\infty), \tag{50.2}$$

where $\Omega(t; 0)$ *is the right-hand maximum solution of the comparison system through the origin in the interval* $[0, +\infty)$. *Let* $a^i(t, X)$, $l^i(t, X)$ $(i = 1, 2, \ldots, m)$ *satisfy Assumptions* A (see § 47) *and let* $\beta^i(t, X)$ *satisfy inequalities*

$$\beta^i(t; X) > 0 \ on \ \Sigma_{a^i} \quad (i = 1, 2, \ldots, m).$$

Under these assumptions the first mixed problem for system (46.7) *with initial conditions* (47.3) *and boundary conditions* (47.4) *admits at most one parabolic* (see § 46), Σ_a*-regular* (see § 47) *solution in* D.

Proof. Suppose that

$$U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big), \quad V(t, X) = \big(v^1(t, X), \ldots, v^m(t, X)\big)$$

are two such solutions. Then they satisfy all the assumptions of Theorem 49.1 with $g^i \equiv f^i$, $\eta_i = B^i = 0$ $(i = 1, 2, \ldots, m)$ and $a_0(0) = +\infty$. Therefore, we have

$$|U(t, X) - V(t, X)| \leqslant \Omega(t - t_0; 0)$$

in D and hence, by (50.2), it follows that

$$U(t, X) \equiv V(t, X)$$

in D, what was to be proved.

THEOREM 50.2. *Let the right-hand sides* $f^i(t, X, U, Q, R)$ $(i = 1, 2, \ldots, m)$ *of system* (46.7) *be defined for* $(t, X) \in D$ *of type* C (see § 33) *and for arbitrary* U, Q, R. *Assume that, for* $t > t_0$,

$$[f^i(t, X, U, Q, R) - f^i(t, X, \tilde{U}, Q, R)]\operatorname{sgn}(u^i - \tilde{u}^i) \leqslant \sigma(t - t_0, \max_l |u^l - \tilde{u}^l|), \tag{50.3}$$

where $\sigma(t, v)$ *is the right-hand side of a comparison equation of type* II (see § 14). *Let* $a^i(t, X)$, $l^i(t, X)$ $(i = 1, 2, \ldots, m)$ *satisfy Assumptions* A (see § 47) *and let* $\beta^i(t, X)$ *satisfy inequalities*

$$\beta^i(t, X) > 0 \ on \ \Sigma_{a^i} \quad (i = 1, 2, \ldots, m).$$

Under these assumptions the first mixed problem for system (46.7) *with initial conditions* (47.3) *and boundary conditions* (47.4) *admits at most one parabolic* (see § 46), Σ_a*-regular* (see § 47) *solution in* D.

Proof. Suppose that $U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big)$ and $V(t, X) = \big(v^1(t, X), \ldots, v^m(t, X)\big)$ are two such solutions. Like in Theorem 48.1 we assume, without loss of generality, that $t_0 = 0$. Then we have

$$U(0, X) = V(0, X) \quad \text{for} \quad X \in S_0, \tag{50.4}$$

and

$$\beta^i(t, X)[u^i(t, X) - v^i(t, X)] - \alpha^i(t, X)\frac{d[u^i - v^i]}{dl^i} = 0$$

(50.5)
$$\text{for} \quad (t, X) \in \Sigma_{\alpha^i},$$
$$u^i(t, X) - v^i(t, X) = 0 \quad \text{for} \quad (t, X) \in \Sigma - \Sigma_{\alpha^i}$$
$$(i = 1, 2, \ldots, m).$$

Put, for $0 \leqslant t < T$,

$$M^i(t) = \max_{X \in S_t} \big(u^i(t, X) - v^i(t, X)\big),$$
$$N^i(t) = \max_{X \in S_t} \big(v^i(t, X) - u^i(t, X)\big) \qquad (i = 1, 2, \ldots, m),$$
$$W(t) = \max_i \{\max_{X \in S_t} |u^i(t, X) - v^i(t, X)|\}.$$

The assertion of our theorem is equivalent with

(50.6) $$W(t) \equiv 0 \quad \text{for} \quad 0 \leqslant t < T.$$

Now, by Theorem 34.1, $W(t)$ is continuous in the interval $[0, T)$ and, by (50.4), we have

$$W(0) = 0.$$

Hence, by the second comparison theorem (see § 14), identity (50.6) will be proved if we show that the differential inequality

(50.7) $$D_- W(t) \leqslant \sigma\big(t, W(t)\big)$$

is satisfied in the set

$$E = \{t \in (0, T):\ W(t) > 0\}.$$

Let $\tilde{t} \in E$; then we have

(50.8) $$W(\tilde{t}) > 0.$$

By Theorem 34.1, there is an index j and a point $\tilde{X} \in S_{\tilde{t}}$ such that either

(50.9) $$W(\tilde{t}) = M^j(\tilde{t}) = u^j(\tilde{t}, \tilde{X}) - v^j(\tilde{t}, \tilde{X}), \qquad D_- W(\tilde{t}) \leqslant D^- M^j(\tilde{t}),$$

or

(50.10) $$W(\tilde{t}) = N^j(\tilde{t}) = v^j(\tilde{t}, \tilde{X}) - u^j(\tilde{t}, \tilde{X}), \qquad D_- W(\tilde{t}) \leqslant D^- N^j(\tilde{t}).$$

Suppose we have, for instance, (50.9); then, in view of (50.5), (50.8) and (50.9) we conclude, by Lemma 47.1, that $(\tilde{t}, \tilde{X})$ is an interior point of D. Hence, relations (49.8) and (49.9) hold true. By Theorem 33.1, 2°, we have

$$D^- M^j(\tilde{t}) \leqslant u^j_t(\tilde{t}, \tilde{X}) - v^j_t(\tilde{t}, \tilde{X}).$$

Therefore, proceeding further like in the proof of Theorem 49.1 and using (49.8) and (50.9) we get

$$D_-W(\tilde{t}) \leqslant [f^j(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), u^j_X(\tilde{t}, \tilde{X}), u^j_{XX}(\tilde{t}, \tilde{X})) - \\ - f^j(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), u^j_X(\tilde{t}, \tilde{X}), v^j_{XX}(\tilde{t}, \tilde{X}))] + \\ + [f^j(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), u^j_X(\tilde{t}, \tilde{X}), v^j_{XX}(\tilde{t}, \tilde{X})) - \\ - f^j(\tilde{t}, \tilde{X}, V(\tilde{t}, \tilde{X}), u^j_X(\tilde{t}, \tilde{X}), v^j_{XX}(\tilde{t}, \tilde{X}))] .$$

The first difference in the brackets is, by the parabolicity of solution $U(t, X)$ (see § 46) and by (49.9), non-positive. Since, by (50.8) and (50.9), we have

$$u^j(\tilde{t}, \tilde{X}) > v^j(\tilde{t}, \tilde{X}) ,$$

inequality (50.3) applied to the second difference in brackets yields

$$D_-W(\tilde{t}) \leqslant \sigma(\tilde{t}, \max_l |u^l(\tilde{t}, \tilde{X}) - v^l(\tilde{t}, \tilde{X})|) .$$

In view of the obvious relation (see (50.9))

$$W(\tilde{t}) = \max_l |u^l(\tilde{t}, \tilde{X}) - v^l(\tilde{t}, \tilde{X})| ,$$

the last inequality is equivalent with (50.7), which completes the proof.

Remark 50.1. The uniqueness criterion contained in Theorem 50.2 is more general than that of Theorem 50.1. This depends on the fact that the right-hand sides of a comparison system of type I (see § 14) are supposed to be continuous for $t = 0$, while the right-hand side of a comparison equation of type II is not. Thus, for instance, the uniqueness of the solution of the first mixed problem for the equation

$$u_t = |\ln(t - t_0)| u + h(t, X, u_X, u_{XX})$$

is a consequence of Theorem 50.2 (see Example 14.2, (γ)), whereas it is not one of Theorem 50.1.

Remark 50.2. It easily follows from the proof of Theorem 50.2 that if we knew that $W'_+(0) = 0$, then we would obtain a still more general uniqueness criterion with $\sigma(t, v)$ in (50.3) being the right-hand side of a comparison equation of type III (see § 14). But, to get relation $W'_+(0) = 0$, we would have to require that the solutions $U(t, X)$ and $V(t, X)$ satisfy system (46.7) for $t = 0$. Therefore, such a criterion would be useful only in particular cases since usually parabolic equations are not satisfied on the lower base of the domain D.

Remark 50.3. In the proofs of Theorems 48.1, 49.1, 50.1 and 50.2 we used, as an essential argument, the following very well known proposition: if a function $\varphi(X) = \varphi(x_1, \ldots, x_n)$ is of class C^2 in the neighborhood of the point X_0 and if it attains local maximum at that point, then

$$\varphi_X(X_0) = 0$$

and the quadratic form in $\lambda_1, \ldots, \lambda_n$

$$\sum_{l,k=1}^{n} \varphi_{x_l x_k}(X_0)\lambda_l\lambda_k$$

is negative. On the other hand, if the function $\varphi(X)$ were even of class C^∞, nothing could be inferred on the behavior of its higher derivatives at X_0 from the fact that it attains local extremum at X_0. This explains why general theorems of the types discussed in §§ 48-50 cannot be expected to hold true for equations of higher order than 2.

Remark 50.4. In the particular case, when the right-hand sides of system (46.7) and (49.1) respectively do not depend on second derivatives, Theorems 48.1, 49.1, 50.1 and 50.2 concern systems of first order partial differential equations. Now, the question arises how these theorems are related with analogous theorems of Chapter VII. In Chapter VII we have more restrictive assumptions on the domain D and on the regularity of the right-hand sides of system, viz. the domain D is a pyramid and the right-hand sides of the system satisfy a Lipschitz condition with regard to the first derivatives of unknown functions (the pyramid depending on the Lipschitz constant); on the other hand, in Chapter VIII we impose boundary conditions for the solution on the side surface of D which are superfluous in theorems of Chapter VII.

§ 51. Continuous dependence of the solution of the first mixed problem on initial and boundary values and on the right-hand sides of system. We now prove

THEOREM 51.1. *Let the right-hand sides* $f^i(t, X, U, Q, R)$ *and* $g^i(t, X, U, Q, R)$ $(i = 1, 2, \ldots, m)$ *of system* (46.7) *and* (49.1) *respectively be defined for* $(t, X) \in D$ *of type C with* $T < +\infty$ (see § 33) *and for arbitrary* U, Q, R. *Suppose* f^i *to satisfy assumptions of Theorem* 50.1. *Let* $\alpha^i(t, X)$, $l^i(t, X)$ $(i = 1, 2, \ldots, m)$ *satisfy Assumptions* A (see § 47) *and* $\beta^i(t, X)$ *inequalities*

$$\beta^i(t, X) > B^i > 0 \quad \textit{for} \quad (t, X) \in \Sigma_{\alpha^i} \quad (i = 1, 2, \ldots, m)\,.$$

Suppose finally that $U(t, X) = (u^1(t, X), \ldots, u^m(t, X))$ *is a parabolic* (see § 46), Σ_α*-regular* (see § 47) *solution of system* (46.7) *in* D *and* $V(t, X) = (v^1(t, X), \ldots, v^m(t, X))$ *is a* Σ_α*-regular solution of system* (49.1) *in* D.

Under these assumptions, to every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever we have

(51.1) $$|f^i(t, X, U, Q, R) - g^i(t, X, U, Q, R)| < \delta \quad (i = 1, 2, \ldots, m),$$

(51.2) $$|U(t_0, X) - V(t_0, X)| < \Delta \quad \textit{for} \quad X \in S_{t_0},$$

$$\left|\beta^i(t, X)[u^i(t, X) - v^i(t, X)] - \alpha^i(t, X)\frac{d[u^i - v^i]}{dl^i}\right| < \delta$$

$$\textit{for} \quad (t, X) \in \Sigma_{\alpha^i},$$

(51.3) $$|u^i(t, X) - v^i(t, X)| < \delta \quad \textit{for} \quad (t, X) \in \Sigma - \Sigma_{\alpha^i}$$

$$(i = 1, 2, \ldots, m),$$

where $\Delta = (\delta, \ldots, \delta)$, then inequality

(51.4) $$|U(t, X) - V(t, X)| < E$$

holds true in D, where $E = (\varepsilon, \ldots, \varepsilon)$.

Proof. In view of Theorem 10.1, to every $\varepsilon > 0$ there is a $\delta_1 > 0$ such that the right-hand maximum solution $\Omega(t; H, \delta_1)$ of the comparison system

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, \ldots, y_m) + \delta_1 \quad (i = 1, 2, \ldots, m)$$

(concerning $\sigma_i(t, Y)$ see the assumptions of Theorem 50.1), passing through $(0, H) = (0, \eta_1, \ldots, \eta_m)$, is defined in the interval $[0, T)$ and satisfies inequality

(51.5) $$\Omega(t; H, \delta_1) < E \quad \text{for} \quad 0 \leqslant t < T,$$

provided that

(51.6) $$0 \leqslant H \leqslant \Delta_1,$$

where $\Delta_1 = (\delta_1, \ldots, \delta_1)$. Let inequalities (51.1)-(51.3) hold true with

$$\delta = \min_i(\delta_1, B^i\delta_1) > 0;$$

then, by (51.2) and (51.3), inequalities (49.3) and (49.4) of Theorem 49.1 are satisfied with $\eta_i = \delta_1$ $(i = 1, 2, \ldots, m)$. On the other hand, by (50.1) and (51.1) we have

$$[f^i(t, X, U, Q, R) - g^i(t, X, \tilde{U}, Q, R)]\operatorname{sgn}(u^i - \tilde{u}^i) \leqslant \sigma_i(t - t_0, |U - \tilde{U}|) + \delta_1$$

$$(i = 1, 2, \ldots, m).$$

Hence, by Theorem 49.1, we get

(51.7) $$|U(t, X) - V(t, X)| \leqslant \Omega(t; \Delta_1, \delta_1) \quad \text{in} \quad D.$$

From (51.5) and (51.7) follows (51.4), what was to be proved.

§ 52. Stability of the solution of the first mixed problem. Let the right-hand sides of system (46.7) be defined for $(t, X) \in D$ of type C with $T = +\infty$ (see § 33) and for arbitrary U, Q, R, and satisfy identities

$$f^i(t, X, 0, 0, 0) \equiv 0 \quad (i = 1, 2, \ldots, m). \tag{52.1}$$

Let $a^i(t, X)$, $l^i(t, X)$ $(i = 1, 2, \ldots, m)$ satisfy Assumptions A (see § 47) and $\beta^i(t, X)$ inequalities

$$\beta^i(t, X) > B^i > 0 \quad \text{for} \quad (t, X) \in \Sigma_{a^i} \quad (i = 1, 2, \ldots, m).$$

Owing to assumption (52.1), $V(t, X) \equiv 0$ is a Σ_a-regular (see § 47) solution of the first mixed problem (47.3), (47.4), with $\Phi(X) \equiv \Psi(t, X) \equiv 0$, for system (46.7).

Definition of stability. Put $E = (\varepsilon, \ldots, \varepsilon)$ and $\Delta = (\delta, \ldots, \delta)$. We say (under the above hypotheses) that the null solution of system (46.7) is *stable* if to every $\varepsilon > 0$ there is a $\delta > 0$ such that for every Σ_a-regular (see § 47) and parabolic (see § 46) solution $U(t, X) = (u^1(t, X), \ldots, u^m(t, X))$ of system (46.7) in D we have

$$|U(t, X)| < E \quad \text{in} \quad D, \tag{52.2}$$

whenever

$$\left\{\begin{aligned} &|U(t_0, X)| < \Delta && \text{for} \quad X \in S_{t_0}, \\ &\left|\beta^i(t, X) u^i(t, X) - a^i(t, X) \frac{du^i}{dl^i}\right| < \delta && \text{for} \quad (t, X) \in \Sigma_{a^i}, \\ &|u^i(t, X)| < \delta && \text{for} \quad (t, X) \in \Sigma - \Sigma_{a^i} \\ & && \qquad (i = 1, 2, \ldots, m). \end{aligned}\right. \tag{52.3}$$

Now, we can prove the following

Theorem 52.1. *Under the assumptions introduced at the beginning of this paragraph suppose that*

$$f^i(t, X, U, 0, 0) \operatorname{sgn} u^i \leqslant \sigma_i(t - t_0, |U|) \quad (i = 1, 2, \ldots, m), \tag{52.4}$$

where $\sigma_i(t, V)$ are the right-hand sides of a comparison system of type I (see § 14). *Assume that*

$$\sigma_i(t, 0) \equiv 0 \quad (i = 1, 2, \ldots, m)$$

and that the null solution of the comparison system is stable (see [7], p. 314). *Then the null solution of system* (46.7) *is stable too.*

Proof. The null solution of the comparison system being stable, to $\varepsilon > 0$ there is a $\delta_1 > 0$ such that whenever

$$0 \leqslant H \leqslant \Delta_1 \quad (\Delta_1 = (\delta_1, \ldots, \delta_1)),$$

then

$$\Omega(t; H) < E \quad \text{for} \quad 0 \leqslant t < +\infty, \tag{52.5}$$

where $\Omega(t; H)$ is the right-hand maximum solution of the comparison system through $(0, H) = (0, \eta_1, \ldots, \eta_m)$. Put

$$\delta = \min_i (\delta_1, B^i \delta_1) > 0$$

and suppose that inequalities (52.3) hold true with the above δ. Then, by (52.3) and (52.4), all the assumptions of Theorem 48.1 are satisfied with $\eta_i = \delta_1$ $(i = 1, 2, \ldots, m)$ and $V(t, X) \equiv 0$. Hence, by Theorem 48.1, we get

$$|U(t, X)| \leqslant \Omega(t; \Delta_1) \quad \text{in} \quad D. \tag{52.6}$$

Inequality (52.2) follows now from (52.5) and (52.6).

EXAMPLE. Let the comparison system be a linear one of the form

$$\frac{dy_i}{dt} = \sum_{k=1}^{m} a_{ik}(t) y_k \qquad (i = 1, 2, \ldots, m), \tag{52.7}$$

where $a_{ik}(t) \geqslant 0$ are continuous for $t \geqslant 0$. Suppose that for

$$\varphi(t) = \max_{i,k} |a_{ik}(t)|$$

we have

$$\int_0^\infty \varphi(t) dt < +\infty.$$

It is well known that under these assumptions the null solution of system (52.7) is a stable one. Hence, if system (46.7) satisfies hypotheses of Theorem 52.1 with inequalities (52.4) of the form

$$f^i(t, X, U, 0, 0) \operatorname{sgn} u^i \leqslant \sum_{k=1}^{m} a_{ik}(t) |u^k|,$$

then the null solution of (46.7) is stable.

§ 53. Preliminary remarks and lemmas referring to the second mixed problem. We are going now to discuss the second mixed problem for systems of the form (46.7). We recall (see § 47) that the second mixed problem consists in determining a Σ_a-regular solution (see § 47) of (46.7) satisfying initial conditions (47.3) and boundary conditions (47.4), where $\beta^i(t, X)$ are functions which—unlike in the first mixed problem—are not supposed to be positive for $(t, X) \in \Sigma_{a^i}$. In order to get analogues of theorems concerning the first mixed problem, we will have to impose some more restrictive conditions on the right-hand sides of system (46.7) and, moreover, we will assume the existence of adequate sign-stabilizing factors. More precisely, we will suppose that there exist functions

$K^i(t, X)$ $(i = 1, 2, \ldots, m)$, such that new unknown functions defined by formulas

$$\tilde{u}^i(t, X) = \frac{u^i(t, X)}{K^i(t, X)} \qquad (i = 1, 2, \ldots, m)$$

satisfy boundary conditions (47.4) with new coefficients $\tilde{\beta}^i(t, X)$, which are positive for $(t, X) \in \Sigma_{a^i}$. In the case of one linear parabolic equation the introduction of the above sign-stabilizing factors is due to M. Krzyżański [18]. We will establish certain sufficient conditions referring to the domain D, the coefficients $a^i(t, X)$ and $\beta^i(t, X)$ and to the directions $l^i(t, X)$ which imply the existence of the above factors.

In what follows we suppose that a region D of type C (see § 33), directions $l^i(t, X)$, and functions $a^i(t, X)$, $\beta^i(t, X)$ $(i = 1, 2, \ldots, m)$ defined on the side surface Σ of D respectively on Σ_{a^i} are given, where $a^i(t, X)$, $l^i(t, X)$ satisfy Assumptions A (see § 47).

Let the functions $K^i(t, X)$ $(i = 1, 2, \ldots, m)$ be positive and of class C^2 in the closure of D and let $U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big)$ be Σ_a-regular (see § 47) in D. Under these assumptions we have the following easy to check

LEMMA 53.1. *Define* $\tilde{U}(t, X) = \big(\tilde{u}^1(t, X), \ldots, \tilde{u}^m(t, X)\big)$ *by the formulas*

$$(53.1) \qquad \tilde{u}^i(t, X) = u^i(t, X)[K^i(t, X)]^{-1} \qquad (i = 1, 2, \ldots, m);$$

then we have the following propositions:

1° $\beta^i u^i - a^i \dfrac{du^i}{dl^i} = K^i\left[\tilde{\beta}^i\tilde{u}^i - a^i\dfrac{d\tilde{u}^i}{dl^i}\right]$ *for* $(t, X) \in \Sigma_{a^i}$ $(i = 1, 2, \ldots, m)$, *where*

$$(53.2) \qquad \tilde{\beta}^i(t, X) = \beta^i(t, X) - a^i(t, X)[K^i(t, X)]^{-1}\frac{dK^i}{dl^i} \quad \textit{for} \quad (t, X) \in \Sigma_{a^i} \quad (i = 1, 2, \ldots, m).$$

2° *If* $U(t, X)$ *satisfies initial conditions* (47.3) *and boundary conditions* (47.4), *then*

$$(53.3) \qquad \tilde{u}^i(t_0, X) = \varphi^i(X)[K^i(t_0, X)]^{-1} \quad \textit{for} \quad X \in S_{t_0} \quad (i = 1, 2, \ldots, m),$$

and

$$(53.4) \qquad \begin{aligned} &\tilde{\beta}^i(t, X)\tilde{u}^i(t, X) - a^i(t, X)\frac{d\tilde{u}^i}{dl^i} = \psi^i(t, X)[K^i(t, X)]^{-1} && \textit{for} \quad (t, X) \in \Sigma_{a^i}, \\ &\tilde{u}^i(t, X) = \psi^i(t, X)[K^i(t, X)]^{-1} && \textit{for} \quad (t, X) \in \Sigma - \Sigma_{a^i} \\ & && (i = 1, 2, \ldots, m), \end{aligned}$$

where $\tilde{\beta}^i(t, X)$ *are given by formulas* (53.2).

The above lemma justifies the following definition.

DEFINITION OF SIGN-STABILIZING FACTORS. Functions $K^i(t, X)$ $(i = 1, 2, \ldots, m)$, which are positive and of class C^2 in the closure of D, will be called *sign-stabilizing factors* if there exist constants B^i $(i = 1, 2, \ldots, m)$ such that

$$\widetilde{\beta}^i(t, X) > B^i \geqslant 0 \quad \text{for} \quad (t, X) \in \Sigma_{a^i} \quad (i = 1, 2, \ldots, m),$$

where $\widetilde{\beta}^i(t, X)$ are defined by formulas (53.2).

Remark 53.1. The existence of sign-stabilizing factors is trivial if we assume that for the original coefficients $\beta^i(t, X)$ we have

$$\beta^i(t, X) > B^i \geqslant 0 \quad \text{for} \quad (t, X) \in \Sigma_{a^i} \quad (i = 1, 2, \ldots, m).$$

Indeed, in that case $K^i(t, X) \equiv 1$ $(i = 1, 2, \ldots, m)$ are obviously sign-stabilizing factors. On the other hand, we will see in § 54 that sign-stabilizing factors may exist also in the case when $\beta^i(t, X)$ take on values which are non-positive. Hence, it follows that the existence of sign-stabilizing factors is an essentially less restrective condition imposed on $\beta^i(t, X)$ than the above inequalities, and that sign-stabilizing factors can be of service in the treatment of the second mixed problem.

Next we state, without proofs, three easy to check lemmas.

LEMMA 53.2. *If* $U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big)$ *is a* Σ_a*-regular* (see § 47) *and parabolic* (see § 46) *solution of system* (46.7) *in* D, *then* $\widetilde{U}(t, X) = \big(\widetilde{u}^1(t, X), \ldots, \widetilde{u}^m(t, X)\big)$ *defined by* (53.1) *is a* Σ_a*-regular and parabolic solution of the transformed system*

$$z^i_t = \widetilde{f}^i(t, X, Z, z^i_X, z^i_{XX}) \quad (i = 1, 2, \ldots, m), \tag{53.5}$$

where

$$\begin{aligned}(53.6)\quad & \widetilde{f}^i(t, X, Z, Q, R) \\ &= [K^i(t, X)]^{-1}\big[f^i\big(t, X, z^1K^1(t, X), \ldots, z^mK^m(t, X), QK^i(t, X) + \\ &\qquad + z^iK^i_X(t, X), \ldots, r_{jk}K^i(t, X) + q_jK^i_{x_k}(t, X) + q_kK^i_{x_j}(t, X) + \\ &\qquad\qquad + z^iK^i_{x_jx_k}(t, X), \ldots\big) - z^iK^i_t(t, X)\big] \quad (i = 1, 2, \ldots, m).\end{aligned}$$

LEMMA 53.3. *Let the functions* $K^i(t, X)$ $(i = 1, 2, \ldots, m)$ *be of class* C^2 *in the closure of* D *and satisfy inequalities*

$$0 < \mu \leqslant K^i(t, X) \leqslant \widetilde{M}, \quad |K^i_t|, |K^i_{x_j}|, |K^i_{x_jx_k}| \leqslant \widetilde{M}; \tag{53.7}$$

put

$$M = n(n+1)\widetilde{M}.$$

Suppose the functions $\sigma_i(t, y_1, \ldots, y_m)$, $\tau_i(t, y)$ $(i = 1, 2, \ldots, m)$ *to be continuous, non-negative and increasing in all variables for* $t \geqslant 0$, $y \geqslant 0$,

$y_i \geqslant 0$ $(i = 1, 2, \ldots, m)$. *Assume finally that the right-hand sides of systems* (46.7) *and* (49.1) *satisfy inequalities*

$$(53.8) \quad [f^i(t, X, U, Q, R) - g^i(t, X, \bar{U}, \bar{Q}, \bar{R})]\operatorname{sgn}(u^i - \bar{u}^i) \leqslant \sigma_i(t - t_0, |U - \bar{U}|) + \tau_i\Big(t - t_0, \sum_j |q_j - \bar{q}_j| + \sum_{j,k} |r_{jk} - \bar{r}_{jk}|\Big) \quad (i = 1, 2, \ldots, m).$$

Under these assumptions the right-hand sides of the transformed system (53.5) *and of the system*

$$(53.9) \qquad z_t^i = \tilde{g}^i(t, X, Z, z_X^i, z_{XX}^i) \quad (i = 1, 2, \ldots, m),$$

obtained by transformation (53.6) *from system* (49.1), *satisfy inequalities*

$$(53.10) \quad [\tilde{f}^i(t, X, U, Q, R) - \tilde{g}^i(t, X, \bar{U}, Q, R)]\operatorname{sgn}(u^i - \bar{u}^i) \leqslant \tilde{\sigma}_i(t - t_0, |U - \bar{U}|) \quad (i = 1, 2, \ldots, m),$$

where

$$(53.11) \quad \tilde{\sigma}^i(t, y_1, \ldots, y_m) = \frac{1}{\mu}\left[\sigma_i\Big(\frac{M}{\mu}t, My_1, \ldots, My_m\Big) + \tau_i\Big(\frac{M}{\mu}t, My_i\Big) + My_i\right] \quad (i = 1, 2, \ldots, m).$$

LEMMA 53.4. *Let* $\sigma_i(t, y_1, \ldots, y_m)$ *and* $\tau_i(t, y)$ $(i = 1, 2, \ldots, m)$ *satisfy assumptions of Lemma* 53.3 *and define* $\tilde{\sigma}_i(t, y_1, \ldots, y_m)$ *by formula* (53.11). *Consider two systems of ordinary differential equations*

$$(53.12) \qquad \frac{dy_i}{dt} = \sigma_i(t, y_1, \ldots, y_m) + \tau_i(t, y_i) + y_i \quad (i = 1, 2, \ldots, m)$$

and

$$(53.13) \qquad \frac{dy_i}{dt} = \tilde{\sigma}_i(t, y_1, \ldots, y_m) \quad (i = 1, 2, \ldots, m).$$

Under the above assumptions we have the following propositions:

1° *Both systems are comparison systems of type* I (see § 14).

2° *If* $\Omega(t; H)$ *is the right-hand maximum solution of system* (53.12) *through* $(0, H) = (0, \eta_1, \ldots, \eta_m)$ *defined on* $[0, +\infty)$, *then*

$$(53.14) \qquad \tilde{\Omega}(t; H) = \frac{1}{M}\Omega\Big(\frac{M}{\mu}t; M\eta_1, \ldots, M\eta_m\Big)$$

is the right-hand maximum solution of system (53.13) *through* $(0, H)$ *defined on* $[0, +\infty)$.

§ 54. Sufficient conditions for the existence of sign-stabilizing factors. It is important to know whether the domain D, the functions $\alpha^i(t, X)$, $\beta^i(t, X)$ and the directions $l^i(t, X)$ being given the existence of sign-stabilizing factors $K^i(t, X)$ (see § 53), satisfying inequalities (53.7), is guaranteed.

We will consider a particular case when the construction of sign-stabilizing factors can be easily achieved. Let D be a cylinder whose axis is parallel to the t-axis and whose basis is a bounded domain G in the plane $t = 0$. Assume the boundary ∂G of G to be a surface given by the equation $G(X) = 0$, where $G(X)$ is of class C^2 in the closure of G. Suppose that

$$|G(X)|, |G_{x_j}(X)|, |G_{x_j x_k}(X)| \leqslant N \quad \text{for} \quad X \in \bar{G},$$
$$\operatorname{grad}^2 G(X) > 0 \quad \text{for} \quad X \in \partial G.$$

Let $\alpha^i(t, X) \equiv 1$ and $\beta^i(t, X) \geqslant b^i$ $(i = 1, 2, \ldots, m)$, where b^i are some negative constants. Assume finally the directions $l^i(t, X)$ to be chosen so that

$$\sum_{j=1}^{m} G_{x_j}(X)\cos\bigl(l^i(t, X), x_j\bigr) \geqslant \Gamma^i > 0 \quad \text{for} \quad (t, X) \in \Sigma \quad (i = 1, 2, \ldots, m).$$

A simple computation shows that under these assumptions the functions

$$K^i(t, X) = e^{-\gamma G(X)} \quad (i = 1, 2, \ldots, m),$$

where

$$\gamma = \max_i \left(\frac{1 - b^i}{\Gamma^i}\right),$$

are sign-stabilizing factors with $B^i = 1$ $(i = 1, 2, \ldots, m)$, satisfying inequalities (53.7) with

$$\mu = e^{-\gamma N}, \quad \widetilde{M} = e^{\gamma N}(\gamma N + 1)^2.$$

§ 55. Analogues of theorems in §§ 48-52 in case of the second mixed problem. Using lemmas of the preceding section we will derive from theorems contained in §§ 48-52 the following results for the second mixed problem: estimates of the solution, estimates of the difference between two solutions, uniqueness criteria, continuous dependence of the solution on initial and boundary values and on the right-hand sides of system and, finally, a stability criterion.

In what follows we will assume, without stating it explicitly in each theorem that

(α) the right-hand sides of systems to be considered are defined for $(t, X) \in D$ of type C (see § 33) and for arbitrary U, Q, R,

(β) functions $\alpha^i(t, X)$ and directions $l^i(t, X)$ $(i = 1, 2, \ldots, m)$ satisfying Assumptions A (see § 47) are given on the side surface Σ of D, as well as functions $\beta^i(t, X)$ on Σ_{α^i} $(i = 1, 2, \ldots, m)$.

THEOREM 55.1. *Suppose that the right-hand sides of system* (46.7) *satisfy inequalities*

$$(55.1)\qquad f^i(t, X, U, Q, R)\operatorname{sgn} u^i \leqslant \sigma_i(t-t_0, |U|) + \tau_i\Big(t-t_0, \sum_j |q_j| + \sum_{j,k} |r_{jk}|\Big) \quad (i = 1, 2, \ldots, m),$$

where $\sigma_i(t, y_1, \ldots, y_m)$ *and* $\tau_i(t, y)$ *are continuous, non-negative and increasing in all variables for* $t \geqslant 0$, $y \geqslant 0$, $y_i \geqslant 0$ $(i = 1, 2, \ldots, m)$. *Denote by* $\Omega(t; H) = \big(\omega_1(t; H), \ldots, \omega_m(t; H)\big)$ *the right-hand maximum solution of system* (53.12) *through* $(0, H) = (0, \eta_1, \ldots, \eta_m)$ *and assume it to be defined on* $[0, +\infty)$. *Suppose there exist sign-stabilizing factors* (see § 53) $K^i(t, X)$ $(i = 1, 2, \ldots, m)$ *satisfying inequalities*

$$(55.2)\qquad 0 < \mu \leqslant K^i(t, X) \leqslant \widetilde{M}, \quad |K_t^i|, |K_{x_j}^i|, |K_{x_j x_k}^i| \leqslant \widetilde{M} \quad (i = 1, 2, \ldots, m;\ j, k = 1, 2, \ldots, n)$$

and some constants B^i *such that*

$$(55.3)\qquad \widetilde{\beta}^i(t, X) > B^i \geqslant 0 \quad \textit{for} \quad (t, X) \in \Sigma_{\alpha^i} \quad (i = 1, 2, \ldots, m),$$

where

$$(55.4)\qquad \widetilde{\beta}^i(t, X) = \beta^i(t, X) - \alpha^i(t, X)[K^i(t, X)]^{-1}\frac{dK^i}{dl^i} \quad \textit{for} \quad (t, X) \in \Sigma_{\alpha^i} \quad (i = 1, 2, \ldots, m).$$

Let $U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big)$ *be a parabolic* (see § 46), Σ_α*-regular* (see § 47) *solution of system* (46.7) *in* D, *satisfying initial inequality*

$$(55.5)\qquad |U(t_0, X)| \leqslant H \quad \textit{for} \quad X \in S_{t_0}$$

and boundary inequalities

$$(55.6)\qquad \begin{aligned} &\left|\beta^i(t, X)u^i(t, X) - \alpha^i(t, X)\frac{du^i}{dl^i}\right| \leqslant B^i\frac{\mu}{M}\omega_i\Big(\frac{M}{\mu}(t-t_0); \frac{M}{\mu}H\Big) \quad \textit{for} \quad (t, X) \in \Sigma_{\alpha^i},\\ &|u^i(t, X)| \leqslant \frac{\mu}{M}\omega_i\Big(\frac{M}{\mu}(t-t_0); \frac{M}{\mu}H\Big) \quad \textit{for} \quad (t, X) \in \Sigma - \Sigma_{\alpha^i} \quad (i = 1, 2, \ldots, m),\end{aligned}$$

where $M = n(n+1)\widetilde{M}$.

Under the above assumptions we have in D

$$(55.7)\qquad |U(t, X)| \leqslant \Omega\Big(\frac{M}{\mu}(t-t_0); \frac{M}{\mu}H\Big).$$

Proof. Put

$$(55.8)\qquad \widetilde{u}^i(t, X) = u^i(t, X)[K^i(t, X)]^{-1} \quad (i = 1, 2, \ldots, m).$$

By Lemma 53.2, $\tilde{U}(t, X) = (\tilde{u}^1(t, X), \ldots, \tilde{u}^m(t, X))$ is a Σ_a-regular and parabolic solution of the transformed system (53.5) and, by Lemma 53.1, inequalities (55.2), (55.5) and (55.6) imply

$$|\tilde{U}(t_0, X)| \leqslant \frac{H}{\mu} \quad \text{for} \quad X \in S_{t_0}, \tag{55.9}$$

and

$$\left|\tilde{\beta}^i(t, X)\tilde{u}^i(t, X) - a^i(t, X)\frac{d\tilde{u}^i}{dl^i}\right| \leqslant B^i \frac{1}{M}\omega_i\left(\frac{M}{\mu}(t-t_0); \frac{M}{\mu}H\right) \quad \text{for} \quad (t, X) \in \Sigma_{a^i},$$

(55.10)

$$|\tilde{u}^i(t, X)| \leqslant \frac{1}{M}\omega_i\left(\frac{M}{\mu}(t-t_0); \frac{M}{\mu}H\right) \quad \text{for} \quad (t, X) \in \Sigma - \Sigma_{a^i} \quad (i = 1, 2, \ldots, m),$$

where $\tilde{\beta}^i(t, X)$ are given by formula (55.4). From (53.6), (55.1) and (55.2) it follows that the right-hand sides of the transformed system (53.5) satisfy inequalities

$$\tilde{f}^i(t, X, U, 0, 0)\operatorname{sgn} u^i \leqslant \tilde{\sigma}_i(t-t_0, |U|) \quad (i = 1, 2, \ldots, m), \tag{55.11}$$

where

$$\tilde{\sigma}_i(t, y_1, \ldots, y_m) = \frac{1}{\mu}\left[\sigma_i\left(\frac{M}{\mu}t, My_1, \ldots, My_m\right) + \tau_i\left(\frac{M}{\mu}t, My_i\right) + My_i\right] \tag{55.12}$$
$$(i = 1, 2, \ldots, m).$$

From (55.3), (55.9), (55.10) and (55.11) we infer that for the transformed system (53.5) and its solution $\tilde{U}(t, X)$ all the hypotheses of Theorem 48.1 are satisfied. Hence, we have in D

$$|\tilde{U}(t, X)| \leqslant \tilde{\Omega}\left(t-t_0; \frac{H}{\mu}\right), \tag{55.13}$$

where $\tilde{\Omega}(t; H)$ is the right-hand maximum solution of system (53.13) through $(0, H)$. But, by Lemma 53.4, we have, for $0 \leqslant t < +\infty$,

$$\tilde{\Omega}(t; H) = \frac{1}{M}\Omega\left(\frac{M}{\mu}t; MH\right). \tag{55.14}$$

Relations (55.2), (55.8), (55.13) and (55.14) imply inequalities (55.7) in D, what completes the proof.

THEOREM 55.2. *Let the right-hand members of systems* (46.7) *and* (49.1) *satisfy inequalities*

$$[f^i(t, X, U, Q, R) - g^i(t, X, \bar{U}, \bar{Q}, \bar{R})]\operatorname{sgn}(u^i - \bar{u}^i)$$
$$\leqslant \sigma_i(t-t_0, |U-\bar{U}|) + \tau_i\left(t-t_0, \sum_j |q_j - \bar{q}_j| + \sum_{j,k} |r_{jk} - \bar{r}_{jk}|\right),$$

where $\sigma_i(t, Y)$ and $\tau_i(t, y)$ satisfy assumptions of Theorem 55.1. *Suppose there exist sign-stabilizing factors* (see § 53) *satisfying inequalities* (55.2) *and constants B^i, such that inequalities* (55.3), *with $\tilde{\beta}^i(t, X)$ defined by* (55.4), *hold true. Assume that* $U(t,X)=\big(u^1(t,X),\dots,u^m(t,X)\big)$ *is a parabolic* (see § 46), Σ_a*-regular* (see § 47) *solution of system* (46.7) *in* D *and* $V(t,X)=\big(v^1(t,X),\dots,v^m(t,X)\big)$ *is a* Σ_a*-regular solution of system* (49.1) *in* D, *their difference satisfying initial inequalities* (55.5) *and boundary inequalities* (55.6).

Under these assumptions the inequality

$$|U(t,X)-V(t,X)|\leqslant \Omega\Big(\frac{M}{\mu}(t-t_0);\ \frac{M}{\mu}H\Big)$$

holds true in D, *where* $\Omega(t;H)$ *is the right-hand maximum solution of system* (53.12) *through* $(0,H)=(0,\eta_1,\dots,\eta_m)$.

Proof. Proceeding like in the proof of Theorem 55.1, we put (55.8) and

$$(55.15)\qquad \tilde v^i(t,X)=v^i(t,X)[K^i(t,X)]^{-1}\qquad (i=1,2,\dots,m)$$

and we check (using Lemmas 53.1-53.3) that for the transformed systems (53.5) and (53.9) and their solutions $\tilde U(t,X)$ and $\tilde V(t,X)$ all the assumptions of Theorem 49.1 are satisfied. Hence, applying Theorem 49.1 and using Lemma 53.4, we get the assertion of our theorem.

THEOREM 55.3. *Let the right-hand sides of system* (46.7) *satisfy the inequalities*

$$[f^i(t,X,U,Q,R)-f^i(t,X,\bar U,\bar Q,\bar R)]\operatorname{sgn}(u^i-\bar u^i)$$
$$\leqslant \sigma_i(t-t_0,|U-\bar U|)+\tau_i\Big(t-t_0,\sum_j|q_j-\bar q_j|+\sum_{j,k}|r_{jk}-\bar r_{jk}|\Big)$$
$$(i=1,2,\dots,m),$$

where $\sigma_i(t,Y)$, $\tau_i(t,y)$ *satisfy assumptions of Theorem* 55.1. *Suppose that*

$$\sigma_i(t,0)\equiv\tau_i(t,0)\equiv 0\qquad (i=1,2,\dots,m)$$

and that

$$\Omega(t;0)\equiv 0\quad \text{in}\quad [0,+\infty),$$

where $\Omega(t;0)$ *is the right-hand maximum solution of system* (53.12), *issued from the origin. Assume, finally, there exist sign-stabilizing factors* (see § 53), *satisfying inequalities* (55.2), *and constants* B^i *such that inequalities* (55.3) *hold true.*

Under these assumptions the second mixed problem for system (46.7) *with initial conditions* (47.3) *and boundary conditions* (47.4) *admits at most one parabolic* (see § 46), Σ_a*-regular* (see § 47) *solution in* D.

Proof. Since two solutions of the problem satisfy assumptions of Theorem 55.2 with $f^i\equiv g^i$ and $\eta_i=B^i=0$, our theorem follows from Theorem 55.2.

THEOREM 55.4. *Assume the right-hand sides of system* (46.7) *to satisfy the inequalities*

$$(55.16)\quad [f^i(t, X, U, Q, R)-f^i(t, X, \bar{U}, \bar{Q}, \bar{R})]\operatorname{sgn}(u^i-\bar{u}^i) \leqslant \sigma(t-t_0, \max_j |u^j-\bar{u}^j|)+\tau\Big(t-t_0, \sum_j |q_j-\bar{q}_j|+\sum_{j,k} |r_{jk}-\bar{r}_{jk}|\Big)$$
$$\text{for}\quad t > t_0 \quad (i = 1, 2, \ldots, m),$$

where $\sigma(t, y)$ and $\tau(t, y)$ are continuous, non-negative and increasing in all variables for $t > 0$, $y \geqslant 0$. Suppose that

$$(55.17)\qquad \frac{dy}{dt} = \sigma(t, y)+\tau(t, y)+y$$

is a comparison equation of type II (see § 14). *Assume, finally, there exist sign-stabilizing factors* (see § 53), *satisfying inequalities* (55.2), *and constants B^i, such that inequalities* (55.3) *hold true.*

Under these assumptions the second mixed problem for system (46.7) *with initial conditions* (47.3) *and boundary conditions* (47.4) *admits at most one parabolic* (see § 46), *Σ_a-regular* (see § 47) *solution in D.*

Proof. It is obvious that it suffices to prove uniqueness of the corresponding problem for the transformed system (53.6) obtained from the given system (46.7) by the mapping (55.8). Now, in view of (55.16), it is easy to check that the right-hand sides of the transformed system satisfy the inequalities

$$[\tilde{f}^i(t, X, U, Q, R)-\tilde{f}^i(t, X, \bar{U}, \bar{Q}, \bar{R})]\operatorname{sgn}(u^i-\bar{u}^i) \leqslant \tilde{\sigma}(t-t_0, \max_j |u^j-\bar{u}^j|) \quad \text{for} \quad t > t_0 \quad (i = 1, 2, \ldots, m),$$

where

$$\tilde{\sigma}(t, y) = \frac{1}{\mu}\left[\sigma\Big(\frac{M}{\mu}t, My\Big)+\tau\Big(\frac{M}{\mu}t, My\Big)+My\right].$$

Equation (55.17) being a comparison one of type II it is not difficult to check that the same is true for the equation

$$\frac{dy}{dt} = \tilde{\sigma}(t, y).$$

The above remarks and inequalities (55.3) imply that for the transformed system (53.5) and the transformed initial and boundary conditions (53.3) and (53.4) all the assumptions of Theorem 50.2 are satisfied. This completes the proof.

THEOREM 55.5. *Let the right-hand sides of system* (46.7) *satisfy assumptions of Theorem* 55.3. *Assume there exist sign-stabilizing factors* (see § 53), *satisfying inequalities* (55.2), *and constants B^i such that inequalities*

$$(55.18)\quad \tilde{\beta}^i(t, X) > B^i > 0 \quad \text{for} \quad (t, X) \in \Sigma_{a^i} \quad (i = 1, 2, \ldots, m)$$

hold true.

Under these assumptions the parabolic and Σ_a-regular solution of the second mixed problem for system (46.7) *depends continuously* (*in the sense specified in Theorem* 51.1) *on initial and boundary values and on the right-hand sides of system.*

Proof. Applying our standard procedure we check that for the transformed problem obtained from the original one by the mapping (55.8) all the hypotheses of Theorem 51.1 are satisfied. Thus, our theorem follows from Theorem 51.1.

In a similar way, from Theorem 52.1 we derive the following

THEOREM 55.6. *Let the right-hand sides of system* (46.7) *satisfy inequalities*

$$f^i(t, X, U, Q, R)\operatorname{sgn} u^i \leqslant \sigma_i(t-t_0, |U|)+\tau_i\Big(t-t_0, \sum_j |q_j|+\sum_{j,k}|r_{jk}|\Big) \qquad (i = 1, 2, \ldots, m),$$

where $\sigma_i(t, Y)$ and $\tau_i(t, y)$ satisfy assumptions of Theorem 55.1. *Suppose that*

$$f^i(t, X, 0, 0, 0) \equiv \sigma_i(t, 0) \equiv \tau_i(t, 0) \equiv 0 \qquad (i = 1, 2, \ldots, m)$$

and that the null solution of system

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, \ldots, y_m)+\tau_i(t, y_i)+y_i \qquad (i = 1, 2, \ldots, m)$$

is stable. Assume the existence of sign-stabilizing factors (see § 53), *satisfying inequalities* (55.2) *and such that inequalities* (55.18) *hold true. This being assumed the null solution of system* (46.7) *is stable* (for the definition of stability, see § 52).

§ 56. Energy estimates for solutions of hyperbolic equations. In this section we consider a system of linear equations of the form

$$(56.1) \qquad H^i[u^i] \equiv \sum_{j,k=1}^{n} a^i_{jk}(X)u^i_{x_jx_k} = \sum_{l=1}^{m}\sum_{j=1}^{n} b^{il}_j(X)u^l_{x_j}+\sum_{l=1}^{m} c^{il}(X)u^l+f^i(X) \qquad (i = 1, 2, \ldots, m),$$

where the ith equation involves second derivatives of u^i only and $a^i_{jk} = a^i_{kj}$. The coefficients of equations (56.1) are supposed to be defined in a region D. Before we define D more precisely, we recall the following notions.

The differential operator $H^i[u]$ is called *hyperbolic* at a point $X \in D$ if $n-1$ eigenvalues of the matrix $\big(a^i_{jk}(X)\big)_{j,k=1,\ldots,n}$ are positive and one is negative.

Let $G(X)$ be of class C^1 in the neighborhood of a point $X_0 \in D$ and suppose that $\mathrm{grad}^2 G(X) > 0$ and $G(X_0) = 0$. Let us write

$$A^i[G] = \sum_{j,k=1}^{n} a^i_{jk}(X) G_{x_j}(X) G_{x_k}(X) . \tag{56.2}$$

The operator H^i being hyperbolic at the point X_0 we say that the orientation with respect to H^i of the surface Σ defined by the equation $G(X) = 0$ is at the point X_0:

(α) *characteristic if* $A^i[G]_{x=x_0} = 0$,

(β) *space-like if* $A^i[G]_{x=x_0} < 0$,

(γ) *time-like if* $A^i[G]_{x=x_0} > 0$.

We introduce now following assumptions concerning the region D in the space $(x_1, \ldots, x_n)$ and the coefficients of system (56.1).

ASSUMPTIONS B. (a) *D is open, contained in the zone $0 < x_n < b < +\infty$, and the intersection of D with any closed zone $0 \leqslant t \leqslant x_n \leqslant t+h < b$ is non-empty and bounded.*

(b) *Π_t denoting the intersection of $\bar{D}$ with the plane $x_n = t$ and $\psi(X)$ being an arbitrary continuous function in $\bar{D}$, the function*

$$\varphi(t) = \iint_{\Pi_t} \psi(x_1, \ldots, x_n) d\sigma \text{ (1)}$$

is continuous on $[0, b)$.

(c) *$a^i_{jk}(X)$ are of class C^1, $b^{il}_j(X)$, $c^{il}(X)$ and $f^i(X)$ are bounded and integrable in $\bar{D}$ and*

$$\mu \sum_{r=1}^{n} \lambda_r^2 \leqslant \sum_{j,k=1}^{n-1} a^i_{jk}(X) \lambda_j \lambda_k - a^i_{nn}(X) \lambda_n^2 \leqslant M \sum_{r=1}^{n} \lambda_r^2 \qquad (i = 1, 2, \ldots, m) \tag{56.3}$$

for $X \in \bar{D}$ and arbitrary $\lambda_1, \ldots, \lambda_n$, where M and μ are positive constants (2).

(d) *The side surface Σ of D, i.e. that part of the boundary of D which is contained in the open zone $0 < x_n < b$, is composed of two $(n-1)$-dimensional surfaces Σ^S and Σ^T (one of them may be empty).*

(e) *Σ^S is the union of a finite number of surfaces of class C^1 whose orientation, with respect to every operator H^i, is characteristic or space-like at every point; moreover, we have*

$$\cos(\bar{n}, x_n) < 0 \quad on \quad \Sigma^S,$$

where $\bar{n}$ denotes the interior orthogonal direction.

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(1) $\int ds$, $\iint d\sigma$, $\iiint dv$ denote $(n-2)$-dimensional, $(n-1)$-dimensional and n-dimensional integrals respectively.

(2) It is easy to check that the left-hand inequality (56.3) implies hyperbolicity of the operator H^i.

(f) *Σ^T is the union of a finite number of surfaces of class C^1 whose orientation, with respect to every operator H^i, is time-like at each point and*

$$\cos(\overline{n}, x_n) > 0 \quad on \quad \Sigma^T;$$

moreover, Σ_t^T denoting the intersection of Σ^T with the plane $x_n = t$ and $\psi(X)$ being an arbitrary continuous function in $\overline{D}$, the function

$$\varkappa(t) = \int_{\Sigma_t^T} \psi(x_1, \ldots, x_n)\,ds$$

is continuous on $[0, b)$.

THEOREM 56.1. *Suppose the Assumptions* B *to hold true, and let the functions $u^i(X) = u^i(x_1, \ldots, x_n)$ $(i = 1, 2, \ldots, m)$ be of class C^2 in D and of class C^1 in the closure of D. Assume $U(X) = (u^1(X), \ldots, u^m(X))$ to satisfy system* (56.1) *in D. For $0 \leqslant t < b$, put*

$$E(t) = \iint_{\Pi_t} \sum_{i=1}^{m} \Big[\sum_{j,k=1}^{n-1} a_{jk}^i u_{x_j}^i u_{x_k}^i - a_{nn}^i (u_{x_n}^i)^2 + (u^i)^2 \Big] d\sigma .$$

Under the above assumptions we have in the interval $[0, b)$

$$D^+E(t) \leqslant LE(t) + g(t)\,, \tag{56.4}$$

where

$$g(t) = \int_{\Sigma_t^T} \sum_{i=1}^{m} \Big[\sum_{j,k=1}^{n-1} a_{jk}^i u_{y_j}^i u_{y_k}^i + (u^i)^2 \Big] \cos(\overline{n}, x_n)\,ds + \iint_{\Pi_t} \sum_{i=1}^{m} (f^i)^2 d\sigma\,, \tag{56.5}$$

and $(y_1, \ldots, y_{n-1})$ are suitably chosen local coordinates on Σ^T; L is a positive constant depending on μ (see (56.3)) *and on the bounds of coefficients b_j^{il}, c^{il} and of the first derivatives of a_{jk}^i, but independent of the solution $U(X)$.*

Proof. It can easily be checked that

$$2H^i[u^i]u_{x_n}^i = 2 \sum_{j,k=1}^{n} a_{jk}^i u_{x_j x_k}^i u_{x_n}^i = 2 \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} (a_{jk}^i u_{x_k}^i u_{x_n}^i) -$$
$$- \sum_{j,k=1}^{n} \frac{\partial}{\partial x_n} (a_{jk}^i u_{x_j}^i u_{x_k}^i) - 2 \sum_{j,k=1}^{n} \frac{\partial a_{jk}^i}{\partial x_j} u_{x_k}^i u_{x_n}^i + \sum_{j,k=1}^{n} \frac{\partial a_{jk}^i}{\partial x_n} u_{x_j}^i u_{x_k}^i .$$

Hence multiplying the equation

$$H^i[u^i] = \sum_{l=1}^{m} \sum_{j=1}^{n} b_j^{il} u_{x_j}^l + \sum_{l=1}^{m} c^{il} u^l + f^i$$

by $2u^i_{x_n}$ we obtain in D the identity

$$(56.6)\qquad 2\sum_{j,k=1}^{n}\frac{\partial}{\partial x_j}(a^i_{jk}u^i_{x_k}u^i_{x_n})-\sum_{j,k=1}^{n}\frac{\partial}{\partial x_n}(a^i_{jk}u^i_{x_j}u^i_{x_k})\equiv 2f^i u^i_{x_n}+F^i_1[u]\,,$$

where F^i_1 is a quadratic form in $u^1, \ldots, u^m$ and their first derivatives. The coefficients of F^i_1 are polynomials of b^{il}_j, c^{il} and of the first derivatives of a^i_{jk}.

For $0 \leqslant t < b$ and $h > 0$ and for any set E in the space $(x_1, \ldots, x_n)$, let us denote by $E_{t,h}$ the intersection of E with the zone $t \leqslant x_n \leqslant t+h$.

Integrating identity (56.6) over the region $D_{t,h}$ and applying Green-Gauss theorem we get

$$(56.7)\qquad \iint\limits_{\partial D_{t,h}}\Big[2\sum_{j,k=1}^{n}a^i_{jk}u^i_{x_k}u^i_{x_n}\cos(\bar{n}, x_j)-\sum_{j,k=1}^{n}a^i_{jk}u^i_{x_j}u^i_{x_k}\cos(\bar{n}, x_n)\Big]d\sigma$$
$$= -\iiint\limits_{D_{t,h}}(F^i_1[u]+2f^i u^i_{x_n})\,dv\,.$$

In virtue of the assumptions (d), (e) and (f), the set

$$(56.8)\qquad \partial D_{t,h}=\Pi_{t+h}\cup\Pi_t\cup\Sigma^S_{t,h}\cup\Sigma^T_{t,h}$$

is the union of a finite number of surfaces, each of which can be described analytically by an equation of the form

$$G(x_1, \ldots, x_n)=0\,,$$

with G of class C^1 and $G_{x_n}\neq 0$ in the neighborhood of the respective surface. Introducing new independent variables

$$y_j = x_j \quad (j=1, 2, \ldots, n-1)\,,\qquad y_n = G(x_1, \ldots, x_n)$$

and using formulas

$$u^i_{x_j}=u^i_{y_j}+u^i_{y_n}G_{x_j}\quad (j=1, 2, \ldots, n-1)\,,\qquad u^i_{x_n}=u^i_{y_n}G_{x_n}\,,$$
$$G_{x_n}\cos(\bar{n}, x_j)=G_{x_j}\cos(\bar{n}, x_n)\quad (j=1, 2, \ldots, n-1)$$

on the corresponding surface, the expression under the sign of integral on the left-hand side of (56.7) can be written in the form

$$\Big[A^i[G](u^i_{y_n})^2-\sum_{j,k=1}^{n-1}a^i_{jk}u^i_{y_j}u^i_{y_k}\Big]\cos(\bar{n}, x_n)\,,$$

where $A^i[G]$ is defined by formula (56.2). Hence, by (56.8) and in view of the fact that on Π_{t+h} we have $G(X)\equiv x_n-(t+h)$ and $\cos(\bar{n}, x_n)=-1$, while on Π_t there is $G(X) = x_n - t$ and $\cos(\bar{n}, x_n) = 1$, formula (56.7) can be rewritten in the following way:

$$(56.9)\quad \iint_{\Pi_{t+h}} \Big[\sum_{j,k=1}^{n-1} a^i_{jk} u^i_{x_j} u^i_{x_k} - a^i_{nn}(u^i_{x_n})^2\Big] d\sigma - \iint_{\Pi_t} \Big[\sum_{j,k=1}^{n-1} a^i_{jk} u^i_{x_j} u^i_{x_k} - a^i_{nn}(u^i_{x_n})^2\Big] d\sigma$$

$$= \iint_{\Sigma^S_{t,h}} \Big[\sum_{j,k=1}^{n-1} a^i_{jk} u^i_{y_j} u^i_{y_k} - A^i[G](u^i_{y_n})^2\Big] \cos(\bar{n}, x_n)\, d\sigma +$$

$$+ \iint_{\Sigma^T_{t,h}} \Big[\sum_{j,k=1}^{n-1} a^i_{jk} u^i_{y_j} u^i_{y_k} - A^i[G](u^i_{y_n})^2\Big] \cos(\bar{n}, x_n)\, d\sigma -$$

$$- \iiint_{D_{t,h}} (F^i_1[u] + 2f^i u^i_{x_n})\, dv\,.$$

Since we have $-2f^i u^i_{x_n} \leqslant (f^i)^2 + (u^i_{x_n})^2$, $A^i[G] \leqslant 0$ on $\Sigma^S_{t,h}$ (space-like or characteristic orientation), $A^i[G] \geqslant 0$ on $\Sigma^T_{t,h}$ (time-like orientation), and, by (c), (e), (f),

$$\sum_{j,k=1}^{n-1} a^i_{jk} u^i_{y_j} u^i_{y_k} \geqslant 0\,,$$

$$\cos(\bar{n}, x_n) < 0 \text{ on } \Sigma^S_{t,h}\,, \qquad \cos(\bar{n}, x_n) > 0 \text{ on } \Sigma^T_{t,h}\,,$$

formula (56.9) yields the following inequality:

$$(56.10)\quad \iint_{\Pi_{t+h}} \Big[\sum_{j,k=1}^{n-1} a^i_{jk} u^i_{x_j} u^i_{x_k} - a^i_{nn}(u^i_{x_n})^2\Big] d\sigma - \iint_{\Pi_t} \Big[\sum_{j,k=1}^{n-1} a^i_{jk} u^i_{x_j} u^i_{x_k} - a^i_{nn}(u^i_{x_n})^2\Big] d\sigma$$

$$\leqslant \iint_{\Sigma^T_{t,h}} \Big[\sum_{j,k=1}^{n-1} a^i_{jk} u^i_{y_j} u^i_{y_k}\Big] \cos(\bar{n}, x_n)\, d\sigma + \iiint_{D_{t,h}} (f^i)^2 dv + \iiint_{D_{t,h}} F^i_2[u]\, dv\,,$$

where F^i_2 is a quadratic form with properties analogous to those of F^i_1. Now, integrating the identity

$$2u^i u^i_{x_n} = \frac{\partial}{\partial x_n}(u^i)^2$$

over the region $D_{t,h}$ and applying, once more, Green-Gauss theorem we obtain

$$\iint_{\Pi_{t+h}} (u^i)^2 d\sigma - \iint_{\Pi_t} (u^i)^2 d\sigma$$

$$= \iint_{\Sigma^S_{t,h}} (u^i)^2 \cos(\bar{n}, x_n)\, d\sigma + \iint_{\Sigma^T_{t,h}} (u^i)^2 \cos(\bar{n}, x_n)\, d\sigma + 2 \iiint_{D_{t,h}} u^i u^i_{x_n}\, dv\,,$$

whence

$$(56.11)\qquad \iint_{\Pi_{t+h}} (u^i)^2 d\sigma - \iint_{\Pi_t} (u^i)^2 d\sigma \leqslant \iint_{\Sigma^T_{t,h}} (u^i)^2 \cos(\bar{n}, x_n)\, d\sigma + \iiint_{D_{t,h}} [(u^i)^2 + (u^i_{x_n})^2]\, dv\,.$$

Adding inequalities (56.10) and (56.11) and then summing over i we get

$$(56.12)\qquad E(t+h) - E(t) \leqslant \iint_{\Sigma^T_{t,h}} \sum_{i=1}^{m} \Big[\sum_{j,k=1}^{n-1} a^i_{jk} u^i_{y_j} u^i_{y_k} + (u^i)^2\Big] \cos(\bar{n}, x_n)\, d\sigma + \\ + \iiint_{D_{t,h}} \sum_{i=1}^{m} (f^i)^2 dv + \iiint_{D_{t,h}} \sum_{i=1}^{m} F^i_3[u]\, dv\,,$$

where F^i_3 is another quadratic form similar to F^i_1. Inequality (56.12) devided by $h > 0$ gives in the limit, when h goes to zero following a suitable sequence,

$$(56.13)\qquad D^+E(t) \leqslant \int_{\Sigma^T_t} \sum_{i=1}^{m} \Big[\sum_{j,k=1}^{n-1} a^i_{jk} u^i_{y_j} u^i_{y_k} + (u^i)^2\Big] \cos(\bar{n}, x_n)\, ds + \\ + \iint_{\Pi_t} \sum_{i=1}^{m} (f^i)^2 d\sigma + \iint_{\Pi_t} \sum_{i=1}^{m} F^i_3[u]\, d\sigma\,.$$

Observe that $\sum_{i=1}^{m} F^i_3[u]$ is a quadratic form in $u^1, \ldots, u^m$ and in their first derivatives, its coefficients being polynomials of b^{il}_j, c^{il} and of the first derivatives of a^i_{jk}. Hence, it is obvious that

$$(56.14)\qquad \sum_{i=1}^{m} F^i_3[u] \leqslant M_1 \sum_{i=1}^{m} \Big[\sum_{j=1}^{n} (u^i_{x_j})^2 + (u^i)^2\Big]\,,$$

where M_1 is a positive constant depending only on the bounds of the coefficients of system (56.1) and of the first derivatives of a^i_{jk}. From (56.3) and (56.14) it follows that

$$\sum_{i=1}^{m} F^i_3[u] \leqslant \frac{M_1}{\mu_1} \sum_{i=1}^{m} \Big[\sum_{j,k=1}^{n-1} a^i_{jk} u^i_{x_j} u^i_{x_k} - a^i_{nn}(u^i_{x_n})^2 + (u^i)^2\Big]$$

where $\mu_1 = \min(1, \mu)$, whence

$$(56.15)\qquad \iint_{\Pi_t} \sum_{i=1}^{m} F^i_3[u]\, d\sigma \leqslant \frac{M_1}{\mu_1} E(t)\,.$$

Putting

$$L = \frac{M_1}{\mu_1},$$

we obtain from (56.13) and (56.15) differential inequality (56.4) with L having the required properties.

THEOREM 56.2. *Under the assumptions of Theorem* 56.1 *we have the energy estimate, for* $0 \leqslant t < b$,

$$\iint\limits_{\Pi_t} \sum_{i=1}^{m} \Big[\sum_{j=1}^{n} (u^i_{x_j})^2 + (u^i)^2\Big] d\sigma \tag{56.16}$$
$$\leqslant \frac{e^{Lt}}{\mu_1}\Big[(M+1)\iint\limits_{\Pi_0}\Big\{\sum_{j=1}^{n}(u^i_{x_j})^2 + (u^i)^2\Big\} d\sigma + \int_0^t e^{-L\tau} g(\tau)\, d\tau\Big],$$

where

$$g(\tau) = \int\limits_{\Sigma^T_\tau} \sum_{i=1}^{m}\Big[\sum_{j,k=1}^{n-1} a^i_{jk} u^i_{y_j} u^i_{y_k} + (u^i)^2\Big]\cos(\bar{n}, x_n)\, ds + \iint\limits_{\Pi_\tau}\sum_{i=1}^{m} (f^i)^2 d\sigma .$$

Proof. From Theorem 56.1 it follows, by Theorem 9.5 (see Example 9.1) that, for $0 \leqslant t < b$,

$$E(t) \leqslant e^{Lt}\Big[E(0) + \int_0^t e^{-L\tau} g(\tau)\, d\tau\Big].$$

Hence, by (56.3) and by the definition of $E(t)$, we get (56.16).

We recall that under the Assumptions B the mixed problem for system (56.1) in the region D consists in finding a solution $U(X) = \big(u^1(X), \ldots, u^m(X)\big)$ of system (56.1), of class C^2 in D and of class C^1 in the closure of D, satisfying initial conditions

$$U(X) = \Phi_0(X), \qquad U_{x_n}(X) = \Phi_1(X) \quad \text{for} \quad X \in \Pi_0$$

and boundary conditions

$$U(X) = \Psi(X) \quad \text{for} \quad X \in \Sigma^T .$$

In the case when Σ^T is empty, the above problem reduces to the Cauchy problem.

The energy estimate (56.16) implies uniqueness of the solution of the mixed problem. Indeed, to show this, it is sufficient to prove that $U(X) \equiv 0$ is the only solution of the homogeneous problem, i.e. of the problem with $\Phi_0(X) \equiv \Phi_1(X) \equiv \Psi(X) = f^i(X) \equiv 0$. Now, let $U(X)$ be a solution of the homogeneous problem and observe that in the variables $y_1, \ldots, y_n$ the surface Σ^T is described by the equation $y_n = 0$ (see the proof of Theorem 56.1). Hence it follows that $U(X)$ being identically zero on Σ^T the first derivatives U_{y_j} $(j = 1, 2, \ldots, n-1)$ vanish on Σ^T. Since the same is true for U and U_{x_k} $(k = 1, 2, ..., n)$ on Π_0, the right-hand side of inequality (56.16) is zero. Hence it follows that $U(X) = 0$ on Π_t for every $0 \leqslant t < b$ and consequently $U(X) \equiv 0$ in D.

COROLLARY 56.1. *Theorems* 56.1 *and* 56.2 *remain true if* $U(X) = \big(u^1(X), ..., u^m(X)\big)$ *is supposed to satisfy—instead of system* (56.1)—*the following system of differential inequalities*

$$(56.17)\qquad \Big|\sum_{j,k=1}^{n} a_{jk}^i(X)\, u_{x_j x_k}^i\Big| \leqslant \sum_{l=1}^{m}\sum_{j=1}^{n} |b_j^{il}(X)|\,|u_{x_j}^l| + \sum_{l=1}^{m} |c^{il}(X)|\,|u^l| + |f^i(X)| \qquad (i = 1, 2, ..., m).$$

Proof. Let ε be an arbitrary positive number and put for $U(X)$ satisfying inequalities (56.17)

$$(56.18)\qquad \varepsilon^i(X) = \frac{\sum\limits_{j,k=1}^{n} a_{jk}^i(X)\, u_{x_j x_k}^i(X)}{\sum\limits_{l=1}^{m}\sum\limits_{j=1}^{n} |b_j^{il}(X)|\,|u_{x_j}^l(X)| + \sum\limits_{l=1}^{m} |c^{il}(X)|\,|u^l(X)| + |f^i(X)| + \varepsilon}.$$

It follows from (56.17) that

$$(56.19)\qquad |\varepsilon^i(X)| \leqslant 1 \qquad (i = 1, 2, ..., m).$$

On the other hand, (56.18) implies that

$$(56.20)\qquad \sum_{j,k=1}^{n} a_{jk}^i(X)\, u_{x_j x_k}^i = \sum_{l=1}^{m}\sum_{j=1}^{n} \tilde{b}_j^{il}(X)\, u_{x_j}^l + \sum_{l=1}^{m} \tilde{c}^{il}(X)\, u^l + \tilde{f}_\varepsilon^i(X),$$

where

$$(56.21)\qquad \begin{cases} \tilde{b}_j^{il}(X) = \varepsilon^i(X)\,|b_j^{il}(X)|\operatorname{sgn} u_{x_j}^l(X), \\ \tilde{c}^{il}(X) = \varepsilon^i(X)\,|c^{il}(X)|\operatorname{sgn} u^l(X), \\ \tilde{f}_\varepsilon^i(X) = \varepsilon^i(X)[|f^i(X)| + \varepsilon]. \end{cases}$$

Thus we see that $U(X)$ satisfies a system (56.20) for which the assumptions of Theorem 56.1 are satisfied. Moreover, by (56.19) and (56.21), it is clear that $\tilde{b}_j^{il}$ and $\tilde{c}^{il}$ have the same bounds as b_j^{il} and c^{il}. Hence it follows, by Theorem 56.1 and 56.2, that the differential inequality (56.4) and the energy estimate (56.16) hold true with f^i in the formula (56.5) replaced by $\tilde{f}_\varepsilon^i$; but, since $\varepsilon > 0$ is arbitrary and

$$\lim_{\varepsilon\to 0} \tilde{f}_\varepsilon^i = f^i,$$

we get in the limit (56.4) and (56.16) what was to be proved.

Remark. Corollary 56.1 is more convenient in applications than Theorem 56.2. Let us consider, for example, an almost linear system

$$(56.22)\qquad \sum_{j,k=1}^{n} a^{i}_{jk}(X)\, u^{i}_{x_j x_k} = h^{i}(t, X, u^{1}, \dots, u^{m}, u^{1}_{x_1}, \dots, u^{1}_{x_n}, \dots, u^{m}_{x_1}, \dots, u^{m}_{x_n})$$
$$(i = 1, 2, \dots, m)\,.$$

By Corollary 56.1, we get the following uniqueness criterion: if the right-hand sides of system (56.22) satisfy a Lipschitz condition with respect to $u^{1}, \dots, u^{m}, u^{1}_{x_1}, \dots, u^{1}_{x_n}, \dots, u^{m}_{x_1}, \dots, u^{m}_{x_n}$, then the mixed problem for system (56.22) admits at most one solution. Indeed, under the above assumptions, the difference of two solutions of system (56.22) satisfies a system (56.17) of differential inequalities with $f^{i} \equiv 0$. Hence the difference of two solutions, having the same initial and boundary values, satisfies the energy estimate (56.16) with the right-hand side identically zero; but, this implies the vanishing of the above difference what was to be proved.

CHAPTER IX

PARTIAL DIFFERENTIAL INEQUALITIES OF FIRST ORDER

This chapter deals with systems of first order partial differential inequalities of the form

$$u_x^i \leqslant f^i(x, y_1, \ldots, y_n, u^1, \ldots, u^m, u_{y_1}^i, \ldots, u_{y_n}^i) \qquad (i = 1, 2, \ldots, m)$$

and, more generally, with over-determined systems of the form

$$u_{x_j}^i \leqslant f_j^i(x_1, \ldots, x_p, y_1, \ldots, y_n, u^1, \ldots, u^m, u_{y_1}^i, \ldots, u_{y_n}^i)$$
$$(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, p)\,,$$

where the ith inequality involves derivatives of u^i only.

In Chapter VII we considered systems of equations of the above form and obtained—among others—estimates of the solution and of the difference between two solutions by means of maximum solutions of adequate comparison systems of ordinary differential equations. Now, the results of the present chapter will enable us to do the same by means of solutions of adequate comparison systems of first order partial differential equations.

We begin by discussing systems of strong inequalities and then we will pass to systems of weak inequalities. We want to stress here that—unlike in the theory of ordinary differential equations—it is useless to introduce the notion of a maximum solution of the Cauchy problem for first order partial differential equations. In fact, the notion of a maximum solution is very useful—as we have seen—but only in the case when some regularity assumptions assure local existence and do not exclude non-uniqueness of solution. Now, this situation does not occur in the theory of first order partial differential equations. The practically least restrictive regularity assumptions which guarantee local existence of solution of the Cauchy problem in the non-linear case, viz. the requirement that the right-hand sides of equations be of class C^1 with first derivatives Lipschitzian, assure at the same time uniqueness (see Theorem 42.1 and Remark 42.1).

§ 57. Systems of strong first order partial differential inequalities. We start by introducing the following definition:

DEFINITION 57.1. A region D in the space $(Z, U, Q) = (z_1, \ldots, z_q, u^1, \ldots, u^m, q_1, \ldots, q_n)$ will be called *positive with respect to* U if whenever $(Z, U, Q) \in D$ and $V \geqslant U$, then $(Z, V, Q) \in D$.

THEOREM 57.1. *Let the functions* $f^i(x, y_1, \ldots, y_n, u^1, \ldots, u^m, q_1, \ldots, q_n) = f^i(x, Y, U, Q)$ $(i = 1, 2, \ldots, m)$ *be defined in a region which is positive with respect to* U *and whose projection on the space* $(x, y_1, \ldots, y_n)$ *contains the pyramid*

$$(57.1)\qquad 0 \leqslant x - x_0 < \gamma\,, \qquad |y_k - \mathring{y}_k| \leqslant a_k - L(x - x_0) \qquad (k = 1, 2, \ldots, m)\,,$$

where $0 \leqslant L < +\infty$, $0 < a_k < +\infty$, $\gamma \leqslant \min\limits_k (a_k/L)$. *Assume the functions* $f^i(x, Y, U, Q)$ $(i = 1, 2, \ldots, m)$ *to satisfy condition* W_+ *with respect to* U (see § 4) *and the Lipschitz condition with regard to* Q

$$(57.2)\qquad |f^i(x, Y, U, Q) - f^i(x, Y, U, \widetilde{Q})| \leqslant L \sum_{k=1}^{n} |q_k - \widetilde{q}_k| \qquad (i = 1, 2, \ldots, m)\,.$$

Let $U(x, Y) = \big(u^1(x, Y), \ldots, u^m(x, Y)\big)$ *and* $V(x, Y) = \big(v^1(x, Y), \ldots, v^m(x, Y)\big)$ *be of class* $\mathfrak{D}$ *in the pyramid* (57.1) (see § 37) *and satisfy initial inequalities*

$$(57.3)\qquad U(x_0, Y) < V(x_0, Y)\,.$$

Denoting by D *the pyramid* (57.1) *put*

$$E^i = \{(x, Y) \in D\colon\ U(x, Y) \overset{i}{\leqslant} V(x, Y)\} \qquad (i = 1, 2, \ldots, m)$$

and suppose that, for every j, *differential inequalities*

$$(57.4)\qquad \begin{aligned} u^j_x(x^*, Y^*) &\leqslant f^j\big(x^*, Y^*, U(x^*, Y^*), u^j_Y(x^*, Y^*)\big)\,, \\ v^j_x(x^*, Y^*) &> f^j\big(x^*, Y^*, V(x^*, Y^*), v^j_Y(x^*, Y^*)\big) \end{aligned}$$

are satisfied whenever $(x^*, Y^*) \in E^j$. *This being assumed inequalities*

$$(57.5)\qquad U(x, Y) < V(x, Y)$$

hold true in the pyramid (57.1) ([1]).

Proof. By (57.3) and by the continuity, the set of $\widetilde{x}$, such that $x_0 \leqslant \widetilde{x} < x_0 + \gamma$ and that (57.5) holds true in the intersection of the

([1]) From the proof it will follow that our theorem remains true under less restrictive assumptions on the regularity of $U(x, Y)$ and $V(x, Y)$. It is sufficient to suppose that $U(x, Y)$ and $V(x, Y)$ are continuous in D and that, for $(x^*, Y^*) \in E^j$, u^j and v^j have first derivatives at (x^*, Y^*) and, moreover, Stolz's differentials if (x^*, Y^*) belongs to the side surface of D.

pyramid (57.1) with the zone $x_0 \leqslant x < \tilde{x}$, is not empty. Let x^* denote its least upper bound. We have to prove that $x^* = x_0 + \gamma$. Suppose it is not true and hence $x^* < x_0 + \gamma$. Then there exists an index j and a point Y^* such that (x^*, Y^*) belongs to the pyramid (57.1) and

$$(57.6) \qquad \begin{aligned} &U(x, Y) \leqslant V(x, Y) \quad \text{for} \quad x_0 \leqslant x \leqslant x^*, \\ &u^j(x^*, Y^*) = v^j(x^*, Y^*). \end{aligned}$$

By the last relations $(x^*, Y^*) \in E^j$ and hence differential inequalities (57.4) hold true. Now, there are two cases to be distinguished.

Case I. Suppose (x^*, Y^*) is an interior point of (57.1). Consider the function $u^j(x^*, Y) - v^j(x^*, Y)$ depending on Y. By (57.6), it attains maximum at Y^* and hence, Y^* being an interior point, we have

$$(57.7) \qquad u^j_Y(x^*, Y^*) = v^j_Y(x^*, Y^*).$$

Similarly, the function $u^j(x, Y^*) - v^j(x, Y^*)$ depending on x attains its maximum in the interval $[x_0, x^*]$ at the point x^*. Therefore

$$(57.8) \qquad u^j_x(x^*, Y^*) - v^j_x(x^*, Y^*) \geqslant 0.$$

On the other hand, by (57.4), (57.6), (57.7) and by condition W_+, we get

$$\begin{aligned} u^j_x(x^*, Y^*) - v^j_x(x^*, Y^*) < f^j\big(x^*, Y^*, U(x^*, Y^*), u^j_Y(x^*, Y^*)\big) - \\ - f^j\big(x^*, Y^*, V(x^*, Y^*), u^j_Y(x^*, Y^*)\big) \leqslant 0, \end{aligned}$$

which contradicts (57.8).

Case II. Suppose (x^*, Y^*) is a point on the side surface of the pyramid (57.1). We can assume (rearranging the indices if necessary) that we have

$$(57.9) \qquad \begin{cases} y_p^* = a_p - L(x^* - x_0) & (p = 1, 2, \ldots, s), \\ y_q^* = -a_q + L(x^* - x_0) & (q = s+1, \ldots, s+r), \\ |y_k^*| < a_k - L(x^* - x_0) & (k = s+r+1, \ldots, n). \end{cases}$$

Fix p and consider the function

$$u^j(x^*, y_1^*, \ldots, y_{p-1}^*, y_p, y_{p+1}^*, \ldots, y_n^*) - v^j(x^*, y_1^*, \ldots, y_{p-1}^*, y_p, y_{p+1}^*, \ldots, y_n^*)$$

depending on y_p in the interval

$$-a_p + L(x^* - x_0) \leqslant y_p \leqslant a_p - L(x^* - x_0).$$

By (57.6) and (57.9) it attains maximum at $y_p^* = a_p - L(x^* - x_0)$, i.e. at the right-hand extremity of the interval. Hence, it follows that

$$(57.10) \qquad u^j_{y_p}(x^*, Y^*) - v^j_{y_p}(x^*, Y^*) \geqslant 0 \quad (p = 1, 2, \ldots, s).$$

By a similar argument, we get

$$\begin{aligned} &u^j_{y_q}(x^*, Y^*) - v^j_{y_q}(x^*, Y^*) \leqslant 0 \qquad (q = s+1, \ldots, s+r), \\ &u^j_{y_k}(x^*, Y^*) - v^j_{y_k}(x^*, Y^*) = 0 \qquad (k = s+r+1, \ldots, n). \end{aligned} \tag{57.11}$$

Now, for $x_0 \leqslant x \leqslant x^*$, put

$$Y(x) = \big(a_p - L(x-x_0), -a_q + L(x-x_0), y_k^*\big)$$

and consider the composite function $u^j\big(x, Y(x)\big) - v^j\big(x, Y(x)\big)$. It attains maximum at x^*, by (57.6) and (57.9), and hence

$$\frac{d}{dx}\big[u^j\big(x, Y(x)\big) - v^j\big(x, Y(x)\big)\big]_{x=x^*} \geqslant 0. \tag{57.12}$$

But, u^j and v^j being of class $\mathfrak{D}$ in the pyramid (57.1) (see § 37) and the point $(x^*, Y^*) = \big(x^*, Y(x^*)\big)$ belonging to the side surface of (57.1), the functions u^j, v^j possess Stolz's differentials at $\big(x^*, Y(x^*)\big)$. Therefore, we can apply to the left-hand side of inequality (57.12) the formula for the derivative of a composite function and thus we get

$$\begin{aligned} &u^j_x(x^*, Y^*) - v^j_x(x^*, Y^*) \\ &\geqslant L\Big[\sum_p \big(u^j_{y_p}(x^*, Y^*) - v^j_{y_p}(x^*, Y^*)\big) - \sum_q \big(u^j_{y_q}(x^*, Y^*) - v^j_{y_q}(x^*, Y^*)\big)\Big]. \end{aligned} \tag{57.13}$$

On the other hand, we have, by (57.4),

$$\begin{aligned} &u^j_x(x^*, Y^*) - v^j_x(x^*, Y^*) \\ &< \big[f^j\big(x^*, Y^*, U(x^*, Y^*), u^j_Y(x^*, Y^*)\big) - f^j\big(x^*, Y^*, V(x^*, Y^*), u^j_Y(x^*, Y^*)\big)\big] + \\ &+ \big[f^j\big(x^*, Y^*, V(x^*, Y^*), u^j_Y(x^*, Y^*)\big) - f^j\big(x^*, Y^*, V(x^*, Y^*), v^j_Y(x^*, Y^*)\big)\big]. \end{aligned}$$

The first difference in the brackets is non-positive, by (57.6) and by condition W_+ (see § 4). To the second difference in brackets we apply inequality (57.2) and thus—taking advantage of (57.10) and (57.11)—we get

$$\begin{aligned} &u^j_x(x^*, Y^*) - v^j_x(x^*, Y^*) \\ &< L\Big[\sum_p \big(u^j_{y_p}(x^*, Y^*) - v^j_{y_p}(x^*, Y^*)\big) - \sum_q \big(u^j_{y_q}(x^*, Y^*) - v^j_{y_q}(x^*, Y^*)\big)\Big], \end{aligned}$$

which contradicts (57.13).

Since in both cases I and II we obtained a contradiction, the theorem is proved.

Remark 57.1. Theorem 57.1 as well as all theorems to be proved in this chapter are true for more general domains than the pyramid (see [49]). Indeed, in the case of Theorem 57.1, for instance, if we assume additionally that the derivatives f_{q_j} exist, then the Lipschitz condition (57.2) has the following geometrical meaning with regard to the pyramid (57.1) which we denote by D:

(α) for any point (x^*, Y^*) on the side surface of D and for every fixed i, the vector

$$\big(1, -f_{q_1}(x^*, Y^*, U, Q), \ldots, -f_{q_n}(x^*, Y^*, U, Q)\big)$$

is either tangent to the side surface of D or points to the exterior of D.

Now, the pyramid (57.1) in Theorem 57.1 can be replaced by an arbitrary region D with the side surface being the union of a finite number of surfaces of class C^1 and having—in case of the existence of the derivatives $f^i_{q_k}$—the geometrical property (α).

§ 58. Overdetermined systems of strong first order partial differential inequalities. Our next theorem will be derived from Theorem 57.1 by means of Mayer's transformation (see § 38).

THEOREM 58.1. *Let the functions* $f^i_k(x_1, \ldots, x_p, y_1, \ldots, y_n, u^1, \ldots, u^m$ $q_1, \ldots, q_n) = f^i_k(X, Y, U, Q)$ $(i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, p)$ *be defined in a region which is positive with respect to* U (see Definition 57.1) *and whose projection on the space* $(x_1, \ldots, x_p, y_1, \ldots, y_n)$ *contains the pyramid*

$$(58.1)\qquad 0 \leqslant x_l - \mathring{x}_l, \quad \sum_{k=1}^{p}(x_k - \mathring{x}_k) < \gamma, \quad |y_r - \mathring{y}_r| \leqslant a_r - L\sum_{k=1}^{p}(x_k - \mathring{x}_k)$$
$$(l = 1, 2, \ldots, p;\ r = 1, 2, \ldots, n),$$

where $0 \leqslant L < +\infty$, $0 < a_k < +\infty$, $\gamma \leqslant \min_r (a_r/L)$. *Suppose that, for every fixed* k, *the functions* $f^i_k(X, Y, U, Q)$ $(i = 1, 2, \ldots, m)$, *satisfy condition* W_+ *with respect to* U (see § 4) *and the Lipschitz condition with regard to* Q

$$(58.2)\qquad |f^i_k(X, Y, U, Q) - f^i_k(X, Y, U, \tilde{Q})| \leqslant L\sum_{r=1}^{n}|q_r - \tilde{q}_r|$$
$$(i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, p).$$

Let $U(X, Y) = \big(u^1(X, Y), \ldots, u^m(X, Y)\big)$ *and* $V(X, Y) = \big(v^1(X, Y), \ldots, v^m(X, Y)\big)$ *be of class* $\mathfrak{D}$ *in the pyramid* (58.1) (see § 37) *and satisfy the initial inequality*

$$(58.3)\qquad U(X_0, Y) < V(X_0, Y).$$

Denoting by D *the pyramid* (58.1) *put*

$$G^i = \{(X, Y) \in D\colon\ U(X, Y) \overset{i}{\leqslant} V(X, Y)\} \qquad (i = 1, 2, \ldots, m)$$

and suppose that, for every fixed j, *the differential inequalities*

$$(58.4)\qquad \begin{aligned} u^j_{x_k} &\leqslant f^j_k\big(X, Y, U(X, Y), u^j_Y(X, Y)\big) \\ v^j_{x_k} &> f^j_k\big(X, Y, V(X, Y), v^j_Y(X, Y)\big) \end{aligned} \qquad (k = 1, 2, \ldots, p)$$

are satisfied for $(X, Y) \in G^j$. *This being assumed, inequalities*

$$U(X, Y) < V(X, Y)$$

hold true in the pyramid (58.1).

Proof. Introduce Mayer's transformation

$$X = X_0 + \Lambda x,$$

where $\Lambda = (\lambda_1, \ldots, \lambda_p)$. For Λ satisfying

$$\lambda_l \geqslant 0 \quad (l = 1, 2, \ldots, p), \quad \sum_{k=1}^{p} \lambda_k = \lambda < \gamma, \tag{58.5}$$

put

$$\begin{aligned} \widetilde{U}(x, Y; \Lambda) &= U(X_0 + \Lambda x, Y), \\ \widetilde{V}(x, Y; \Lambda) &= V(X_0 + \Lambda x, Y). \end{aligned} \tag{58.6}$$

It is obvious that, for Λ satisfying (58.5), the functions (58.6) are of class $\mathfrak{D}$ (see § 37) in the pyramid

$$0 \leqslant x < \frac{\gamma}{\lambda}, \quad |y_r - \mathring{y}_r| \leqslant a_r - \lambda L x \quad (r = 1, 2, \ldots, n), \tag{58.7}$$

where

$$\frac{\gamma}{\lambda} > 1. \tag{58.8}$$

By (58.3), functions $\widetilde{U}(x, Y; \Lambda) = \big(\widetilde{u}^1(x, Y; \Lambda), \ldots, \widetilde{u}^m(x, Y; \Lambda)\big)$, $\widetilde{V}(x, Y; \Lambda) = \big(\widetilde{v}^1(x, Y; \Lambda), \ldots, \widetilde{v}^m(x, Y; \Lambda)\big)$ satisfy initial inequality $\widetilde{U}(0, Y; \Lambda) < \widetilde{V}(0, Y; \Lambda)$. The functions $U(X, Y)$ and $V(X, Y)$ being of class $\mathfrak{D}$ they possess Stolz's differentials with regard to X; therefore, we have

$$\widetilde{U}_x = \sum_{j=1}^{p} \lambda_j U_{x_j}(X_0 + \Lambda x, Y), \quad \widetilde{V}_x = \sum_{j=1}^{p} \lambda_j V_{x_j}(X_0 + \Lambda x, Y).$$

Denoting by D_λ the pyramid (58.7), put

$$E_\lambda^i = \{(x, Y) \in D_\lambda \colon \widetilde{U}(x, Y; \Lambda) \overset{i}{\leqslant} \widetilde{V}(x, Y; \Lambda)\} \quad (i = 1, 2, \ldots, m).$$

Fix an index j and suppose that $(x, Y) \in E_\lambda^j$. Then $(X_0 + \Lambda x, Y) \in G^j$ and hence it follows, by (58.4) and (58.5), that

$$\begin{aligned} &\widetilde{u}_x^j \leqslant F^j\big(x, Y, \widetilde{U}(x, Y), \widetilde{u}_Y^j(x, Y); \Lambda\big), \\ &\widetilde{v}_x^j > F^j\big(x, Y, \widetilde{V}(x, Y), \widetilde{v}_Y^j(x, Y); \Lambda\big), \end{aligned}$$

for $(x, Y) \in E^j_\lambda$ where

$$(58.9)\quad F^i(x, Y, U, Q; \Lambda) = \sum_{k=1}^{p} \lambda_k f^i_k(X_0 + \Lambda_x, Y, U, Q) \quad (i = 1, 2, \ldots, m).$$

In virtue of the hypotheses of our theorem we check, by (58.2), that

$$|F^i(x, Y, U, Q; \Lambda) - F^i(x, Y, U, \tilde{Q}; \Lambda)| \leqslant \lambda L \sum_{r=1}^{n} |q_r - \tilde{q}_r| \qquad (i = 1, 2, \ldots, m)$$

and that the functions $F^i(x, Y, U, Q; \Lambda)$ $(i = 1, 2, \ldots, m)$ satisfy condition W_+ with regard to U. Thus we see that $\tilde{U}(x, Y; \Lambda)$, $\tilde{V}(x, Y; \Lambda)$ and $F^i(x, Y, U, Q; \Lambda)$ satisfy, for every fixed Λ, subject to conditions (58.5), all the assumptions of Theorem 57.1 in the pyramid (58.7). Hence, we have in the pyramid (58.7)

$$\tilde{U}(x, Y; \Lambda) < \tilde{V}(x, Y; \Lambda)$$

and in particular, by (58.8),

$$(58.10)\qquad \tilde{U}(1, Y; \Lambda) < \tilde{V}(1, Y; \Lambda).$$

Now, let (X, Y) be an arbitrary point in the pyramid (58.1); then $\Lambda = X - X_0 = (x_1 - \mathring{x}_1, \ldots, x_p - \mathring{x}_p)$ satisfies conditions (58.5) and, by (58.6) and (58.10), we get

$$U(X, Y) = \tilde{U}(1, Y; X - X_0) < \tilde{V}(1, Y; X - X_0) = V(X, Y),$$

what was to be proved.

§ 59. Systems of weak first order partial differential inequalities. In this section we deal with weak differential inequalities (see [42]). Unlike in §§ 57-58, we will have to make more restrictive assumptions on the right-hand sides of the differential inequalities, viz. assumptions which imply right-sided uniqueness of the solution of the Cauchy problem for the corresponding system of equations (see Corollary 60.1).

THEOREM 59.1. *Let the functions* $f^i(x, Y, U, Q)$ $(i = 1, 2, \ldots, m)$ *be defined in a region which is positive with respect to* U (see Definition 57.1) *and whose projection on the space of points* (x, Y) *contains the pyramid* (57.1). *Assume the functions* $f^i(x, Y, U, Q)$ *to satisfy condition* W_+ *with regard to* U (see § 4) *and the inequalities*

$$(59.1)\quad f^i(x, Y, U, Q) - f^i(x, Y, \tilde{U}, \tilde{Q}) \leqslant \sigma_i(x - x_0, U - \tilde{U}) + L \sum_{r=1}^{n} |q_r - \tilde{q}_r| \qquad (i = 1, 2, \ldots, m),$$

whenever $U \geqslant \tilde{U}$, *where* $\sigma_i(t, U)$ *are the right-hand sides of a comparison system of type* I (see § 14). *Concerning the comparison system we suppose that*

$$\sigma_i(t, 0) \equiv 0 \quad (i = 1, 2, \ldots, m)$$

and that for its right-hand maximum solution $\Omega(t; 0)$ *through the origin we have*

$$\Omega(t; 0) \equiv 0 \,. \tag{59.2}$$

Let $U(x, Y) = \big(u^1(x, Y), \ldots, u^m(x, Y)\big)$ *and* $V(x, Y) = \big(v^1(x, Y), \ldots$ $\ldots, v^m(x, Y)\big)$ *be continuous in the pyramid* (57.1) *and satisfy initial inequalities*

$$U(x_0, Y) \leqslant V(x_0, Y) \,. \tag{59.3}$$

Denoting by D *the pyramid* (57.1) *put*

$$E^i = \{(x, Y) \in D \colon u^i(x, Y) > v^i(x, Y)\} \quad (i = 1, 2, \ldots, m) \,.$$

Assume that for every fixed j, *whenever* $(x, Y) \in E^j$, *then* u^j *and* v^j *possess first derivatives at* (x, Y) *and, moreover, Stolz's differentials if* (x, Y) *belongs to the side surface of* D, *and satisfy at* (x, Y) *differential inequalities*

$$\begin{aligned} u^j_x(x, Y) &\leqslant f^j\big(x, Y, U(x, Y), u^j_Y(x, Y)\big) \,, \\ v^j_x(x, Y) &\geqslant f^j\big(x, Y, V(x, Y), v^j_Y(x, Y)\big) \,. \end{aligned} \tag{59.4}$$

Under these assumptions inequality

$$U(x, Y) \leqslant V(x, Y) \tag{59.5}$$

is satisfied in the pyramid (57.1).

Proof. Denote by S_t the projection on $(y_1, \ldots, y_n)$ of the intersection of the pyramid (57.1) with the plane $x = x_0 + t$ and put, for $0 \leqslant t < \gamma$,

$$M^i(t) = \max_{Y \in S_t} [u^i(x_0+t, Y) - v^i(x_0+t, Y)] \,, \qquad \tilde{M}^i(t) = \max\big(0, M^i(t)\big)$$
$$(i = 1, 2, \ldots, m) \,.$$
$$\tilde{M}(t) = \big(\tilde{M}^1(t), \ldots, \tilde{M}^m(t)\big) \,.$$

It is obvious that (59.5) is equivalent with

$$\tilde{M}(t) \leqslant 0 \quad \text{in} \quad [0, \gamma) \,. \tag{59.6}$$

Now, relation (59.6) will be proved by means of the first comparison theorem from the theory of ordinary differential inequalities (see § 14). By (59.3), we have

$$\tilde{M}(0) \leqslant 0 \,. \tag{59.7}$$

From Theorem 33.1 it follows that $\widetilde{M}^i(t)$ are continuous on $[0, \gamma)$. By Theorem 35.1, for every index j and $t^* \in (0, \gamma)$ there is a point $Y^* \in S_{t^*}$ such that

$$M^j(t^*) = u^j(x_0+t^*, Y^*) - v^j(x_0+t^*, Y^*) \tag{59.8}$$

and whenever u^j and v^j possess first derivatives at (x_0+t^*, Y^*) and, moreover, Stolz's differentials if (x_0+t^*, Y^*) belongs to the side surface of D, then

$$\begin{aligned} D^- M^j(t^*) \leqslant u_x^j(x_0+t^*, Y^*) - v_x^j(x_0+t^*, Y^*) - \\ -L \sum_{r=1}^{n} |u_{y_r}^j(x_0+t^*, Y^*) - v_{y_r}^j(x_0+t^*, Y^*)| . \end{aligned} \tag{59.9}$$

Put

$$\widetilde{E}_i = \{t \in (0, \gamma) \colon \widetilde{M}^i(t) > 0\} \qquad (i = 1, 2, \ldots, m) .$$

Fix an index j and suppose that $t^* \in \widetilde{E}_j$. Then, obviously, we have

$$\widetilde{M}^j(t^*) = M^j(t^*) , \qquad D^- \widetilde{M}^j(t^*) = D^- M^j(t^*) \tag{59.10}$$

and consequently, by (59.8), there is a point $Y^* \in S_{t^*}$ such that

$$\widetilde{M}^j(t^*) = u^j(x_0+t^*, Y^*) - v^j(x_0+t^*, Y^*) . \tag{59.11}$$

Since $\widetilde{M}^j(t^*) > 0$, we conclude that $(x_0+t^*, Y^*) \in E^j$ and hence inequalities (59.4) hold true at (x_0+t^*, Y^*); moreover, u^j and v^j have at (x_0+t^*, Y^*) that regularity which implies (59.9). By (59.9) and (59.10), we get

$$\begin{aligned} D^- \widetilde{M}^j(t^*) \leqslant u_x^j(x_0+t^*, Y^*) - v_x^j(x_0+t^*, Y^*) - \\ -L \sum_{r=1}^{n} |u_{y_r}^j(x_0+t^*, Y^*) - v_{y_r}^j(x_0+t^*, Y^*)| . \end{aligned}$$

From the last inequality and from (59.4) it follows that

$$\begin{aligned} D^- \widetilde{M}^j(t^*) \leqslant f^j\big(x_0+t^*, Y^*, U(x_0+t^*, Y^*), u_Y^j(x_0+t^* Y^*)\big) - \\ -f^j\big(x_0+t^*, Y^*, V(x_0+t^*, Y^*), v_Y^j(x_0+t^*, Y^*)\big) - \\ -L \sum_{r=1}^{n} |u_{y_r}^j(x_0+t^*, Y^*) - v_{y_r}^j(x_0+t^*, Y^*)| . \end{aligned} \tag{59.12}$$

Observe now that, by the definition of $\widetilde{M}(t)$ and by (59.11), we have (see § 4)

$$U(x_0+t^*, Y^*) \overset{j}{\leqslant} V(x_0+t^*, Y^*) + \widetilde{M}(t^*) .$$

By the last inequalities and by condition W_+ (see § 4) imposed on the functions $f^i(x, Y, U, Q)$, it follows from (59.12) that

$$D^-\widetilde{M}^j(t^*) \leqslant f^j\big(x_0+t^*, Y^*, V(x_0+t^*, Y^*)+\widetilde{M}(t^*), u^j_Y(x_0+t^* Y^*)\big) - \\ -f^j\big(x_0+t^*, Y^*, V(x_0+t^*, Y^*), v^j_Y(x_0+t^*, Y^*)\big) - \\ -L\sum_{r=1}^{n}|u^j_{y_r}(x_0+t^*, Y^*)-v^j_{y_r}(x_0+t^*, Y^*)|\,.$$

Since $\widetilde{M}(t^*) \geqslant 0$, we get from the last inequalities, by (59.1), that

$$D^-\widetilde{M}^j(t^*) \leqslant \sigma_j\big(t^*, \widetilde{M}(t^*)\big)\,. \tag{59.13}$$

Thus we have proved that, for every j, inequality (59.13) holds true whenever $t^* \in \widetilde{E}^j$. Hence and by (59.2) and (59.7), inequalities (59.6) follow from the first comparison theorem (see § 14). This completes the proof.

Remark 59.1 [1]. Theorem 59.1 can be derived from Theorem 57.1 without having recourse to the first comparison theorem. Indeed, for $\varepsilon > 0$, denote by $\Omega(t; \varepsilon) = \big(\omega_1(t; \varepsilon), \ldots, \omega_m(t; \varepsilon)\big)$ the right-hand maximum solution through the point $(0, \varepsilon, \ldots, \varepsilon)$ of the comparison system

$$\frac{dw_i}{dt} = \sigma_i(t, w_1, \ldots, w_m)+\varepsilon \quad (i = 1, 2, \ldots, m)\,.$$

Since, by (59.2), $\Omega(t, 0) \equiv 0$, we infer, by Theorem 10.1, that, for $\varepsilon > 0$ sufficiently small, $\Omega(t; \varepsilon)$ is defined on $[0, \gamma)$ and

$$\lim_{\varepsilon\to 0}\Omega(t; \varepsilon) = 0 \quad \text{on} \quad [0, \gamma)\,. \tag{59.14}$$

Consider now the function

$$\widetilde{V}(x, Y) = \Omega(x-x_0; \varepsilon)+V(x, Y) = \big(\widetilde{v}^1(x, Y), \ldots, \widetilde{v}^m(x, Y)\big)$$

in the pyramid (57.1), which we denote by D, and put

$$\widetilde{\widetilde{E}}^i = \{(x, Y) \in D\colon\ U(x, Y) \overset{i}{\leqslant} \widetilde{V}(x, Y)\} \quad (i = 1, 2, \ldots, m)\,.$$

Fix an index j and let $(x^*, Y^*) \in \widetilde{\widetilde{E}}^j$; then, since $\omega_j(x^*-x_0; \varepsilon) > 0$, we have $(x^*, Y^*) \in E^j$ and hence, by the second inequality (59.4), we get

$$\widetilde{v}^j_x(x^*, Y^*) \geqslant f^j\big(x^*, Y^*, V(x^*, Y^*), v^j_Y(x^*, Y^*)\big)+\omega'_j(x^*-x_0; \varepsilon) \\ = f^j\big(x^*, Y^*, V(x^*, Y^*), v^j_Y(x^*, Y^*)\big)+\sigma_j\big(x^*-x_0, \Omega(x^*-x_0; \varepsilon)\big)+\varepsilon \\ = f^j\big(x^*, Y^*, \widetilde{V}(x^*, Y^*), \widetilde{v}^j_Y(x^*, Y^*)\big)+ \\ +\big[f^j\big(x^*, Y^*, V(x^*, Y^*), \widetilde{v}^j_Y(x^*, Y^*)\big)- \\ -f^j\big(x^*, Y^*, \widetilde{V}(x^*, Y^*), \widetilde{v}^j_Y(x^*, Y^*)\big)+ \\ +\sigma_j\big(x^*-x_0, \widetilde{V}(x^*, Y^*)-V(x^*, Y^*)\big)\big]+\varepsilon.$$

[1] This remark is due to P. Besala.

Since $\widetilde{V}(x^*, Y^*)-V(x^*, Y^*) = \Omega(x^*-x_0; \varepsilon) > 0$, it follows from the last inequality, by (59.1), that

$$\tilde{v}^j_x(x^*, Y^*) > f^j(x^*, Y^*, \widetilde{V}(x^*, Y^*), \tilde{v}^j_Y(x^*, Y^*)) . \tag{59.15}$$

By (59.3), we have

$$\widetilde{V}(x_0, Y) = V(x_0, Y)+\Omega(x-x_0; \varepsilon) > V(x_0, Y) \geqslant U(x_0, Y) ,$$

and hence, by the first inequality (59.4) and by (59.15), we get from Theorem 57.1 that

$$U(x, Y) < \widetilde{V}(x, Y) = V(x, Y)+\Omega(x-x_0; \varepsilon)$$

in the pyramid (57.1). From the above inequality and from (59.14) we obtain in the limit (letting ε tend to 0) inequalities (59.5).

The usefulness of Theorem 59.1 with weak assumptions concerning the regularity of functions u^j and v^j and differential inequalities in the set E^j, will appear in the proof of Theorem 61.1.

Example 59.1. Suppose $u(x, Y)$ to be of class $\mathfrak{D}$ in the pyramid (57.1) and to satisfy there the differential inequality

$$u_x \leqslant L \sum_{r=1}^{n} |u_{y_r}| \quad \left(u_x \geqslant -L \sum_{r=1}^{n} |u_{y_r}|\right) ,$$

where $L > 0$, and the initial inequality

$$u(x_0, Y) \leqslant \eta \quad (u(x_0, Y) \geqslant \eta) ,$$

where η is a constant. Then we have in the pyramid (57.1)

$$u(x, Y) \leqslant \eta \quad (u(x, Y) \geqslant \eta) .$$

This follows immediately from Theorem 59.1 (for $m = 1$) if we put $v(x, Y) \equiv \eta$.

Remark 59.2. Theorem 59.1 remains true if inequalities (59.1) are replaced by somewhat less restrictive ones, viz.

$$f^i(x, Y, U, Q)-f^i(x, Y, \widetilde{U}, \widetilde{Q}) \leqslant \sigma\big(x-x_0, \max_l (u^l-\tilde{u}^l)\big)+L \sum_{k=1}^{n} |q_k-\tilde{q}_k| \quad (i = 1, 2, \dots, m) ,$$

whenever $U \geqslant \widetilde{U}$, where $\sigma(t, v)$ is the right-hand side of a comparison equation of type II (see § 14). The proof of this variant of Theorem 59.1 is quite similar to that of Theorem 59.1 and is carried out by applying the second comparison theorem (see § 14) to the function $\widetilde{\widetilde{M}}(t) = \max_j \widetilde{M}^j(t)$, where $\widetilde{M}^j(t)$ are defined like in the proof of Theorem 59.1.

In a natural way the question arises whether in Theorem 59.1 strong initial inequalities (59.3) imply strong inequalities (59.5) in the pyramid (57.1). We are going to answer this question in the case $m = 1$, introducing some additional more restrictive hypotheses. We start by recalling a definition from the theory of first order partial differential equations.

Consider a first order partial differential equation

$$u_x = f(x, Y, u, u_{y_1}, \ldots, u_{y_n}) \tag{59.16}$$

and suppose $f(x, Y, u, Q)$ to be of class C^1 in some region whose projection on the space (x, Y) contains the pyramid (57.1). The characteristic equations, corresponding to (59.16), are of the form (40.5). Its solutions are called *characteristic strips*. Let $u(x, Y)$ be an arbitrary function having first derivatives in the pyramid (57.1). We say that $u(x, Y)$ *is generated by characteristics of equation* (59.16) if, for every point $(x^*, Y^*) = (x^*, y_1^*, \ldots, y_n^*)$ in the pyramid (57.1), there is a characteristic strip

$$Y(x) = \big(y_1(x), \ldots, y_n(x)\big), \qquad Q(x) = \big(q_1(x), \ldots, q_n(x)\big), \qquad u(x)$$

defined on the interval $[x_0, x^*]$, such that

$$\begin{cases} Y(x^*) = Y^*, \\ |y_k(x) - \mathring{y}_k| \leqslant a_k - L(x - x_0) \quad \text{for} \quad x_0 \leqslant x \leqslant x^* \quad (k = 1, 2, \ldots, n), \\ q_k(x) = u_{y_k}\big(x, Y(x)\big) \quad (k = 1, 2, \ldots, n), \qquad u(x) = u\big(x, Y(x)\big). \end{cases} \tag{59.17}$$

It is a well-known fact that a function of class C^1 generated by characteristics is necessarily a solution of (59.16).

We are now able to state the next theorem, whose proof resembles that of Theorem 57.1.

THEOREM 59.2. *Suppose $f(x, Y, u, Q)$ to be of class C^1 in some region, whose projection on the space (x, Y) covers the pyramid* (57.1) *with $L > 0$, and to satisfy the Lipschitz condition*

$$|f(x, Y, u, Q) - f(x, Y, u, \tilde{Q})| < L \sum_{k=1}^{n} |q_k - \tilde{q}_k| \qquad \textit{for} \quad \sum_{k=1}^{n} |q_k - \tilde{q}_k| > 0\,. \tag{59.18}$$

Suppose that solutions of system (40.5) *are uniquely determined by initial data. Let $u(x, Y)$ and $v(x, Y)$ be of class $\mathfrak{D}$ in the pyramid* (57.1) (see § 37) *and satisfy there initial inequality*

$$u(x_0, Y) < v(x_0, Y) \tag{59.19}$$

and differential inequalities

$$u_x \leqslant f(x, Y, u, u_Y), \qquad v_x \geqslant f(x, Y, v, v_Y). \tag{59.20}$$

Assume finally that both $u(x, Y)$ and $v(x, Y)$ are generated by characteristics [1].

Under these assumptions we have

$$u(x, Y) < v(x, Y) \tag{59.21}$$

in the pyramid (57.1).

Proof. By (59.19) and by the continuity, there is an $\tilde{x}$ $(x_0 < \tilde{x} < x_0 + \gamma)$ such that (59.21) holds true in the pyramid (57.1) for $x_0 \leqslant x < \tilde{x}$. Denote by x^* the least upper bound of such numbers $\tilde{x}$. We have to prove that $x^* = x_0 + \gamma$. Suppose it is not true and hence $x^* < x_0 + \gamma$. Then there is obviously a point Y^* such that (x^*, Y^*) belongs to the pyramid and

$$u(x^*, Y^*) = v(x^*, Y^*). \tag{59.22}$$

Now, there are two cases to be distinguished.

Case I. Suppose (x^*, Y^*) is an interior point of (57.1). Then—like in the proof of Theorem 57.1—we have

$$u_{y_k}(x^*, Y^*) = v_{y_k}(x^*, Y^*) \qquad (k = 1, 2, \ldots, m). \tag{59.23}$$

By (59.22), (59.23) and by the uniqueness of solutions of system (40.5) the characteristic strip corresponding to $u(x, Y)$ and satisfying (59.17) is identical on the interval $[x_0, x^*]$ with that corresponding to $v(x, Y)$. Hence, for $x = x_0$ in particular, we have

$$u\big(x_0, Y(x_0)\big) = v\big(x_0, Y(x_0)\big),$$

which contradicts (59.19).

Case II. Suppose (x^*, Y^*) is a point on the side surface of the pyramid (57.1). We can assume—like in the proof of Theorem 57.1—that we have (57.9). Then, by a similar argument, we get

$$\begin{cases} u_{y_p}(x^*, Y^*) - v_{y_p}(x^*, Y^*) \geqslant 0 & (p = 1, 2, \ldots, s), \\ u_{y_q}(x^*, Y^*) - v_{y_q}(x^*, Y^*) \leqslant 0 & (q = s+1, \ldots, s+r), \\ u_{y_k}(x^*, Y^*) - v_{y_k}(x^*, Y^*) = 0 & (k = s+r+1, \ldots, n), \end{cases} \tag{59.24}$$

and

$$\begin{aligned} & u_x(x^*, Y^*) - v_x(x^*, Y^*) \\ & \quad \geqslant L\Big[\sum_p \big(u_{y_p}(x^*, Y^*) - v_{y_p}(x^*, Y^*)\big) - \sum_q \big(u_{y_q}(x^*, Y^*) - v_{y_q}(x^*, Y^*)\big)\Big]. \end{aligned} \tag{59.25}$$

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[1] This last assumption implies that if $u(x, Y)$ and $v(x, Y)$ are of class C^1, then they are solutions of equation (59.16).

On the other hand, by (59.20) and (59.22), we have

$$u_x(x^*, Y^*) - v_x(x^*, Y^*)$$
$$\leqslant f\big(x^*, Y^*, u(x^*, Y^*), u_Y(x^*, Y^*)\big) - f\big(x^*, Y^*, u(x^*, Y^*), v_Y(x^*, Y^*)\big) .$$

We can assume that

$$\sum_{r=1}^{n} |u_{y_r}(x^*, Y^*) - v_{y_r}(x^*, Y^*)| > 0 ,$$

since otherwise we would have (59.23) and we would reach contradiction like in case I. Now, from the last inequality we obtain, by (59.18) and (59.24)

$$u_x(x^*, Y^*) - v_x(x^*, Y^*)$$
$$< L\Big[\sum_{p} \big(u_{y_p}(x^*, Y^*) - v_{y_p}(x^*, Y^*)\big) - \sum_{q} \big(u_{y_q}(x^*, Y^*) - v_{y_q}(x^*, Y^*)\big)\Big]$$

what contradicts (59.25). Since in both cases we have reached a contradiction, the theorem is proved.

§ 60. Overdetermined systems of weak first order partial differential inequalities. The theorem of this section will be derived from Theorem 59.1 by means of Mayer's transformation. Its proof is patterned on that of Theorem 58.1.

THEOREM 60.1. *Let the functions* $f_l^i(x_1, \ldots, x_p, y_1, \ldots, y_n, u^1, \ldots, u^m, q_1, \ldots, q_n) = f_l^i(X, Y, U, Q)$ $(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p)$ *be defined in a region which is positive with regard to* U (see Definition 57.1) *and whose projection on the space of points* (X, Y) *contains the pyramid* (58.1). *Assume that, for every fixed* l, *the functions* $f_l^i(X, Y, U, Q)$ $(i = 1, 2, \ldots, m)$ *satisfy condition* W_+ *with regard to* U (see § 4) *and the inequalities*

$$(60.1)\qquad f_l^i(X, Y, U, Q) - f_l^i(X, Y, \tilde{U}, \tilde{Q})$$
$$\leqslant \sigma_i\Big(\sum_{r=1}^{p} (x_r - \mathring{x}_r), U - \tilde{U}\Big) + L\sum_{k=1}^{n} |q_k - \tilde{q}_k|$$
$$(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p) ,$$

whenever $U \geqslant \tilde{U}$, *where* $\sigma_i(t, V)$ *are the right-hand sides of a comparison system of type* I (see § 14). *What concerns the comparison system, we suppose that*

$$\sigma_i(t, 0) \equiv 0 \qquad (i = 1, 2, \ldots, m)$$

and that for its right-hand maximum solution through the origin $\Omega(t; 0)$ *we have*

$$(60.2)\qquad \Omega(t; 0) \equiv 0 .$$

Let $U(X, Y) = \big(u^1(X, Y), \ldots, u^m(X, Y)\big)$ *and* $V(X, Y) = \big(v^1(X, Y), \ldots$ $\ldots, v^m(X, Y)\big)$ *be continuous in the pyramid* (58.1) *and satisfy initial inequality*

$$U(X_0, Y) \leqslant V(X_0, Y) \tag{60.3}$$

Denote by D *the pyramid* (58.1) *and put*

$$G^i = \{(X, Y) \in D\colon\ u^i(X, Y) > v^i(X, Y)\} \quad (i = 1, 2, \ldots, m)\,.$$

Assume that for every fixed j, *whenever* $(X, Y) \in G^j$, *then* $u^j(X, Y)$ *and* $v^j(X, Y)$ *possess first derivatives with respect to* Y *and Stolz's differentials with regard to* X *at* (X, Y) *and, moreover, Stolz's differentials with respect to all variables if* (X, Y) *belongs to the side surface of* D, *and satisfy at* (X, Y) *differential inequalities*

$$\begin{aligned} &u^j_{x_l} \leqslant f^j_l(X, Y, u^1, \ldots, u^m, u^j_{y_1}, \ldots, u^j_{y_n}) \\ &v^j_{x_l} \geqslant f^j_l(X, Y, v^1, \ldots, v^m, v^j_{y_1}, \ldots, v^j_{y_n})\,. \end{aligned} \quad (l = 1, 2, \ldots, p) \tag{60.4}$$

This being assumed, inequality

$$U(X, Y) \leqslant V(X, Y)$$

holds true in the pyramid (58.1).

Proof. Proceeding like as in the proof of Theorem 58.1 define, for $\Lambda = (\lambda_1, \ldots, \lambda_p)$ satisfying (58.5), $\widetilde{U}(x, Y; \Lambda)$, $\widetilde{V}(x, Y; \Lambda)$ and $F^i(x, Y, U, Q; \Lambda)$ by formulas (58.6) and (58.9) respectively. Then $\widetilde{U}(x, Y; \Lambda) = \big(\widetilde{u}^1(x, Y; \Lambda), \ldots, \widetilde{u}^m(x, Y; \Lambda)\big)$ and $\widetilde{V}(x, Y; \Lambda) = \big(\widetilde{v}^1(x, Y; \Lambda), \ldots$ $\ldots, \widetilde{v}^m(x, Y; \Lambda)\big)$ are continuous in the pyramid (58.7), where γ/λ satisfies (58.8) and the functions F^i satisfy condition W_+ with regard to U. By (58.9) and (60.1), we have

$$F^i(x, Y, U, Q; \Lambda) - F^i(x, Y, \overline{U}, \overline{Q}; \Lambda) \leqslant \lambda\sigma_i(\lambda x, U - \overline{U}) + \lambda L \sum_{k=1}^{n} |q_k - \overline{q}_k| \qquad (i = 1, 2, \ldots, m)\,,$$

whenever $U \geqslant \overline{U}$. Notice that for the comparison system of type I with right-hand sides $\lambda\sigma_i(\lambda t, U)$ the right-hand maximum solution through the origin is, by Theorem 36.1, $\Omega(\lambda t; 0)$ and, therefore, by (60.2), it is identically zero. In virtue of (60.3), the functions $\widetilde{U}$ and $\widetilde{V}$ satisfy initial inequality

$$\widetilde{U}(0, Y; \Lambda) \leqslant \widetilde{V}(0, Y; \Lambda)\,.$$

Denote by D_λ the pyramid (58.7) and put

$$E^i_\lambda = \{(x, Y) \in D_\lambda\colon\ \widetilde{u}^i(x, Y; \Lambda) > \widetilde{v}^i(x, Y; \Lambda)\} \quad (i = 1, 2, \ldots, m)\,.$$

Fix an index j and let $(x, Y) \in E^i_\lambda$; then, obviously, we have $(X_0 + \Lambda x, Y) \in G^j$ and hence u^j and v^j have at $(X_0 + \Lambda x, Y)$ that regularity which was assumed at points of G^j and they satisfy inequalities (60.4) at $(X_0 + \Lambda x, Y)$. From this we infer that, for $(x, Y) \in E^j_\lambda$, the functions $\tilde{u}^j(x, Y; \Lambda)$ and $\tilde{v}^j(x, Y; \Lambda)$ have at (x, Y) the regularity required in Theorem 59.1 and that they satisfy differential inequalities

$$\tilde{u}^j_x \leqslant F^j(x, Y, \tilde{U}, \tilde{u}^j_Y; \Lambda), \qquad \tilde{v}^j_x \geqslant F^j(x, Y, \tilde{V}, \tilde{v}^j_Y; \Lambda)$$

at points of E^j_λ. Thus we see that, for Λ subject to conditions (58.5), the functions $\tilde{U}(x, Y; \Lambda)$, $\tilde{V}(x, Y; \Lambda)$ and $F^i(x, Y, U, Q; \Lambda)$ satisfy all the assumptions of Theorem 59.1 in the pyramid (58.7). Hence we have in this pyramid

$$\tilde{U}(x, Y; \Lambda) \leqslant \tilde{V}(x, Y; \Lambda)$$

and in particular, by (58.8),

$$\tilde{U}(1, Y; \Lambda) \leqslant \tilde{V}(1, Y; \Lambda). \tag{60.5}$$

Now let (X, Y) be an arbitrary point in the pyramid (58.1); then $\Lambda = X - X_0$ satisfies conditions (58.5) and, by (58.6) and (60.5), we get

$$U(X, Y) = \tilde{U}(1, Y; X - X_0) \leqslant \tilde{V}(1, Y; X - X_0) = V(X, Y),$$

what was to be proved.

Since non-overdetermined systems of equations or inequalities are particular cases of overdetermined ones, from now on we will formulate and prove theorems only for overdetermined systems.

From Theorem 60.1 immediately follows the next corollary on the right-sided uniqueness of the solution of the Cauchy problem.

COROLLARY 60.1. *If the right-hand members of the system of equations*

$$u^i_{x_l} = f^i_l(X, Y, u^1, \ldots, u^m, u^i_{y_1}, \ldots, u^i_{y_n}) \tag{60.6}$$
$$(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p)$$

satisfy assumptions of Theorem 60.1, *then the Cauchy problem for system* (60.6), *with initial data set on* $X = X_0$, *admits at most one solution of class* $\mathfrak{D}$ (see § 37) *in the pyramid* (58.1).

§ 61. Comparison systems of first order partial differential equations. A system of equations

$$v^i_{\xi_l} = h^i_l(\xi_1, \ldots, \xi_p, Y, v^1, \ldots, v^m, v^i_{y_1}, \ldots, v^i_{y_n}) \tag{61.1}$$
$$(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p)$$

will be called *comparison system of partial differential equations* if the following conditions are satisfied:

1° $h_l^i(\Xi, Y, V, Q)$ $(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p)$ are defined and non-negative for $V \geqslant 0$ and $Q \geqslant 0$ and for (Ξ, Y) in the pyramid

$$(61.2)\qquad \begin{aligned} &0 \leqslant \xi_l, \quad \sum_{j=1}^{p} \xi_j < \gamma \quad (l = 1, 2, \ldots, p),\\ &|y_k - \mathring{y}_k| \leqslant a_k - L\sum_{j=1}^{p} \xi_j \quad (k = 1, 2, \ldots, n), \end{aligned}$$

where $0 \leqslant L < +\infty$, $0 < a_k < +\infty$, $\gamma \leqslant \min_k (a_k/L)$;

2° for every fixed l the functions $h_l^i(\Xi, Y, V, Q)$ $(i = 1, 2, \ldots, m)$ satisfy condition W_+ with respect to V;

3° inequalities

$$(61.3)\qquad h_l^i(\Xi, Y, V, Q) - h_l^i(\Xi, Y, \widetilde{V}, \widetilde{Q}) \leqslant \sigma_i\Big(\sum_{r=1}^{p} \xi_r, V - \widetilde{V}\Big) + L\sum_{k=1}^{n} |q_k - \widetilde{q}_k|$$
$$(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p)$$

are satisfied whenever $V \geqslant \widetilde{V}$, where $\sigma_i(t, V)$ are the right-hand sides of a comparison system of type I (see § 14) with $\sigma_i(t, 0) \equiv 0$ $(i = 1, 2, \ldots, m)$ and with the right-hand maximum solution through the origin $\Omega(t; 0) \equiv 0$.

By a *solution of the comparison system* (61.1) we will mean a sequence of non-negative functions $V(\Xi, Y) = \big(v^1(\Xi, Y), \ldots, v^m(\Xi, Y)\big)$ of class $\mathfrak{D}$ in the pyramid (61.2) (see § 37), satisfying equations (61.1), and such that

$$(61.4)\qquad v_Y^i(\Xi, Y) \geqslant 0 \quad (i = 1, 2, \ldots, m).$$

Using the above defined comparison system we will prove the following theorem on absolute value estimates:

THEOREM 61.1. *Let a comparison system of partial differential equations* (61.1) *be given. Suppose that the functions* $U(X, Y) = \big(u^1(X, Y), \ldots$ $\ldots, u^m(X, Y)\big)$ *are of class* $\mathfrak{D}$ (see § 37) *in the pyramid*

$$(61.5)\qquad \sum_{l=1}^{p} |x_l - \mathring{x}_l| < \gamma, \quad |y_k - \mathring{y}_k| \leqslant a_k - L\sum_{l=1}^{p} |x_l - \mathring{x}_l| \quad (k = 1, 2, \ldots, n)$$

and satisfy differential inequalities

$$(61.6)\qquad |u_{x_l}^i| \leqslant h_l^i(|X - X_0|, Y, |U|, |u_Y^i|) \quad (i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p),$$

where $X_0 = (\mathring{x}_1, \ldots, \mathring{x}_p)$. *Let finally* $V(\Xi, Y) = \big(v^1(\Xi, Y), \ldots, v^m(\Xi, Y)\big)$ *be a solution of the comparison system* (61.1) *such that*

$$|U(X_0, Y)| \leqslant V(0, Y). \tag{61.7}$$

Under these assumptions we have in the pyramid (61.5)

$$|U(X, Y)| \leqslant V(|X - X_0|, Y). \tag{61.8}$$

Proof. It is clear that the assumptions of Theorem 61.1 are invariant under the transformation

$$\bar{x}_l - \mathring{x}_l = \varepsilon_l(x_l - \mathring{x}_l) \qquad (l = 1, 2, \ldots, p),$$

where $|\varepsilon_l| = 1$. Hence, it suffices to prove (61.8) in the right-hand pyramid (58.1). Put

$$\bar{U}(\Xi, Y) = |U(X_0 + \Xi, Y)|, \qquad \bar{h}_l^i(\Xi, Y, V, Q) = h_l^i(\Xi, Y, V, |Q|) \tag{61.9}$$
$$(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p).$$

It is obvious, by (61.3), that

$$\bar{h}_l^i(\Xi, Y, V, Q) - \bar{h}_l^i(\Xi, Y, \tilde{V}, \tilde{Q}) \leqslant \sigma_i\Big(\sum_{r=1}^{p} \xi_r, V - \tilde{V}\Big) + L \sum_{k=1}^{n} |q_k - \tilde{q}_k|$$
$$(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p),$$

whenever $V \geqslant \tilde{V}$. By (61.7), we have

$$\bar{U}(0, Y) \leqslant V(0, Y).$$

Denoting the pyramid (61.2) by D, put

$$G^i = \{(\Xi, Y) \in D\colon \bar{u}^i(\Xi, Y) > v^i(\Xi, Y)\} \qquad (i = 1, 2, \ldots, m).$$

Fix an index j and suppose that $(\Xi^*, Y^*) \in G^j$. Since $u^j(X, Y)$ is of class $\mathfrak{D}$ in the pyramid (58.1) and for $(\Xi^*, Y^*) \in G^j$ we have

$$|u^j(X_0 + \Xi^*, Y^*)| = \bar{u}^j(\Xi^*, Y^*) > v^j(\Xi^*, Y^*) \geqslant 0,$$

it follows that the function $\bar{u}^j(\Xi, Y)$ has at (Ξ^*, Y^*) first derivatives with respect to Y and Stolz's differential with regard to Ξ and, moreover, Stolz's differential with respect to all variables if (Ξ^*, Y^*) belongs to the side surface of D. Further we have at $(\Xi^*, Y^*) \in G^j$

$$\bar{u}^j_{x_l} \leqslant |u^j_{x_l}| \qquad (l = 1, 2, \ldots, p), \qquad |\bar{u}^j_Y| = |u^j_Y|.$$

Hence, by (61.6) and (61.9), we get for $(\Xi^*, Y^*) \in G^j$

$$\bar{u}^j_{x_l}(\Xi^*, Y^*) \leqslant \bar{h}_l^j\big(\Xi^*, Y^*, \bar{U}(\Xi^*, Y^*), \bar{u}^j_Y(\Xi^*, Y^*)\big) \qquad (l = 1, 2, \ldots, p).$$

On the other hand, $V(\Xi, Y)$ being a solution of system (61.1) we have, by (61.4) and (61.9),

$$v^j_{x_l}(\Xi^*, Y^*) = \bar{h}^j_l(\Xi^*, Y^*, V(\Xi^*, Y^*), v^j_Y(\Xi^*, Y^*)) \qquad (l = 1, 2, \ldots, p).$$

Thus we see that the functions $\bar{U}$, V and h^i_l satisfy all the assumptions of Theorem 60.1 in the pyramid (61.2) and therefore inequality

$$\bar{U}(\Xi, Y) \leqslant V(\Xi, Y)$$

holds true in the pyramid (61.2). But this is equivalent with (61.8) in the right-hand pyramid (58.1), what was to be proved.

§ 62. Estimates of solutions of first order partial differential equations and a uniqueness criterion. In this section we deal with analogues of Theorems 37.1 and 38.1 in the case when, instead of a comparison system of ordinary differential equations, we use a comparison system of partial differential equations. The next theorem is an immediate consequence of Theorem 61.1.

THEOREM 62.1. *Let the right-hand sides* $f^i_l(X, Y, U, Q)$ $(i = 1, 2, \ldots, m;$ $l = 1, 2, \ldots, p)$ *of system* (60.6) *be defined in a region whose projection on the space of points* (X, Y) *contains the pyramid* (61.5). *Suppose the inequalities*

$$|f^i_l(X, Y, U, Q)| \leqslant h^i_l(|X - X_0|, Y, |U|, |Q|)$$
$$(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p)$$

to be satisfied, where $h^i_l(\Xi, Y, V, Q)$ *are the right-hand sides of a comparison system of partial differential equations* (see § 61). *Let* $U(X, Y) = (u^1(X, Y), \ldots, u^m(X, Y))$ *be a solution of system* (60.6) *of class* $\mathfrak{D}$ (see § 37) *in the pyramid* (61.5). *Suppose that* $V(\Xi, Y) = (v^1(\Xi, Y), \ldots, v^m(\Xi, Y))$ *is a solution of the comparison system* (61.1) (see § 61) *such that*

$$|U(X_0, Y)| \leqslant V(0, Y).$$

Under these assumptions we have

$$|U(X, Y)| \leqslant V(|X - X_0|, Y)$$

in the pyramid (61.5).

The example we give below shows that, in general, the estimate obtained by means of Theorem 62.1 is sharper than that given in Theorem 37.1.

EXAMPLE. Consider an equation

$$u_x = f(x, y, u, u_y) \tag{62.1}$$

and let its right-hand side be defined in a region whose projection on the plane of points (x, y) contains the pyramid

$$(62.2)\qquad |x-x_0| < \gamma\,,\qquad \left|y-\frac{\pi}{4}\right| \leqslant \frac{\pi}{4} - L|x-x_0|\,,\qquad \gamma \leqslant \frac{\pi}{4L}\,.$$

Suppose that

$$(62.3)\qquad |f(x, y, u, q)| \leqslant K|u| + L|q| + C\,,$$

where $K > 0$, $C \geqslant 0$. Let $u(x, y)$ be a solution of (62.1) of class $\mathfrak{D}$ (see § 37) in the pyramid (62.2) and satisfying the initial condition

$$(62.4)\qquad u(x_0, y) = \sin y\,.$$

It follows from (62.4) that

$$(62.5)\qquad |u(x_0, y)| \leqslant \sup_{|y-\pi/4|\leqslant\pi/4} |u(x_0, y)| = 1\,.$$

If, in order to get an estimate of $|u(x, y)|$, we want to apply Theorem 37.1, then the comparison equation of type I (see § 14) is

$$\frac{dv}{dt} = Kv + C$$

and its only solution through $(0, 1)$ is

$$\omega(t) = e^{Kt}\left(1 + \frac{C}{K}\right) - \frac{C}{K}\,.$$

Hence, by Theorem 37.1, we get the estimate

$$(62.6)\qquad |u(x, y)| \leqslant e^{K|x-x_0|}\left(1 + \frac{C}{K}\right) - \frac{C}{K}$$

in the pyramid (62.2). Now, if we apply Theorem 62.1, the comparison partial differential equation is

$$v_\xi = Kv + Lv_y + C$$

and its only solution $v(\xi, y)$ in the pyramid

$$0 \leqslant \xi < \gamma\,,\qquad \left|y-\frac{\pi}{4}\right| \leqslant \frac{\pi}{4} - L\xi\,,$$

satisfying the initial condition

$$v(0, y) = |u(x_0, y)| = \sin y\,,$$

is

$$v(\xi, y) = e^{K\xi}\left[\sin(y + L\xi) + \frac{C}{K}\right] - \frac{C}{K}\,.$$

Therefore, by Theorem 62.1, we obtain the estimate which is obviously sharper than the estimate (62.6).

THEOREM 62.2. *Suppose the right-hand sides of system* (60.6) *and of system*

$$(62.7)\qquad u^i_{x_l} = g^i_l(X, Y, U, u^i_Y) \qquad (i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p)$$

are defined in a region, whose projection on the space of points (X, Y) *contains the pyramid* (61.5), *and satisfy the inequalities*

$$|f^i_l(X, Y, U, Q) - g^i_l(X, Y, \tilde{U}, \tilde{Q})| \leqslant h^i_l(|X-X_0|, Y, |U-\tilde{U}|, |Q-\tilde{Q}|)$$
$$(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p),$$

where $h^i_l(\Xi, Y, V, Q)$ *are the right-hand sides of a comparison system of partial differential equations* (see § 61). *Let* $\tilde{U}(X, Y)$ *and* $\tilde{\tilde{U}}(X, Y)$ *be two solutions of system* (60.6) *and of system* (62.7) *respectively, of class* $\mathfrak{D}$ (see § 37) *in the pyramid* (61.5). *Suppose finally that* $V(\Xi, Y)$ *is a solution of the comparison system* (61.1) *such that*

$$|\tilde{U}(X_0, Y) - \tilde{\tilde{U}}(X_0, Y)| \leqslant V(0, Y).$$

This being assumed, we have

$$|\tilde{U}(X, Y) - \tilde{\tilde{U}}(X, Y)| \leqslant V(|X-X_0|, Y)$$

in the pyramid (61.5).

Proof. Theorem 62.2 follows from Theorem 61.1 when we put there

$$U(X, Y) = \tilde{U}(X, Y) - \tilde{\tilde{U}}(X, Y).$$

From the last theorem we derive the following uniqueness criterion.

COROLLARY 62.1. *Suppose the right-hand sides of system* (60.6) *are defined in a region whose projection on the space of points* (X, Y) *covers the pyramid* (61.5), *and satisfy the inequalities*

$$|f^i_l(X, Y, U, Q) - f^i_l(X, Y, \tilde{U}, \tilde{Q})| \leqslant h^i_l(|X-X_0|, Y, |U-\tilde{U}|, |Q-\tilde{Q}|)$$
$$(i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p),$$

where $h^i_l(\Xi, Y, V, Q)$ *are the right-hand members of a comparison system of partial differential equations* (see § 61). *Assume that*

$$(62.8)\qquad h^i_l(\Xi, Y, 0, 0) \equiv 0 \qquad (i = 1, 2, \ldots, m;\ l = 1, 2, \ldots, p).$$

This being supposed, the Cauchy problem for system (60.6) *with initial data given on* $X = X_0$ *admits at most one solution of class* $\mathfrak{D}$ (see § 37) *in the pyramid* (61.5).

Proof. Observe first that, by (62.8), $V(\Xi, Y) \equiv 0$ is a solution of the comparison system (61.1), satisfying the initial condition $V(0, Y) = 0$. Hence, if $\tilde{U}(X, Y)$ and $\tilde{\tilde{U}}(X, Y)$ are two solutions of system (60.6), of class $\mathfrak{D}$ in the pyramid (61.5) and satisfying the same initial conditions, i.e.

$$\tilde{U}(X_0, Y) - \tilde{\tilde{U}}(X_0, Y) = 0\,,$$

then, by Theorem 62.2, we have

$$\tilde{U}(X, Y) - \tilde{\tilde{U}}(X, Y) \equiv 0$$

in the pyramid (61.5), what was to be proved.

CHAPTER X

SECOND ORDER PARTIAL DIFFERENTIAL INEQUALITIES OF PARABOLIC TYPE

In this chapter we investigate systems of parabolic partial differential inequalities of the form (see [55])

$$u_t^i \leqslant f^i(t, x_1, \ldots, x_n, u^1, \ldots, u^m, u^i_{x_1}, \ldots, u^i_{x_n}, u^i_{x_1x_1}, u^i_{x_1x_2}, \ldots, u^i_{x_nx_n})$$
$$(i = 1, 2, \ldots, m).$$

We also discuss maximum solution and Chaplygin's method for parabolic equations (see [26]).

We use here notions and assumptions introduced in Chapter VIII.

§ 63. Strong partial differential inequalities of parabolic type. In this section we give a generalization of the Nagumo-Westphal theorem. We first recall assumptions introduced in § 47.

ASSUMPTIONS A. *A region $D \subset (t, x_1, \ldots, x_n)$ of type C* (see § 33) *being given let the functions $a^i(t, X)$ $(i = 1, 2, \ldots, m)$ be defined and non-negative on its side surface Σ. Denote by Σ_{a^i} the subset of Σ on which $a^i(t, X) \neq 0$. For every $(t, X) \in \Sigma_{a^i}$, let a direction $l^i(t, X)$ $(i = 1, 2, \ldots, m)$ be given, so that l^i is orthogonal to the t-axis and some segment starting at (t, X) of the straight half-line from (t, X) in the direction l^i is contained in the closure of D.*

A parabolic and regular or Σ_a-regular solution of a system of differential inequalities is defined in the same way as it was for a system of equations in §§ 46 and 47.

THEOREM 63.1. *Assume the functions $f^i(t, X, U, Q, R) = f^i(t, x_1, \ldots, x_n, u^1, \ldots, u^m, q_1, \ldots, q_n, r_{11}, r_{12}, \ldots, r_{nn})$ $(i = 1, 2, \ldots, m)$ to be defined for $(t, X) \in D$ of type C* (see § 33) *and for arbitrary U, Q, R and to satisfy condition W_+ with respect to U* (see § 4). *Let the functions $a^i(t, X)$ and the directions $l^i(t, X)$ $(i = 1, 2, \ldots, m)$ satisfy Assumptions A on the side surface of Σ. Suppose $\beta^i(t, X)$ $(i = 1, 2, \ldots, m)$ are defined and positive on Σ_{a^i}. Let $U(t, X) = (u^1(t, X), \ldots, u^m(t, X))$ and $V(t, X) = (v^1(t, X), \ldots, v^m(t, X))$*

be Σ_a-regular (see § 47) *in D and suppose that every function f^i is elliptic with respect to the sequence $U(t, X)$* (see § 46). *Put*

$$G^i = \{(t, X) \in D\colon\ U(t, X) \overset{i}{\leqslant} V(t, X)\} \qquad (i = 1, 2, \dots, m)$$

and suppose that, for every fixed j, we have

$$u_t^j(t^*, X^*) < f^j\big(t^*, X^*, U(t^*, X^*), u_X^j(t^*, X^*), u_{XX}^j(t^*, X^*)\big)\,, \tag{63.1}$$

$$v_t^j(t^*, X^*) \geqslant f^j\big(t^*, X^*, V(t^*, X^*), v_X^j(t^*, X^*), v_{XX}^j(t^*, X^*)\big)\,, \tag{63.2}$$

whenever $(t^*, X^*) \in G^j$. Suppose finally that the initial inequalities

$$U(t_0, X) < V(t_0, X) \qquad \textit{for} \qquad X \in S_{t_0} \tag{63.3}$$

and boundary inequalities of first type

$$\begin{aligned} &\beta^i(t, X)[u^i(t, X) - v^i(t, X)] - a^i(t, X)\frac{d[u^i - v^i]}{dl^i} < 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \textit{for} \quad (t, X) \in \Sigma_{a^i}, \\ &\qquad\qquad u^i(t, X) - v^i(t, X) < 0 \quad \textit{for} \quad (t, X) \in \Sigma - \Sigma_{a^i} \\ &\qquad\qquad\qquad\qquad\qquad\qquad (i = 1, 2, \dots, m) \end{aligned} \tag{63.4}$$

hold true.

Under the above assumptions we have

$$U(t, X) < V(t, X) \tag{63.5}$$

in D.

Proof. Since the set of points (t_0, X), such that $X \in S_{t_0}$, is compact, there is, by (63.3) and by the continuity, a $\tilde{t}$ $(t_0 < \tilde{t} < t_0 + T)$, so that (63.5) holds true in the intersection of $\bar{D}$ with the zone $t_0 \leqslant t < \tilde{t}$. Denote by t^* the least upper bound of such $\tilde{t}$. We have to prove that $t^* = t_0 + T$. Suppose the contrary, i.e. $t^* < t_0 + T$. Then we have in $\bar{D}$

$$U(t, X) \leqslant V(t, X) \qquad \text{for} \qquad t_0 \leqslant t \leqslant t^* \tag{63.6}$$

and for some index j and some $X^* \in S_{t^*}$

$$u^j(t^*, X^*) = v^j(t^*, X^*)\,. \tag{63.7}$$

Indeed, by the definition of t^*, inequalities

$$U(t, X) < V(t, X)$$

hold true in $\bar{D}$ for $t_0 \leqslant t < t^*$. Now, for any point $(t^*, X) \in \bar{D}$, there is—by property (c) of the region D of type C (see § 33)—a sequence $(t_\nu, X_\nu) \in \bar{D}$, so that $t_0 < t_\nu < t^*$ and $(t_\nu, X_\nu) \to (t^*, X)$. Since

$$U(t_\nu, X_\nu) < V(t_\nu, X_\nu)\,,$$

it follows, by the continuity, that

$$U(t^*, X) \leqslant V(t^*, X).$$

Thus inequalities (63.6) are proved. If (63.7) were not true, we would have, for every $X \in S_{t^*}$,

$$U(t^*, X) < V(t^*, X),$$

and hence, the set of points (t^*, X), such that $X \in S_{t^*}$, being compact, inequalities (63.5) would be true, by continuity, in $\bar{D}$ for $t_0 \leqslant t < t^{**}$, where t^{**} is some number greater than t^*. But, this contradicts the definition of t^*. From (63.6) and (63.7) it follows that

$$\max_{X \in S_{t^*}} [u^j(t^*, X) - v^j(t^*, X)] = u^j(t^*, X^*) - v^j(t^*, X^*) = 0$$

and hence, by (63.4) and by Lemma 47.1, we conclude that (t^*, X^*) is an interior point of D. Moreover, by (63.6) and (63.7), we have $(t^*, X^*) \in G^j$, and consequently inequalities (63.1) and (63.2) hold true. The difference $u^j(t^*, X) - v^j(t^*, X)$ is of class C^2 and attains its maximum at the interior point X^*. Therefore, we have

$$u^j_X(t^*, X^*) = v^j_X(t^*, X^*) \tag{63.8}$$

and the quadratic form in $\lambda_1, \ldots, \lambda_n$

$$\sum_{l,k=1}^{n} [u^j_{x_l x_k}(t^*, X^*) - v^j_{x_l x_k}(t^*, X^*)]\lambda_l \lambda_k \quad \text{is negative.} \tag{63.9}$$

Now, from (63.1), (63.2) and (63.8) it results that

$$u^j_t(t^*, X^*) - v^j_t(t^*, X^*) < f^j\big(t^*, X^*, U(t^*, X^*), u^j_X(t^*, X^*), u^j_{XX}(t^*, X^*)\big) - \\ - f^j\big(t^*, X^*, V(t^*, X^*), u^j_X(t^*, X^*), v^j_{XX}(t^*, X^*)\big).$$

By (63.6), (63.7) and by the condition W_+ (see § 4), we get from the last inequality

$$u^j_t(t^*, X^*) - v^j_t(t^*, X^*) < f^j\big(t^*, X^*, U(t^*, X^*), u^j_X(t^*, X^*), u^j_{XX}(t^*, X^*)\big) - \\ - f^j\big(t^*, X^*, U(t^*, X^*), u^j_X(t^*, X^*), v^j_{XX}(t^*, X^*)\big).$$

Owing to the ellipticity of f^j (see § 46) with regard to $U(t, X)$ and by (63.9), the right-hand side of the last inequality is non-positive and consequently we have

$$u^j_t(t^*, X^*) - v^j_t(t^*, X^*) < 0. \tag{63.10}$$

On the other hand, the function

$$u^j(t, X^*) - v^j(t, X^*)$$

of one variable t attains, by (63.6) and (63.7), its maximum at the right-hand extremity t^* of the interval $[t_0, t^*]$. Hence it follows that

$$u_t^j(t^*, X^*) - v_t^j(t^*, X^*) \geqslant 0 ,$$

what contradicts (63.10). This completes the proof.

Remark. Theorem 63.1 as well as the next Theorem 63.2 are true if, instead of the ellipticity with regard to $U(t, X)$, we assume the ellipticity with respect to $V(t, X)$.

Now we are going to prove a similar theorem with boundary inequalities of second type, i.e. with inequalities (63.4) without the assumption that $\beta^i(t, X)$ be positive. Like in § 53 we will assume the existence of sign-stabilizing factors.

THEOREM 63.2. *Let the assumptions of Theorem 63.1 be satisfied with the exception of $\beta^i(t, X)$ $(i = 1, 2, \ldots, m)$ being positive. Suppose, instead, that there exist sign-stabilizing factors, i.e. positive functions $K^i(t, X)$ $(i = 1, 2, \ldots, m)$ of class C^2 in the closure of D, such that*

$$\tilde{\beta}^i(t, X) > 0 \quad \textit{for} \quad (t, X) \in \Sigma_{a^i} \quad (i = 1, 2, \ldots, m) ,$$

where

$$(63.11) \quad \tilde{\beta}^i(t, X) = \beta^i(t, X) - a^i(t, X)[K^i(t, X)]^{-1}\frac{dK^i}{dl^i} \quad \textit{for} \quad (t, X) \in \Sigma_{a^i}$$
$$(i = 1, 2, \ldots, m) .$$

Under these assumptions inequalities (63.5) *hold true in D.*

Proof. We put, like in § 53,

$$\tilde{u}^i(t, X) = u^i(t, X)[K^i(t, X)]^{-1} , \quad \tilde{v}^i(t, X) = v^i(t, X)[K^i(t, X)]^{-1}$$
$$(i = 1, 2, \ldots, m) .$$

The new functions $\tilde{U}(t, X) = (\tilde{u}^1(t, X), \ldots, \tilde{u}^m(t, X))$, $\tilde{V}(t, X) = (\tilde{v}^1(t, X), \ldots, \tilde{v}^m(t, X))$ satisfy, by (63.3), initial inequalities

$$\tilde{U}(t_0, X) < \tilde{V}(t_0, X) \quad \text{for} \quad X \in S_{t_0}$$

and, by Lemma 53.1 and by (63.4), boundary inequalities

$$\tilde{\beta}^i(t, X)[\tilde{u}^i(t, X) - \tilde{v}^i(t, X)] - a^i(t, X)\frac{d[\tilde{u}^i - \tilde{v}^i]}{dl^i} < 0 \quad \text{for} \quad (t, X) \in \Sigma_{a^i} ,$$

$$\tilde{u}^i(t, X) - \tilde{v}^i(t, X) < 0 \quad \text{for} \quad (t, X) \in \Sigma - \Sigma_{a^i}$$
$$(i = 1, 2, \ldots, m) ,$$

where $\tilde{\beta}^i$ are defined by formulas (63.11) and are supposed positive. By Lemma 53.2, every function $\tilde{f}^i(t, X, U, Q, R)$, defined by formula (53.6), is elliptic with respect to $\widetilde{U}(t, X)$; moreover, $\widetilde{U}(t, X)$ and $\widetilde{V}(t, X)$ are Σ_a-regular in D and $\widetilde{f}^i$ satisfy condition W_+ with regard to U. Put

$$\widetilde{G}^i = \{(t, X) \in D:\ \widetilde{U}(t, X) \overset{i}{\leqslant} \widetilde{V}(t, X)\} \quad (i = 1, 2, \ldots, m)\,.$$

Fix an index j and let $(t^*, X^*) \in \widetilde{G}^j$; then obviously $(t^*, X^*) \in G^j$ and, by (63.1) and (63.2), we have (see Lemma 53.2)

$$\widetilde{u}^j_t(t^*, X^*) < \widetilde{f}^j\big(t^*, X^*, \widetilde{U}(t^*, X^*), \widetilde{u}^j_X(t^*, X^*), \widetilde{u}^j_{XX}(t^*, X^*)\big)\,,$$

$$\widetilde{v}^j_t(t^*, X^*) \geqslant \widetilde{f}^j\big(t^*, X^*, \widetilde{V}(t^*, X^*), \widetilde{v}^j_X(t^*, X^*), \widetilde{v}^j_{XX}(t^*, X^*)\big)\,.$$

Thus $\widetilde{U}$, $\widetilde{V}$, $\widetilde{f}^i$ and $\widetilde{\beta}^i$ satisfy all the assumptions of Theorem 63.1 and hence we have in D

$$\widetilde{U}(t, X) < \widetilde{V}(t, X)\,,$$

what implies (63.5).

We close this section by proving an analogue of Theorem 63.1 with a different kind of non-linear boundary inequalities (see [32]).

THEOREM 63.3. *Let all the assumptions of Theorem* 63.1 *be satisfied with* $a^i(t, X) \equiv 1$ $(i = 1, 2, \ldots, m)$ *and with the boundary inequalities* (63.4) *substituted by*

$$\text{(63.12)}\qquad \begin{aligned} \varphi^i(u^1, \ldots, u^m) &< \frac{du^i}{dl^i} \\ \varphi^i(v^1, \ldots, v^m) &\geqslant \frac{dv^i}{dl^i} \end{aligned} \qquad \textit{on } \Sigma \quad (i = 1, 2, \ldots, m)\,,$$

where the functions $\varphi^i(u^1, \ldots, u^m)$ $(i = 1, 2, \ldots, m)$ *satisfy condition* W_- (see § 4).

This being assumed, inequalities (63.5) *hold true in* D.

Proof. Notice that, in the proof of Theorem 63.1, boundary inequalities (63.4) were taken advantage of merely to show that if for some index j and some point $(t^*, X^*) \in \bar{D}$ we have (63.6) and (63.7), then (t^*, X^*) is an interior point of D. Hence Theorem 63.3 will be proved if we show that (63.6), (63.7) and (63.12) imply that (t^*, X^*) is an interior point of D. Suppose that $(t^*, X^*) \in \Sigma$. Now, from (63.6) and (63.7) it follows that the function

$$\psi(\tau) = u^j\big(t^*, X^* + \tau \operatorname{vers} l^j(t^*, X^*)\big) - v^j\big(t^*, X^* + \tau \operatorname{vers} l^j(t^*, X^*)\big)$$

—which, by Assumption A, is defined for non-negative τ sufficiently close to zero—attains its maximum at $\tau = 0$. Hence we get that

$$\text{(63.13)}\qquad \psi'(0) = \frac{d[u^j - v^j]}{dl^j}\bigg|_{(t^*, X^*)} \leqslant 0\,.$$

On the other hand, inequalities (63.6) and (63.7) and condition W_- imply that

$$\varphi^j\big(U(t^*, X^*)\big) \geqslant \varphi^j\big(V(t^*, X^*)\big) .$$

From the last inequality and by (63.12) we obtain

$$\frac{d[u^j - v^j]}{dl^j}\bigg|_{(t^*, X^*)} > 0 ,$$

what contradicts (63.13). This contradiction completes the proof.

§ 64. Weak partial differential inequalities of parabolic type. In order to obtain a theorem on weak inequalities we apply in the present section methods similar to those used in § 59. In particular, we will have to introduce more restrictive assumptions than in Theorem 63.1, which imply (see Corollary 64.1) uniqueness of solution of the corresponding mixed problem.

THEOREM 64.1. *Let the functions* $f^i(t, X, U, Q, R) = f^i(t, x_1, \ldots, x_n, u^1, \ldots, u^m, q_1, \ldots, q_n, r_{11}, r_{12}, \ldots, r_{nn})$ $(i = 1, 2, \ldots, m)$ *be defined for* $(t, X) \in D$ *of type* C (see § 33) *and for arbitrary* U, Q, R *and to satisfy condition* W_+ *with respect to* U (see § 4). *Suppose further that*

$$f^i(t, X, U, Q, R) - f^i(t, X, \tilde{U}, Q, R) \leqslant \sigma_i(t - t_0, U - \tilde{U}) \tag{64.1}$$
$$(i = 1, 2, \ldots, m),$$

whenever $U \geqslant \tilde{U}$, *where* $\sigma_i(t, V)$ *are the right-hand sides of a comparison system of type* I (see § 14). *As to the comparison system we assume that*

$$\sigma_i(t, 0) \equiv 0 \qquad (i = 1, 2, \ldots, m)$$

and that for its right-hand maximum solution through the origin $\Omega(t; 0)$ *we have*

$$\Omega(t; 0) \equiv 0 . \tag{64.2}$$

Let the functions $\alpha^i(t, X)$ *and the directions* $l^i(t, X)$ $(i = 1, 2, \ldots, m)$ *satisfy Assumptions* A (see § 63) *on the side surface* Σ *of* D. *Suppose* $\beta^i(t, X)$ *is positive on* Σ_{α^i} $(i = 1, 2, \ldots, m)$. *Let* $U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big)$ *and* $V(t, X) = \big(v^1(t, X), \ldots, v^m(t, X)\big)$ *be* Σ_α*-regular in* D (see § 47) *and suppose that every function* $f^i(t, X, U, Q, R)$ *is elliptic with regard to* $U(t, X)$ (see § 46). *Assume that the initial inequality*

$$U(t_0, X) \leqslant V(t_0, X) \quad \text{for} \quad X \in S_{t_0} \tag{64.3}$$

and boundary inequalities

$$\beta^i(t,X)[u^i(t,X)-v^i(t,X)]-\alpha^i(t,X)\frac{d[u^i-v^i]}{dl^i}\leqslant 0$$

(64.4) $$\text{for}\quad (t,X)\in\Sigma_{a^i},$$

$$u^i(t,X)-v^i(t,X)\leqslant 0\quad \text{for}\quad (t,X)\in\Sigma-\Sigma_{a^i}\quad (i=1,2,\dots,m)$$

are satisfied. Write

$$E^i=\{(t,X)\in D\colon\ u^i(t,X)>v^i(t,X)\}\quad (i=1,2,\dots,m)$$

and suppose that for every fixed j

(64.5) $$u^j_t(t^*,X^*)\leqslant f^j\big(t^*,X^*,U(t^*,X^*),u^j_X(t^*,X^*),u^j_{XX}(t^*,X^*)\big),$$

(64.6) $$v^j_t(t^*,X^*)\geqslant f^j\big(t^*,X^*,V(t^*,X^*),v^j_X(t^*,X^*),v^j_{XX}(t^*,X^*)\big),$$

whenever $(t^*,X^*)\in E^j$.

This being assumed, we have in D

(64.7) $$U(t,X)\leqslant V(t,X).$$

Proof. Since the assumptions of our theorem are invariant under the mapping $\tau=t-t_0$, we may assume, without loss of generality, that $t_0=0$. Put, for $0\leqslant t<T$,

$$M^i(t)=\max_{X\in S_t}[u^i(t,X)-v^i(t,X)],\qquad \widetilde{M}^i(t)=\max\big(0,M^i(t)\big)$$
$$(i=1,2,\dots,m),$$
$$\widetilde{M}(t)=\big(\widetilde{M}^1(t),\dots,\widetilde{M}^m(t)\big).$$

It is clear that the assertion of our theorem is equivalent with the inequality

(64.8) $$\widetilde{M}(t)\leqslant 0\quad\text{on}\quad [0,T).$$

We are going to prove relation (64.8) by means of the first comparison theorem (see § 14). By (64.3), we have $\widetilde{M}(0)\leqslant 0$ and, by Theorem 33.1, the functions $\widetilde{M}^i(t)$ are continuous on $[0,T)$. Therefore, writing

$$\widetilde{E}^i=\{t\in(0,T)\colon\ \widetilde{M}^i(t)>0\}\quad (i=1,2,\dots,m),$$

inequality (64.8) will be proved by the first comparison theorem (see § 14), if we show that

$$D_-\widetilde{M}^i(t)\leqslant\sigma_i\big(t,\widetilde{M}(t)\big)\quad\text{for}\quad t\in\widetilde{E}^i.$$

Now, fix an index j and let $t^*\in\widetilde{E}^j$. By Theorem 33.1, there is a point $X^*\in S_{t^*}$ such that

(64.9) $$M^j(t^*)=u^j(t^*,X^*)-v^j(t^*,X^*).$$

Since, by the assumption that $t^* \in \widetilde{E}^j$, inequality $\widetilde{M}^j(t^*) > 0$ holds true, we have obviously

$$(64.10) \qquad \widetilde{M}^j(t^*) = M^j(t^*) , \qquad D^- \widetilde{M}^j(t^*) = D^- M^j(t^*)$$

and consequently, by (64.9),

$$(64.11) \qquad \widetilde{M}^j(t^*) = u^j(t^*, X^*) - v^j(t^*, X^*) > 0 .$$

From the last inequality and from (64.4) it follows, by Lemma 47.1, that (t^*, X^*) is an interior point of D. Hence, the function $u^j(t^*, X) - v^j(t^*, X)$ attaining, by (64.9), its maximum at the interior point X^*, we have relations (63.8) and (63.9). By Theorem 33.1 and by (64.10), we have moreover

$$(64.12) \qquad D_- \widetilde{M}^j(t^*) \leqslant u_t^j(t^*, X^*) - v_t^j(t^*, X^*) .$$

Inequality (64.11) implies that $(t^*, X^*) \in E^j$ and consequently, by (64.5), (64.6) and (63.8), we get

$$\begin{aligned}(64.13) \qquad & u_t^j(t^*, X^*) - v_t^j(t^*, X^*) \\ & \leqslant f^j\big(t^*, X^*, U(t^*, X^*), u_X^j(t^*, X^*), u_{XX}^j(t^*, X^*)\big) - \\ & \qquad - f^j\big(t^*, X^*, V(t^*, X^*), u_X^j(t^*, X^*), v_{XX}^j(t^*, X^*)\big) .\end{aligned}$$

Observe that, by the definition of $\widetilde{M}^j(t)$ and by (64.11), (see § 4)

$$U(t^*, X^*) \overset{j}{\leqslant} V(t^*, X^*) + \widetilde{M}(t^*) .$$

By the last inequalities and by condition W_+ (see § 4), it follows from (64.12) and (64.13)

$$\begin{aligned}(64.14) \qquad D_- \widetilde{M}^j(t^*) \leqslant & \big[f^j\big(t^*, X^*, U(t^*, X^*), u_X^j(t^*, X^*), u_{XX}^j(t^*, X^*)\big) - \\ & - f^j\big(t^*, X^*, U(t^*, X^*), u_X^j(t^*, X^*), v_{XX}^j(t^*, X^*)\big)\big] + \\ & + \big[f^j\big(t^*, X^*, V(t^*, X^*) + \widetilde{M}(t^*), u_X^j(t^*, X^*), v_{XX}^j(t^*, X^*)\big) - \\ & - f^j\big(t^*, X^*, V(t^*, X^*), u_X^j(t^*, X^*), v_{XX}^j(t^*, X^*)\big)\big] .\end{aligned}$$

The first difference in brackets is—owing to (63.9) and to ellipticity of f^j with regard to $U(t, X)$—non-positive. To the second difference we apply inequality (64.1) and finally we obtain

$$(64.15) \qquad D_- \widetilde{M}^j(t^*) \leqslant \sigma_j\big(t^*, \widetilde{M}(t^*)\big) .$$

Thus we have shown that inequality (64.15) holds true for any $t^* \in \widetilde{E}^j$; but, this completes the proof.

As an immediate consequence of Theorem 64.1 we obtain the following corollaries.

COROLLARY 64.1 (Uniqueness criterion). *Suppose that the right-hand sides of the system of differential equations*

$$(64.16) \qquad u_t^i = f^i(t, X, U, u_X^i, u_{XX}^i) \quad (i = 1, 2, \dots, m)$$

satisfy all the assumptions of Theorem 64.1. *Then the first mixed problem* (see § 47) *for system* (64.16) *admits in D at most one parabolic, Σ_a-regular* (see §§ 46, 47) *solution.*

COROLLARY 64.2 (Maximum principle). *Let the functions $f^i(t, X, U, Q, R)$ $(i = 1, 2, \dots, m)$ satisfy all the hypotheses of Theorem* 64.1. *Assume that for $U \geqslant 0$ we have*

$$f^i(t, X, U, 0, 0) \leqslant 0 \quad (i = 1, 2, \dots, m).$$

Suppose $U(t, X) = \big(u^1(t, X), \dots, u^m(t, X)\big)$ to be a Σ_a-regular (see § 47) *and parabolic* (see § 46) *solution of the system of differential inequalities*

$$u_t^i \leqslant f^i(t, X, U, u_X^i, u_{XX}^{ii}) \quad (i = 1, 2, \dots, m)$$

in D and to satisfy initial inequalities

$$U(t_0, X) \leqslant M = (m_1, \dots, m_m) \quad \textit{for} \quad X \in S_{t_0},$$

where m_i are non-negative constants, and boundary inequalities

$$\beta^i(t, X)u^i(t, X) - a^i(t, X)\frac{du^i}{dl^i} \leqslant m_i\beta^i(t, X) \quad \textit{for} \quad (t, X) \in \Sigma_{a^i},$$
$$u^i(t, X) \leqslant m_i \quad \textit{for} \quad (t, X) \in \Sigma - \Sigma_{a^i}$$
$$(i = 1, 2, \dots, m),$$

where a^i, l^i satisfy Assumptions A (see § 63) *and β^i are positive.*
Under these assumptions we have in D

$$U(t, X) \leqslant M.$$

Proof. We check immediately that $U(t, X)$ and $V(t, X) = M = \text{const} \geqslant 0$ satisfy all the assumptions of Theorem 64.1.

Remark 64.2 (¹). Theorem 64.1 can be derived from Theorem 63.1 without having recourse to the first comparison theorem (see § 14). In

(¹) This remark is due to P. Besala. Similar arguments were used, in some particular cases, by K. Nickel (see [36]).

this case we use arguments similar to those applied in the proof of Remark 59.1.

The theorem to be proved now involves somewhat less restrictive assumptions under which the first comparison theorem (see § 14), used in the proof of Theorem 64.1, cannot be taken advantage of, whereas the second comparison theorem (see § 14) is applicable.

THEOREM 64.2. *Under the assumptions of Theorem* 64.1 *with inequalities* (64.1) *replaced by*

$$f^i(t, X, U, Q, R) - f^i(t, X, \tilde{U}, Q, R) \leqslant \sigma\big(t - t_0, \max_l (u^l - \tilde{u}^l)\big) \quad (i = 1, 2, \ldots, m) \tag{64.17}$$

for $U \geqslant \tilde{U}$ *and* $t > t_0$, *where* $\sigma(t, y)$ *is the right-hand side of a comparison equation of type* II (see § 14), *inequality* (64.7) *holds true in* D.

Proof. Like in the proof of Theorem 64.1, we assume that $t_0 = 0$. Put, for $0 \leqslant t < T$,

$$\tilde{W}(t) = \max_i \tilde{M}^i(t),$$

where $\tilde{M}^i(t)$ were introduced in the proof of Theorem 64.1. It is obvious that inequality (64.7) is equivalent with

$$\tilde{W}(t) \leqslant 0 \quad \text{on} \quad [0, T). \tag{64.18}$$

Inequality (64.18) will be proved by means of the second comparison theorem (see § 14). By (64.3), we have

$$\tilde{W}(0) \leqslant 0$$

and, by Theorem 33.1, the function $\tilde{W}(t)$ is continuous on $[0, T)$. Therefore, writing

$$E = \{t \in (0, T) \colon \tilde{W}(t) > 0\},$$

inequality (64.18) will be proved, by the second comparison theorem (see § 14), if we show that

$$D_-\tilde{W}(t) \leqslant \sigma\big(t, \tilde{W}(t)\big) \quad \text{for} \quad t \in E.$$

Now, suppose that $t^* \in E$. Obviously there is an index j, so that (see the proof of Theorem 34.1)

$$\tilde{W}(t^*) = \tilde{M}^j(t^*), \quad D_-\tilde{W}(t^*) \leqslant D^-\tilde{M}^j(t^*). \tag{64.19}$$

Since $t^* \in E$, we have, by (64.19), $\widetilde{M}^j(t^*) > 0$, and hence relations (64.10) and (64.11) are satisfied. Therefore, like in the proof of Theorem 64.1, we get inequality (64.13) and consequently, by (64.19), we have

$$D_-\widetilde{W}(t^*) \leqslant \big[f^j\big(t^*, X^*, U(t^*, X^*), u^j_X(t^*, X^*), u^j_{XX}(t^*, X^*)\big) - \\ -f^j\big(t^*, X^*, U(t^*, X^*), u^j_X(t^*, X^*), v^j_{XX}(t^*, X^*)\big)\big] + \\ +\big[f^j(t^*, X^*, V(t^*, X^*)+\widetilde{M}(t^*), u^j_X(t^*, X^*), v^j_{XX}(t^*, X^*)\big) - \\ -f^j\big(t^*, X^*, V(t^*, X^*), u^j_X(t^*, X^*), v^j_{XX}(t^*, X^*)\big)\big]\ .$$

The first difference in brackets is—like in the preceding proof—non-positive, whereas to the second difference we apply inequality (64.17) and get

$$D_-\widetilde{W}(t^*) \leqslant \sigma\big(t^*, \widetilde{W}(t^*)\big)\ ,$$

what was to be proved.

The next corollary is an immediate consequence of Theorem 64.2.

COROLLARY 64.3 (Uniqueness criterion). *If the right-hand sides of the system of equations* (64.16) *satisfy all the assumptions of Theorem* 64.2, *then the first mixed problem* (see § 47) *for the above system admits in D at most one parabolic, Σ_a-regular solution* (see §§ 46,47).

Remark 64.3. Unlike Theorem 64.1, Theorem 64.2 cannot be derived from Theorem 63.1 without having recourse to the second comparison theorem. This depends on the fact that the right-hand side of a comparison equation of type II (see § 14), appearing in inequality (64.17), is not supposed to be continuous for $t = 0$, and consequently Theorem 10.1 can not be applied to its solutions issued from the points $(0, \varepsilon)$.

We turn now to analogues of Theorems 64.1 and 64.2 in the case of boundary inequalities of second type, i.e. when $\beta^i(t, X)$ $(i = 1, 2, \ldots, m)$ are not supposed to be positive. Like in Theorem 63.2 we will have to assume, instead, the existence of sign-stabilizing factors (see § 53).

THEOREM 64.3. *Let the functions $f^i(t, X, U, Q, R)$ $(i = 1, 2, \ldots, m)$ be defined for $(t, X) \in D$ of type C* (see § 33) *and for arbitrary U, Q, R and satisfy condition W_+ with respect to U* (see § 4). *Suppose that, whenever $U \geqslant \overline{U}$, the inequalities*

$$(64.20)\qquad f^i(t, X, U, Q, R) - f^i(t, X, \overline{U}, \overline{Q}, \overline{R}) \\ \leqslant \sigma_i(t-t_0, U-\overline{U}) + \tau_i\Big(t-t_0, \sum_j |q_j-\overline{q}_j| + \sum_{j,k} |r_{jk}-\overline{r}_{jk}|\Big) \\ (i = 1,\ 2, \ldots, m)$$

hold true, where $\sigma_i(t, y_1, \ldots, y_m)$, $\tau_i(t, y)$ are continuous, non-negative and increasing in all variables for $t \geqslant 0$, $y \geqslant 0$, $y_j \geqslant 0$ $(j = 1, 2, \ldots, m)$ and satisfy identities

$$\sigma_i(t, 0) \equiv \tau_i(t, 0) \equiv 0 \qquad (i = 1, 2, \ldots, m)\ .$$

Suppose further that the right-hand maximum solution through the origin of the comparison system

$$\frac{dy_i}{dt} = \sigma_i(t, y_1, \ldots, y_n) + \tau_i(t, y_i) + y_i \qquad (i = 1, 2, \ldots, m)$$

is identically zero. Let the functions $\alpha^i(t, X)$ *and the directions* $l^i(t, X)$ $(i = 1, 2, \ldots, m)$ *satisfy Assumptions* A (see § 63) *on the side surface of* D. *Suppose that* $\beta^i(t, X)$ *is defined on* Σ_{α^i} $(i = 1, 2, \ldots, m)$ *(without being necessarily positive), and there exist sign-stabilizing factors, i.e. positive functions* $K^i(t, X)$ $(i = 1, 2, \ldots, m)$ *of class* C^2 *in the closure of* D, *so that*

$$\widetilde{\beta}^i(t, X) > 0 \quad \textit{for} \quad (t, X) \in \Sigma_{\alpha^i} \quad (i = 1, 2, \ldots, m),$$

where $\widetilde{\beta}^i$ *are given by formulas* (63.11). *Assume, moreover, that*

$$0 < \mu \leqslant K^i(t, X) \leqslant \widetilde{M}, \qquad |K_t^i|, |K_{x_j}^i|, |K_{x_j x_k}^i| \leqslant \widetilde{M}.$$

Let, finally, $U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big)$ *and* $V(t, X) = \big(v^1(t, X), \ldots, v^m(t, X)\big)$ *satisfy assumptions of Theorem* 64.1. *This being assumed, inequality*

(64.21) $$U(t, X) \leqslant V(t, X)$$

holds true in D.

Proof. Like in the proof of Theorem 63.2, we put

$$\widetilde{u}^i(t, X) = u^i(t, X)[K^i(t, X)]^{-1}, \qquad \widetilde{v}^i(t, X) = v^i(t, X)[K^i(t, X)]^{-1} \qquad (i = 1, 2, \ldots, m)$$

and check that the new functions are Σ_α-regular in D and satisfy, by (64.3), initial inequalities

$$\widetilde{U}(t_0, X) \leqslant \widetilde{V}(t_0, X) \quad \text{for} \quad X \in S_{t_0},$$

and, by (64.4) and by Lemma 53.1, boundary inequalities

$$\widetilde{\beta}^i(t, X)[\widetilde{u}^i(t, X) - \widetilde{v}^i(t, X)] - \alpha^i(t, X)\frac{d[\widetilde{u}^i - \widetilde{v}^i]}{dl^i} \leqslant 0 \quad \textit{for} \quad (t, X) \in \Sigma_{\alpha^i},$$

$$\widetilde{u}^i(t, X) - \widetilde{v}^i(t, X) \leqslant 0 \quad \textit{for} \quad (t, X) \in \Sigma - \Sigma_{\alpha^i} \quad (i = 1, 2, \ldots, m),$$

where $\widetilde{\beta}^i$ are defined by formula (63.11) and are supposed positive. By Lemma 53.2, all functions $\widetilde{f}^i(t, X, U, Q, R)$, defined by formula (53.6), are elliptic with respect to $\widetilde{U}(t, X)$ and satisfy, by (64.20) and by Lemma 53.3, inequalities

$$\widetilde{f}^i(t, X, U, Q, R) - \widetilde{f}^i(t, X, \overline{U}, Q, R) \leqslant \widetilde{\sigma}_i(t - t_0, U - \overline{U})$$

whenever $U \geqslant \bar{U}$, where $\tilde{\sigma}_i(t, y_1, \ldots, y_m)$ are given by formulas (53.11); moreover, by Lemma 53.4, $\sigma_i(t, Y)$ are the right-hand sides of a comparison system of type I (see § 14) and satisfy the assumptions of Theorem 64.1. The functions $\tilde{f}^i$ satisfy condition W_+ with respect to U. Put

$$\tilde{E}^i = \{(t, X) \in D: \tilde{u}^i(t, X) > \tilde{v}^i(t, X)\} \qquad (i = 1, 2, \ldots, m).$$

Fix an index j and let $(t^*, X^*) \in \tilde{E}^j$; then, obviously, $(t^*, X^*) \in E^j$ (see Theorem 64.1) and hence, by (64.5) and (64.6), we have (see Lemma 53.2)

$$\tilde{u}^j_t(t^*, X^*) \leqslant \tilde{f}^j\big(t^*, X^*, \tilde{U}(t^*, X^*), \tilde{u}^j_X(t^*, X^*), \tilde{u}^j_{XX}(t^*, X^*)\big),$$

$$\tilde{v}^j_t(t^*, X^*) \geqslant \tilde{f}^j\big(t^*, X^*, \tilde{V}(t^*, X^*), \tilde{v}^j_X(t^*, X^*), \tilde{v}^j_{XX}(t^*, X^*)\big).$$

Thus we see that $\tilde{u}^i$, $\tilde{v}^i$, $\tilde{f}^i$ and $\tilde{\beta}^i$ satisfy all the hypotheses of Theorem 64.1 and, therefore, we have in D

$$\tilde{U}(t, X) \leqslant \tilde{V}(t, X)$$

what implies (64.21).

In a similar way we derive from Theorem 64.2 the next theorem.

THEOREM 64.4. *Let the assumptions of Theorem* 64.3 *hold true with inequalities* (64.20) *substituted by*

$$f^i(t, X, U, Q, R) - f^i(t, X, \bar{U}, \bar{Q}, \bar{R}) \leqslant \sigma\big(t - t_0, \max_l (u^l - \bar{u}^l)\big) + \\ + \tau\Big(t - t_0, \sum_j |q_j - \bar{q}_j| + \sum_{j,k} |r_{jk} - \bar{r}_{jk}|\Big) \qquad (i = 1, 2, \ldots, m)$$

for $U \geqslant \bar{U}$ *and* $t > t_0$, *where* $\sigma(t, y)$ *and* $\tau(t, y)$ *are continuous, non-negative functions, increasing in all variables for* $t > 0$, $y \geqslant 0$, *such that* $\sigma(t, y) + \tau(t, y) + y$ *is the right-hand side of a comparison equation of type* II (see § 14). *This being supposed, inequality* (64.21) *is satisfied in* D.

We close this section by deriving from Theorem 64.1 (resp. 64.2) a theorem [5] involving in thesis absolute value estimates.

THEOREM 64.5. *Let* $f^i(t, X, U, Q, R)$, $a^i(t, X)$, $l^i(t, X)$ *and* $\beta^i(t, X)$ $(i = 1, 2, \ldots, m)$ *satisfy all the assumptions of Theorem* 64.1 (*resp.* 64.2) *and suppose additionally that*

$$f^i(t, X, -U, -Q, -R) = -f^i(t, X, U, Q, R) \quad (i = 1, 2, \ldots, m). \tag{64.22}$$

Let $U(t, X)$ *and* $V(t, X) \geqslant 0$ *be* Σ_a-*regular in* D (see § 47) *and satisfy initial inequalities*

$$|U(t_0, X)| \leqslant V(t_0, X) \tag{64.23}$$

and boundary inequalities

$$(64.24)\qquad \begin{aligned} &\left|\beta^i(t,X)u^i(t,X)-\alpha^i(t,X)\frac{du^i}{dl^i}\right| \leqslant \beta^i(t,X)v^i(t,X)-\alpha^i(t,X)\frac{dv^i}{dl^i} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \textit{for}\quad (t,X)\in\Sigma_{\alpha^i}, \\ &\qquad\qquad |u^i(t,X)|\leqslant v^i(t,X)\quad \textit{for}\quad (t,X)\in\Sigma-\Sigma_{\alpha^i} \\ &\qquad\qquad\qquad\qquad\qquad\qquad (i=1,2,\ldots,m). \end{aligned}$$

Suppose that all the functions $f^i(t, X, U, Q, R)$ *are elliptic with regard to* $U(t, X)$ (see § 47). *Put*

$$\hat{E}^i = \{(t, X)\in D\colon |u^i(t,X)| > v^i(t,X)\} \qquad (i=1,2,\ldots,m)$$

and assume that, for every fixed j,

$$(64.25)\qquad u^j_t(t^*,X^*) = f^j\big(t^*,X^*,U(t^*,X^*),u^j_X(t^*,X^*),u^j_{XX}(t^*,X^*)\big),$$

$$(64.26)\qquad v^j_t(t^*,X^*) \geqslant f^j\big(t^*,X^*),V(t^*,X^*),v^j_X(t^*,X^*),v^j_{XX}(t^*,X^*)\big)$$

whenever $(t^*, X^*)\in\hat{E}^j$. *This being supposed, inequality*

$$(64.27)\qquad |U(t,X)|\leqslant V(t,X)$$

is satisfied in D.

Proof. If we put

$$E^i_+ = \{(t,X)\in D\colon u^i(t,X) > v^i(t,X)\} \qquad (i=1,2,\ldots,m),$$

then, since $v^i(t,X)\geqslant 0$, it is obvious that $(t^*,X^*)\in E^j_+$ implies $(t^*,X^*)\in\hat{E}^j$ and hence, by the assumptions of our theorem, $(t^*,X^*)\in E^j_+$ implies (64.25) and (64.26). Therefore, owing to (64.23) and (64.24), $U(t,X)$ and $V(t,X)$ satisfy all the assumptions of Theorem 64.1 (resp. 64.2) and consequently we have in D

$$(64.28)\qquad U(t,X)\leqslant V(t,X).$$

Now, if we put

$$E^i_- = \{(t,X)\in D\colon -v^i(t,X) > u^i(t,X)\} \qquad (i=1,2,\ldots,m),$$

then—like in the preceding case—we check that $(t^*,X^*)\in E^j_-$ implies $(t^*,X^*)\in\hat{E}^j$ and consequently $(t^*,X^*)\in E^j_-$ implies (64.25) and (64.26). But, from (64.22) and (64.26) it follows that

$$(64.29)\qquad -v^j_t(t^*,X^*) \leqslant f^j\big(t^*,X^*),-V(t^*,X^*),-v^j_X(t^*,X^*),-v^j_{XX}(t^*,X^*)\big).$$

Thus we see that $(t^*,X^*)\in E^j_-$ implies (64.25) and (64.29). Hence, owing to (64.23) and (64.24), $-V(t,X)$ and $U(t,X)$ satisfy all the assumptions of Theorem 64.1 (resp. 64.2) (with $U(t, X)$ replaced by $-V(t, X)$ and $V(t, X)$ by $U(t, X)$). Therefore, we have in D

$$U(t, X) \geqslant -V(t, X)$$

what together with (64.28) gives (64.27).

Remark. A theorem similar to Theorem 64.5 can be derived from Theorems 64.3 and 64.4.

§ 65. Parabolic differential inequalities in unbounded regions. We are going to prove in this section an analogue of Theorem 64.1 in the case when D is an unbounded region specified below (see [3]).

DEFINITION OF THE REGION OF TYPE C^*. A region D in the space of points $(t, x_1, \ldots, x_n)$ will be called *region of type* C^* if following conditions are satisfied:

(α) D is open and contained in the zone $t_0 < t < t_0+T \leqslant +\infty$.

(β) For any t_1, $t_0 \leqslant t_1 < t_0+T$, the intersection σ_{t_1} of the closure of D with the plane $t = t_1$ is non-void and unbounded.

(γ) For any t_1, σ_{t_1} (see (β)) is identical with the intersection of the plane $t = t_1$ with the closure of that part of D which is contained in the zone $t_0 \leqslant t \leqslant t_1$.

Like in the case of a region of type C (see § 47), we denote by Σ that part of the boundary of D which is contained in the open zone $t_0 < t < t_0+T$.

Since we will have to impose certain bounds on the growth at infinity of the functions involved, we introduce the following definition:

DEFINITION OF THE CLASS E_2. Two positive constants M and K being given, a function $\varphi(t, X)$, defined in a region of type C^*, is said to be of *class* $E_2(M, K)$ if

$$|\varphi(t, X)| \leqslant Me^{KX^2}, \tag{65.1}$$

where $|X| = \sqrt{\sum_{i=1}^{n} x_i^2}$. A function $\varphi(t, X)$ is said to be of *class* E_2 if there exist some positive constants M and K, so that (65.1) holds true.

We are able now to formulate and prove the following theorem:

THEOREM 65.1. *Let the functions $f^i(t, X, U, Q, R)$ $(i = 1, 2, \ldots, m)$ be defined for $(t, X) \in D$ of type C^* and for arbitrary U, Q, R, and satisfy condition W_+ with respect to U* (see § 4). *Suppose further that inequalities*

$$\begin{aligned} &[f^i(t, X, U, Q, R) - f^i(t, X, \tilde{U}, \tilde{Q}, \tilde{R})]\operatorname{sgn}(u^i - \tilde{u}^i) \\ &\leqslant L_0 \sum_{l,k} |r_{lk} - \tilde{r}_{lk}| + (L_1|X| + L_2) \sum_l |q_l - \tilde{q}_l| + (L_3|X|^2 + L_4) \sum_r |u^r - \tilde{u}^r| \\ &\qquad (i = 1, 2, \ldots, m) \end{aligned} \tag{65.2}$$

hold true, where L_s $(s = 0, 1, 2, 3, 4)$ *are positive constants. Let* $U(t, X) = (u^1(t, X), \ldots, u^m(t, X))$ *and* $V(t, X) = (v^1(t, X), \ldots, v^m(t, X))$ *be regular* (see § 47) *and of class* E_2 *in* D *and satisfy initial inequality*

$$(65.3) \qquad U(t_0, X) \leqslant V(t_0, X) \quad \textit{for} \quad (t_0, X) \in \sigma_{t_0}$$

and boundary inequalities of first type

$$(65.4) \qquad U(t, X) \leqslant V(t, X) \quad \textit{for} \quad (t, X) \in \Sigma .$$

Suppose all the functions $f^i(t, X, U, Q, R)$ *are elliptic with respect to* $U(t, X)$ (see § 46). *Put*

$$E^i = \{(t, X) \in D\colon u^i(t, X) > v^i(t, X)\} \qquad (i = 1, 2, \ldots, m)$$

and assume that, for every fixed j, *whenever* $(t^*, X^*) \in E^j$, *we have*

$$(65.5) \qquad u^j_t(t^*, X^*) \leqslant f^j\big(t^*, X^*, U(t^*, X^*), u^j_X(t^*, X^*), u^j_{XX}(t^*, X^*)\big) ,$$

$$(65.6) \qquad v^j_t(t^*, X^*) \geqslant f^j\big(t^*, X^*, V(t^*, X^*), v^j_X(t^*, X^*), v^j_{XX}(t^*, X^*)\big) .$$

Under all these assumptions we have in D

$$(65.7) \qquad U(t, X) \leqslant V(t, X) .$$

Proof. Let $U(t, X)$ and $V(t, X)$ be of class $E_2(M, K)$, i.e.

$$(65.8) \qquad |u^i(t, X)|, |v^i(t, X)| \leqslant Me^{K|X|^2} \qquad (i = 1, 2, \ldots, m) .$$

We introduce the growth damping factor

$$H(t, X) = \exp\left[\frac{(K+1)|X|^2}{1-\mu(t-t_0)} + \nu t\right] ,$$

where

$$(65.9) \qquad \begin{aligned} \nu &= 4[2n(K+1)(L_0+L_2) + mL_4 + 1] , \\ \mu &= 4n^2(K+1)L_0 + 2n(L_1+L_2) + \frac{mL_3}{K+1} , \end{aligned}$$

and new functions

$$\tilde{u}^i(t, X) = u^i(t, X)[H(t, X)]^{-1} , \qquad \tilde{v}^i(t, X) = v^i(t, X)[H(t, X)]^{-1}$$
$$(i = 1, 2, \ldots, m) .$$

Obviously, (65.7) is equivalent with

$$(65.10) \qquad \tilde{U}(t, X) \leqslant \tilde{V}(t, X)$$

in D. Now, we will prove first that (65.10) holds true in D^h, D^h denoting the intersection of D with the closed zone

$$(65.11) \qquad t_0 \leqslant t \leqslant t_0 + h$$

where

$$h = \frac{1}{2\mu} . \tag{65.12}$$

For any set E, denote by E_r^h the intersection of E, of the zone (65.11) and of the cylinder $|X| \leqslant r$. It is clear that, in order to prove (65.10) in D^h, it suffices to show that, for any $\varepsilon > 0$, there is a $r_0 > 0$, so that inequalities

$$\tilde{u}^i(t, X) - \tilde{v}^i(t, X) \leqslant \varepsilon \qquad (i = 1, 2, \ldots, m) \tag{65.13}$$

are satisfied in D_r^h, whenever $r > r_0$. Let ε be an arbitrary positive number; there is a positive r_0 such that $r > r_0$ implies

$$[H(t, X)]^{-1} 2M \exp K|X|^2 = \frac{2M \exp K|X|^2}{\exp\left\{\dfrac{(K+1)|X|^2}{1-\mu(t-t_0)} + \nu t\right\}} \leqslant \varepsilon \tag{65.14}$$

for $(t, X) \in C_r^h$, where C_r^h denotes the intersection of the surface $|X| = r$ with the zone (65.11). We will prove that inequalities (65.13) hold true in D_r^h for $r > r_0$, with r_0 chosen above. Let $r > r_0$; there is an index j and a point $(t^*, X^*) \in \overline{D_r^h}$, so that

$$\tilde{u}^j(t^*, X^*) - \tilde{v}^j(t^*, X^*) = \max_l \{\max_{\overline{D_r^h}} [\tilde{u}^l(t, X) - \tilde{v}^l(t, X)]\} .$$

Suppose that inequalities (65.13) are not true in D_r^h; then, we would have

$$\tilde{u}^j(t^*, X^*) - \tilde{v}^j(t^*, X^*) > \varepsilon > 0 . \tag{65.15}$$

We claim that $(t^*, X^*) \in D_r^h$. Indeed, we have

$$\overline{D_r^h} = D_r^h \cup (\sigma_{t_0})_r^h \cup \Sigma_r^h \cup C_r^h .$$

Owing to (65.3) and (65.15), the point (t^*, X^*) does not belong to $(\sigma_{t_0})_r^h$. By (65.4) and (65.15), it does not belong to Σ_r^h either. Finally, by (65.8) and (65.14), we have for $(t, X) \in C_r^h$

$$\tilde{u}^j(t, X) - \tilde{v}^j(t, X) \leqslant \frac{2M \exp K|X|^2}{\exp\left\{\dfrac{(K+1)|X|^2}{1-\mu(t-t_0)} + \nu t\right\}} \leqslant \varepsilon ,$$

and consequently, because of (65.15), the point (t^*, X^*) is not in C_r^h. Therefore, we must have $(t^*, X^*) \in D_r^h$. Then, by (65.15), $(t^*, X^*) \in E^j$ and hence inequalities (65.5) and (65.6) are satisfied. Since the function of one variable t, $\tilde{u}^j(t, X^*) - \tilde{v}^j(t, X^*)$, attains for $t = t^*$ its maximum in the interval $t_0 \leqslant t \leqslant t^*$, we have

$$\tilde{u}_t^j(t^*, X^*) - \tilde{v}_t^j(t^*, X^*) \geqslant 0 . \tag{65.16}$$

Similarly, the function of the point X, $\tilde{u}^j(t^*, X)-\tilde{v}^j(t^*, X)$, attaining its maximum at the interior point X^*, we get that the quadratic form in $\lambda_1, \ldots, \lambda_n$

$$(65.17)\qquad \sum_{l,k=1}^{n}[\tilde{u}^j_{x_lx_k}(t^*, X^*)-\tilde{v}^j_{x_lx_k}(t^*, X^*)]\lambda_l\lambda_k \quad \text{is negative}$$

and

$$(65.18)\qquad \tilde{u}^j_{x_k}(t^*, X^*)=\tilde{v}^j_{x_k}(t^*, X^*) \quad (k=1,2,\ldots,m)\,.$$

Now, substituting in (65.5) and (65.6)

$$u^i=\tilde{u}^iH\,,\quad v^i=\tilde{v}^iH \quad (i=1,2,\ldots,m)$$

and subtracting (65.6) from (65.5) we obtain at the point (t^*, X^*)

$$(65.19)\qquad (\tilde{u}^j_t-\tilde{v}^j_t)H+(\tilde{u}^j-\tilde{v}^j)H_t$$
$$\leqslant \big[f^j\big(t^*, X^*, \tilde{U}(t^*, X^*)H, Q^{\tilde{u}}, R^{\tilde{u}}\big)-f^j\big(t^*, X^*, \tilde{U}(t^*, X^*)H, Q^{\tilde{u}}, R^{\tilde{u},\tilde{v}}\big)\big]+$$
$$+\big[f^j\big(t^*, X^*, \tilde{U}(t^*, X^*)H, Q^{\tilde{u}}, R^{\tilde{u},\tilde{v}}\big)-f^j\big(t^*, X^*, \tilde{V}(t^*, X^*)H, Q^{v}, R^{\tilde{v}}\big)\big]\,,$$

where

$$Q^{\tilde{u}}=\{\tilde{u}^j_{x_k}(t^*, X^*)H(t^*, X^*)+\tilde{u}^j(t^*, X^*)H_{x_k}(t^*, X^*)\}_{k=1}^{n}\,,$$
$$Q^{\tilde{v}}=\{\tilde{v}^j_{x_k}(t^*, X^*)H(t^*, X^*)+\tilde{v}^j(t^*, X^*)H_{x_k}(t^*, X^*)\}_{k=1}^{n}\,,$$

and similarly at the point (t^*, X^*)

$$R^{\tilde{u}}=\{\tilde{u}^j_{x_lx_k}H+\tilde{u}^j_{x_l}H_{x_k}+\tilde{u}^j_{x_k}H_{x_l}+\tilde{u}^jH_{x_lx_k}\}_{l,k=1}^{n}\,,$$
$$R^{\tilde{v}}=\{\tilde{v}^j_{x_lx_k}H+\tilde{v}^j_{x_l}H_{x_k}+\tilde{v}^j_{x_k}H_{x_l}+\tilde{v}^jH_{x_lx_k}\}_{l,k=1}^{n}\,,$$
$$R^{\tilde{u},\tilde{v}}=\{\tilde{v}^j_{x_lx_k}H+\tilde{u}^j_{x_l}H_{x_k}+\tilde{u}^j_{x_k}H_{x_l}+\tilde{u}^jH_{x_lx_k}\}_{l,k=1}^{n}\,.$$

By the ellipticity of $f^j(t, X, U, Q, R)$ with respect to $U(t, X)=\tilde{U}(t, X)H$ (see § 46) and by (65.17), the first difference in the brackets on the right-hand side of inequality (65.19) is non-positive. As to the second difference in brackets we rewrite it in the form

$$(65.20)\qquad \big[f^j\big(t^*, X^*, \tilde{U}(t^*, X^*)H, Q^{\tilde{u}}, R^{\tilde{u},\tilde{v}}\big)-f^j\big(t^*, X^*, W(t^*, X^*)H, Q^{\tilde{v}}, R^{\tilde{v}}\big)\big]+$$
$$+\big[f^j\big(t^*, X^*, W(t^*, X^*)H, Q^{\tilde{v}}, R^{\tilde{v}}\big)-f^j\big(t^*, X^*, \tilde{V}(t^*, X^*)H, Q^{\tilde{v}}, R^{\tilde{v}}\big)\big]\,,$$

where

$$W(t, X)=\big(w^1(t, X), \ldots, w^m(t, X)\big)\,,\quad w^l(t, X)=\min[\tilde{u}^l(t, X), \tilde{v}^l(t, X)] \quad (l=1,2,\ldots,m)\,.$$

Since, by (65.15) (see § 4),

$$W(t^*, X^*) \overset{j}{\leqslant} \widetilde{V}(t^*, X^*)\,,$$

the second difference (65.20) is non-positive, by the condition W_+ with respect to U (see § 4). To the first difference (65.20) we apply inequality (65.2). Taking advantage of (65.18) and remembering that, by the definition of $W(t, X)$ and by (65.15),

$$\begin{aligned}|\widetilde{u}^l(t^*, X^*) - w^l(t^*, X^*)| &= \widetilde{u}^l(t^*, X^*) - w^l(t^*, X^*)\\ &\leqslant \max\,[0, \widetilde{u}^l(t^*, X^*) - \widetilde{v}^l(t^*, X^*)]\\ &\leqslant \widetilde{u}^j(t^*, X^*) - \widetilde{v}^j(t^*, X^*)\\ &\qquad\qquad (l = 1, 2, \dots, m)\end{aligned}$$

we finally get from (65.2) and (65.19)

$$(65.21)\qquad [\widetilde{u}^j_t(t^*, X^*) - \widetilde{v}^j_t(t^*, X^*)]H \leqslant [\widetilde{u}^j(t^*, X^*) - \widetilde{v}^j(t^*, X^*)]\,F[H]$$

where

$$F[H] = L_0 \sum_{l,k=1}^{n} |H_{x_l x_k}| + (L_1|X| + L_2) \sum_{k=1}^{n} |H_{x_k}| + m(L_3|X|^2 + L_4)H - H_t\,.$$

Computing the derivatives of $H(t, X)$ we find that

$$\begin{aligned}F[H] \leqslant H\Big\{&\frac{4(K+1)^2 L_0}{[1-\mu(t-t_0)]^2} \sum_{l,k=1}^{n} |x_l x_k| + \frac{2(K+1)nL_0}{1-\mu(t-t_0)} +\\ &+ \frac{2(K+1)}{1-\mu(t-t_0)}(L_1|X| + L_2) \sum_{k=1}^{n} |x_k| + (L_3|X|^2 + L_4)m - \frac{\mu(K+1)|X|^2}{[1-\mu(t-t_0)]^2} - \nu\Big\}\,.\end{aligned}$$

Since in D_r^h we have, by (65.12),

$$(65.22)\qquad \tfrac{1}{2} \leqslant 1 - \mu(t-t_0) \leqslant 1\,,$$

and since, obviously,

$$|x_l| \leqslant |X|\,, \qquad |X| \leqslant |X|^2 + 1\,,$$

we get further

$$\begin{aligned}F[H] \leqslant \frac{H}{[1-\mu(t-t_0)]^2}\Big\{&(K+1)|X|^2\Big[4(K+1)L_0 n^2 + 2(L_1+L_2)n + \frac{mL_3}{K+1} - \mu\Big] +\\ &+ [2(K+1)n(L_0+L_2) + mL_4] - \nu[1-\mu(t-t_0)]^2\Big\}\,.\end{aligned}$$

Hence, by (65.9) and (65.22), it follows that

$$F[H] \leqslant -4H$$

and consequently, by (65.15) and (65.21),

$$\tilde{u}_l^j(t^*, X^*) - \tilde{v}_l^j(t^*, X^*) \leqslant -4[\tilde{u}^j(t^*, X^*) - \tilde{v}^j(t^*, X^*)] < 0 ,$$

which contradicts (65.16). This contradiction completes the proof of inequalities (65.7) in D^h, where h is given by formulas (65.9) and (65.12). In particular, we have inequalities (65.7) in the intersection of the closure of D^h with the plane $t = t_0 + h$; but, since this intersection is—by property (γ) of the region of type C^*—identical with σ_{t_0+h}, we have (65.7) for $(t, X) \in \sigma_{t_0+h}$. Therefore, we can repeat our argument starting from the plane $t = t_0 + h$, instead of the plane $t = t_0$, and thus obtain inequalities (65.7) in the intersection of D with the zone

$$t_0 + h \leqslant t \leqslant t_0 + 2h .$$

In this way we prove inequalities (65.7) in any point of D after a finite number of steps.

As an immediate consequence of Theorem 65.1 we obtain the following uniqueness criterion.

COROLLARY 65.1. *Let the right-hand sides of the system of differential equations* (64.16) *satisfy all the assumptions of Theorem* 65.1 *for* $(t, X) \in D$ *of type* C^* *and for arbitrary* U, Q, R. *Then the first Fourier's problem* (see § 47) *for system* (64.16) *admits in* D *at most one parabolic, regular* (see §§ 46, 47) *solution of class* E_2.

Remark. In particular, when D of type C^* is the half-space $t > t_0$, then Σ is empty and the first Fourier's problem reduces to the so-called *reduced Cauchy problem*. This problem consists in finding a regular and parabolic solution in the half-space $t > t_0$, satisfying a given initial condition for $t = t_0$. In this case Corollary 65.1 gives a uniqueness criterion for the solution of the reduced Cauchy problem.

§ 66. The Chaplygin method for parabolic equations. This section deals with the Chaplygin method for the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(t, x, u) . \tag{66.1}$$

We consider here the first Fourier's problem (see § 47). We assume always that $(t, x) \in \{(t, x)\colon\ 0 \leqslant t \leqslant T,\ a \leqslant x \leqslant b\} = R$. The interior of R is denoted by R°, the boundary by ∂R. Γ stands here for the plane set composed of points $(0, x)$ with $a \leqslant x \leqslant b$ and $(t, a), (t, b)$ with $0 \leqslant t \leqslant T$. By a *regular function* in R we mean a function u which is continuous on R, continuously differentiable in t to $\partial u/\partial t$ and twice in x to $\partial^2 u/\partial x^2$ for $0 < t \leqslant T$, $x \in (a, b)$.

Theorem 64.1 implies

LEMMA 66.1. *If $u(t, x)$, $v(t, x)$ are regular in R and $\partial f(t, x, u)/\partial u$ is continuous and*

$$\frac{\partial u}{\partial t} \leqslant \frac{\partial^2 u}{\partial x^2} + f\big(t, x, u(t, x)\big), \tag{66.2}$$

$$\frac{\partial v}{\partial t} \geqslant \frac{\partial^2 v}{\partial x^2} + f\big(t, x, v(t, x)\big) \tag{66.3}$$

on R^0 and $u(t, x) \leqslant v(t, x)$ on Γ, then $u(t, x) \leqslant v(t, x)$ on R.

If u (v) satisfies (66.2) ((66.3)), then u (v) is called a *lower* (*upper*) *function*. Let $f(t, x, u)$ be differentiable in u to $f_u(t, x, u)$. Assume that $f(t, x, u)$ and $f_u(t, x, u)$ are continuous and locally Hölder continuous (exponent $\leqslant 1$) in all variables for $t > 0$. Suppose now that the function $u(t, x)$ is Hölder continuous in R. Then the composite functions $f\big(t, x, u(t, x)\big)$, $f_u\big(t, x, u(t, x)\big)$ are locally Hölder continuous. It is a classical result that there is a unique solution $z(t, x)$ of the equation

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + f\big(t, x, u(t, x)\big) + f_u\big(t, x, u(t, x)\big)\big(z - u(t, x)\big) \tag{66.4}$$

with the boundary condition

$$z = \varphi \quad \text{on} \quad \Gamma, \tag{66.5}$$

where φ is continuous on Γ. The functions f, φ being fixed, the function z is uniquely determined by u. Hence, we have here the transformation law $u \to z$, in symbols $z = Cu$. We form the sequence

$$z_0 = u\,, \qquad z_{n+1} = Cz_n$$

which is the Chaplygin sequence for equation (66.1) with boundary data (66.5). First we will prove

THEOREM 66.1. *Suppose that $u_0(t, x)$ is lower and $v_0(t, x)$ upper and let $f(t, x, u)$ be continuously differentiable in u to $f_u(t, x, u)$. We assume that $f(t, x, u)$, $f_u(t, x, u)$ are continuous and locally Hölder continuous for $t > 0$. Let φ be continuous on Γ and suppose that $u_0 \leqslant \varphi \leqslant v_0$ on Γ.*

If $f_u(t, x, u)$ increases in u, then the Chaplygin sequence

$$z_0 = u_0\,, \qquad z_{n+1} = Cz_n$$

satisfies the following conditions:

$$\frac{\partial z_{n+1}}{\partial t} = \frac{\partial^2 z_{n+1}}{\partial x^2} + f(t, x, z_n) + f_u(t, x, z_n)(z_{n+1} - z_n)\,, \tag{66.6}$$

$$\frac{\partial z_n}{\partial t} \leqslant \frac{\partial^2 z_n}{\partial x^2} + f(t, x, z_n)\,, \tag{66.7}$$

$$z_n = \varphi \quad on \quad \Gamma, \tag{66.8}$$

$$u_0 \leqslant z_n \leqslant z_{n+1} \leqslant v_0 \quad on \quad R. \tag{66.9}$$

Proof. The fact that z_n is well defined is a consequence of the previous discussion and of the regularity of u_0. Conditions (66.6) and (66.8) follow from the definition of the Chaplygin sequence. Suppose now that (66.7) holds for $n = k$. Consider the equation

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + g(t, x, z)\,, \tag{66.10}$$

where

$$g(t, x, z) = f(t, x, z_k) + f_u(t, x, z_k)(z - z_k)\,. \tag{66.11}$$

The solution of (66.10) with the boundary condition $z = \varphi$ on Γ is z_{k+1}. Hence

$$\frac{\partial z_{k+1}}{\partial t} \geqslant \frac{\partial^2 z_{k+1}}{\partial x^2} + g(t, x, z_{k+1})\,.$$

But $g(t, x, z_k) = f(t, x, z_k)$ and consequently, by the inductive assumption

$$\frac{\partial z_k}{\partial t} \leqslant \frac{\partial^2 z_k}{\partial x^2} + g(t, x, z_k)\,.$$

The last two inequalities and Lemma 66.1 imply that

$$z_k(t, x) \leqslant z_{k+1}(t, x) \quad \text{in} \quad R\,. \tag{66.12}$$

Formula (66.12) and the convexity of $f(t, x, u)$ in u imply

$$f(t, x, z_k) + f_u(t, x, z_k)(z_{k+1} - z_k) \leqslant f(t, x, z_{k+1})$$

which by (66.6) proves (66.7) for $n = k+1$. (66.7) being proved for arbitrary n, the above reasoning proves (66.12) for any k. This completes the proof.

COROLLARY. *The assumptions of Theorem* 66.1 *imply that the solution* $z(t, x)$ *of* (66.1), (66.5) *exists and by Lemma* 66.1

$$u_0(t, x) \leqslant z(t, x) \leqslant v_0(t, x) \quad on \quad R\,.$$

It follows then from Lemma 66.1 that $z(t, x)$ is the unique solution of the considered boundary value problem. One can prove under our assumptions that $\{z_n\}$ is compact in sup norm and by its monotonicity it must be uniformly convergent. Simple limit passages show that $\lim\limits_{n\to\infty} z_n = z$. For other extensions of the Chaplygin method for parabolic equations, see [26].

The Lusin type [20] estimates for $\{z_n\}$ are given in the following theorem.

THEOREM 66.2. *Let* u_0, v_0, f *satisfy the assumptions of Theorem* 66.1 *and suppose that*

$$|f_u(t, x, \bar{u}) - f_u(t, x, \bar{\bar{u}})| \leqslant \sigma(t, |\bar{u} - \bar{\bar{u}}|)$$

for $u_0(t, x) \leqslant \bar{u}, \bar{\bar{u}} \leqslant v_0(t, x)$.

It is assumed that $\sigma(t, u) \geqslant 0$ is continuous for $0 \leqslant t \leqslant T$, $u \geqslant 0$ and increases in u. Let

$$\max_{a \leqslant x \leqslant b} \{v_0(t, x) - u_0(t, x)\} \leqslant \tau_0(t), \quad 0 \leqslant t \leqslant T,$$

and define

$$\tau_{n+1}(t) = \int_0^t e^{K(t-s)} \sigma\big(s, \tau_n(s)\big) \tau_n(s)\, ds,$$

where

$$K = \sup |f_u(t, x, u)|, \quad (t, x) \in R, \ u_0 \leqslant u \leqslant v_0.$$

Then $|z_n(t, x) - z(t, x)| \leqslant \tau_n(t)$ on R.

The proof for the above theorem is modelled after the proof of Theorem 32.2. Instead of Theorem 9.5 for ordinary differential inequalities one applies Theorem 64.1 of § 64.

§ 67. Maximum solution of the parabolic equation. We will use in this section notation and definitions of § 66. Theorem 63.1 implies

LEMMA 67.1. *Let the regular functions $u_0(t, x)$, $v_0(t, x)$ satisfy*

$$\frac{\partial u_0}{\partial t} \overset{(<)}{\leqslant} \frac{\partial^2 u_0}{\partial x^2} + g\big(t, x, u_0(t, x)\big),$$

$$\frac{\partial v_0}{\partial t} \overset{(\geqslant)}{>} \frac{\partial^2 v_0}{\partial x^2} + g\big(t, x, v_0(t, x)\big)$$

on R^0 and $u_0(t, x) < v_0(t, x)$ on Γ. Then $u_0(t, x) < v_0(t, x)$ on R.

Suppose that the functions $u(t, x)$, $g(t, x, z)$ and $\varphi(t, x)$ are continuous in R,

$$Q = \{(t, x, z) \colon (t, x) \in R, z \text{ arbitrary}\}$$

and Γ respectively. We define

$$r(t, x) = \frac{1}{2\sqrt{\pi}} \int_0^t \int_a^b \frac{\exp\left(-\dfrac{(x-\xi)^2}{4(t-\tau)}\right)}{\sqrt{t-\tau}} g\big(\tau, \xi, u(\tau, \xi)\big)\, d\xi\, d\tau.$$

Let $q(t, x)$ be the solution of the equation $z_t = z_{xx}$ such that $q = \varphi - r$ on Γ. We put

$$v(t, x) \overset{\mathrm{df}}{=} q(t, x) + r(t, x)$$

and denote by $T(u; g, \varphi)$ the transformation $u \to v$. Hence $v = T(u; g, \varphi)$. One can prove [26] that if $u_n \underset{R}{\Rightarrow} u$, $g_n \underset{Q}{\Rightarrow} g$, $\varphi_n \underset{\Gamma}{\Rightarrow} \varphi$, then $v_n = T(u_n; g_n, \varphi_n) \Rightarrow$ $\Rightarrow v = T(u; g, \varphi)$ on R.

If u_n, g_n, φ_n are bounded in sup norm, then $\{v_n\}$ is compact.

If $g(t, x, z)$ is continuous in (t, x, z) and Hölder continuous in x and z, then the solution z of the equation

$$z = T(z; g, \varphi)$$

is a regular solution of

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + g(t, x, z)\,, \tag{67.1}$$

$$z = \varphi \quad \text{on} \quad \Gamma\ (^1). \tag{67.2}$$

The following theorem is due to Prodi [41]:

THEOREM 67.1. *Let $u_0(t, x)$, $v_0(t, x)$ satisfy the assumptions of Lemma* 67.1 *and $u_0 < \varphi < v_0$ on Γ where φ is continuous on Γ. It is supposed that $g(t, x, z)$ is continuous in Q and Hölder continuous in x and z. Then the problem* (67.1), (67.2) *has at least one regular solution.*

We say that the regular solution $u(t, x)$ of (67.1), (67.2) is a *maximum solution* (*minimum solution*) of that problem, if for every other solution of the problem $v(t, x)$ the inequality $v(t, x) \leqslant u(t, x)$ $(v(t, x) \geqslant u(t, x))$ holds in R.

Next we prove

THEOREM 67.2. *Let u_0, v_0, g, φ satisfy the assumptions of theorem* 67.1. *Then* (67.1), (67.2) *has a maximum solution $\bar{u}(t, x)$ and a minimum one $\underline{u}(t, x)$.*

If $u(t, x)$ is regular in R and

$$\frac{\partial u}{\partial t} \leqslant \frac{\partial^2 u}{\partial x^2} + g\big(t, x, u(t, x)\big) \quad \Big(\frac{\partial u}{\partial t} \geqslant \frac{\partial^2 u}{\partial x^2} + g\big(t, x, u(t, x)\big)\Big) \quad in \quad R^0$$

and

$$u(t, x) \leqslant \varphi(t, x) \quad (u(t, x) \geqslant \varphi(t, x)) \quad on \quad \Gamma\,,$$

then

$$u(t, x) \leqslant \bar{u}(t, x) \quad (u(t, x) \geqslant \underline{u}(t, x)) \quad on \quad R\,.$$

Proof. We start with the following definition:

$$g^*(t, x, z) = \begin{cases} g\big(t, x, u_0(t, x)\big) & \text{if } z < u_0(t, x)\,, \\ g(t, x, z) & \text{if } u_0(t, x) \leqslant z \leqslant v_0(t, x)\,, \\ g\big(t, x, v_0(t, x)\big) & \text{if } z > v_0(t, x)\,. \end{cases}$$

The function g^* is bounded and if $\sup|g^*| < M$ and $\sup|\varphi| < K$, then the functions $\tilde{u}_0 = -Mt - K$, $\tilde{v}_0 = Mt + K$ satisfy assumptions of lemma 67.1 with $\tilde{u}_0 = u_0$, $\tilde{v}_0 = v_0$, $g = g^*$. It is easy to check that g^* is

(1) For references, see [26].

Hölder continuous in x and z. Applying the theorem of Prodi we get that there is a solution z_n of the problem

$$(67.3)\qquad \begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial^2 z}{\partial x^2} + g^*(t, x, z) + \frac{1}{n}, \\ z &= \varphi + \frac{1}{n} \quad \text{on} \quad \Gamma \end{aligned}$$

for n sufficiently large. By Lemma 67.1

$$(67.4)\qquad z_{n+1} < z_n \quad \text{in} \quad R.$$

Obviously

$$(67.5)\qquad z_n = T\left(z_n;\ g^* + \frac{1}{n}, \varphi + \frac{1}{n}\right).$$

Hence $\{z_n\}$ is compact. (67.4) implies then that $z_n \underset{R}{\Rightarrow} z$. By a limit passage in (67.5) we get $z = T(z; g^*, \varphi)$. It follows then that $z(t, x)$ is a solution of the problem

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial^2 z}{\partial x^2} + g^*(t, x, z), \\ z &= \varphi \quad \text{on} \quad \Gamma. \end{aligned}$$

But

$$\begin{aligned} g^*\big(t, x, u_0(t, x)\big) &= g\big(t, x, u_0(t, x)\big), \\ g^*\big(t, x, v_0(t, x)\big) &= g\big(t, x, v_0(t, x)\big). \end{aligned}$$

Hence the triples (u_0, z, g^*), (v_0, z, g^*) satisfy the assumptions of Lemma 67.1 and consequently

$$u_0(t, x) < z(t, x) < v_0(t, x) \quad \text{in} \quad R.$$

It follows then from the definition of g^* that

$$g^*\big(t, x, z(t, x)\big) = g\big(t, x, z(t, x)\big).$$

This proves that z is a solution of (67.1), (67.2).

We will now prove that if a regular function u satisfies

$$\begin{aligned} \frac{\partial u}{\partial t} &\leqslant \frac{\partial^2 u}{\partial x^2} + g\big(t, x, u(t, x)\big) \quad \text{in} \quad R^0; \\ u(t, x) &\leqslant \varphi(t, x) \quad \text{on} \quad \Gamma, \end{aligned}$$

then $u(t, x) \leqslant z(t, x)$. This being proved we get the conclusion that $z(t, x)$ is the maximum solution and simultaneously the second part of the assertion follows.

Suppose that $u(t, x)$ satisfies the above inequalities and let $u(t, x) = z_n(t, X)$ for a point $(t, x) \in R$. Lemma 67.1 implies

$$u_0(t, x) < z_n(t, x), \quad u(t, x) < v_0(t, x).$$

Hence, at (t, x),

$$u_0(t, x) < u(t, x) = z_n(t, x) < v_0(t, x)$$

and by definition of g^*

$$g^*\big(t, x, z_n(t, x)\big) = g\big(t, x, z_n(t, x)\big) \quad \text{at} \quad (t, x).$$

It follows then that at (t, x)

$$\frac{\partial u}{\partial t} < \frac{\partial^2 u}{\partial x^2} + g\big(t, x, u(t, x)\big) + \frac{1}{n}$$

and

$$\frac{\partial z_n}{\partial t} \geqslant \frac{\partial^2 z_n}{\partial x^2} + g\big(t, x, z_n(t, x)\big) + \frac{1}{n}.$$

By Theorem 63.1 we conclude therefore that $u(t, x) < z_n(t, x)$ in R, which by a limit passage proves that $u(t, x) \leqslant z(t, x)$, q.e.d. The proof for the minimum solution is quite similar and can be omitted.

The following example (see [31]) shows that the assumptions of Theorem 67.2 do not imply the uniqueness of problem (67.1), (67.2). Moreover, it shows that it can really happen that

$$\bar{u}(t, x) \not\equiv \underline{u}(t, x).$$

Example. We put in the definition of R:

$$T = \frac{\pi}{4}, \quad a = -\frac{\pi}{2}, \quad b = \frac{\pi}{2}$$

and define g by

$$g(t, x, u) = \begin{cases} -\sqrt{\cos^2 x - u^2} + u & \text{if} \quad |u| \leqslant \cos x, \\ u & \text{if} \quad |u| \geqslant \cos x. \end{cases}$$

It is easy to prove (see [31]) that g satisfies locally Hölder conditions in x and u with an exponent $\frac{1}{2}$. On the other hand, the functions

$$u_0(t, x) = -3e^t + 1, \quad v_0(t, x) = 3e^t - 1$$

satisfy the inequalities of lemma 67.1 with the above defined g. Notice now that the functions $z_1 = \cos x \cdot \cos t$, $z_2 = \cos x$ satisfy the same boundary conditions on Γ and both are solutions of the equation $z_t = z_{xx} + g(t, x, z)$ in R. Moreover, $u_0 < z_1 = z_2 < v_0$ on Γ. Hence, all the assumptions of Theorem 67.2 are satisfied for $\varphi = z_1 = z_2$ on Γ, but there are two different solutions $z_1 \neq z_2$ of the same problem. It follows then that the maximum solution $\bar{u}$ is different from minimum solution $\underline{u}$.

CHAPTER XI

DIFFERENTIAL INEQUALITIES IN LINEAR SPACES

The present chapter attempts to give some general theorems concerning differential inequalities, when treated by methods of functional analysis. There are two basic concepts: the generalized mean value theorems and the generalized Bendixson equation. Strictly speaking, the second step is the systematic use of the so called method of first integrals. This method enters here through the Bendixson equation, which in classical form was implicitly used in the integration of a linear ordinary differential inequality. In case of non-linear inequalities the Bendixson equation was used in [50] as a method of proof of Theorem 13.2. For generalized mean value theorems see [1], [23] and [63].

§ 68. Convex sets in linear topological spaces. Let E be a real linear space. We denote by $x, y, \ldots$ the elements of E. Suppose we have introduced in E the topology in which the operations of addition and of multiplication by real scalars are jointly continuous; then, E becomes a linear topological space. If the topology is induced by a system of convex neighborhoods of the zero vector, then E is called a *locally convex linear space*. Let E' denote the adjoint of E. E' consists of all linear and continuous real valued functionals defined over E. The adjoint space E' is non-trivial if and only if E contains an open, convex set different from E. In what follows we assume always that E' is non-trivial, i.e. that it contains non-zero functionals. This certainly happens for locally convex spaces and consequently for Banach spaces.

Let $\xi \in E'$. We introduce the following notation:

$$H(\xi, \alpha) = \{x \in E\colon\ \xi x - \alpha\}\,.$$

In geometrical terms, $H(\xi, \alpha)$ stands for a hyperplane determined by its "gradient" ξ and a scalar α. Following the geometrical terminology we define the closed half-space by

$$K(\xi, \alpha) = \{x \in E\colon\ \xi x \leqslant \alpha\}\,.$$

The open half space is defined by

$$K^0(\xi, \alpha) = \{x \in E\colon\ \xi x < \alpha\}\,.$$

The set $V \subset E$ is a *convex body* if it is convex, closed and has some interior points. The last definition is the following one: the hyperplane $H(\xi, a)$ is *tangent* to the set Z if $Z \subset K(\xi, a)$ and if there is an x_0 belonging to the boundary of Z such that $\xi x_0 = a$.

A nice part of geometrical properties of convex sets known for Euclidean spaces apply to convex subsets of linear topological spaces. Due to classical theorems of Mazur and Eidelheit we can apply the separation theorems in pretty general situations in such spaces. We list below as lemmas some theorems concerning the structure of convex subsets of linear topological spaces. For references see [8].

LEMMA 68.1. *Let E be a real linear topological space and let $V \subset E$ be a convex body. Then for every $x \in \partial V$* (1) *there is a $\xi \in E'$ such that $V \subset K(\xi, \xi x)$. Moreover, the set V is equal to the intersection of all closed half-spaces which include V, i.e.*

$$V = \bigcap_{V \subset K(\xi, a)} K(\xi, a)\,.$$

The interior of V is the intersection of all open half-spaces $K^0(\xi, \xi x)$ with $x \in \partial V$ which include V.

In the case of a locally convex space a similar property holds for arbitrary closed and convex sets:

LEMMA 68.2. *Let E be a locally convex linear topological space. Suppose that V is a closed, convex subset of E. Then $V = \bigcap_{V \subset K(\xi, a)} K(\xi, a)$.*

The above lemmas take on a simple form if V is a cone (with vertex zero). We say that the closed set V is a *cone* if it satisfies the following two conditions:

(68.1) If $x \in V$ and $y \in V$, then $x + y \in V$,

(68.2) If $x \in V$ and $\lambda \geqslant 0$, then $\lambda x \in V$.

V being a cone, its dual V' is defined by

$$V' = \{\xi \in E' : \xi x \geqslant 0 \text{ for every } x \in V\}\,.$$

The elements of V' are called *positive functionals.*

Given a cone V we define in E a partial order $\leqslant$ by

$$x \leqslant y \equiv (y - x) \in V\,.$$

The partial order defined above is called a *partial order induced by V.*

It follows from the definition of a cone that:

(68.3) $x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$,

(68.4) $x \leqslant y$ implies $x + z \leqslant y + z$ for arbitrary $z \in E$,

(68.5) $x \leqslant y$ and $\lambda \geqslant 0$ imply $\lambda x \leqslant \lambda y$,

(68.6) If $Z \subset V$, then its closure $\bar{Z} \subset V$.

(1) ∂V stands for the boundary of V.

On the other hand, if the partial order $\leqslant$ satisfies (68.3), (68.4) and (68.5) and the set $V = \{x: \theta \leqslant x\}$ is closed, then V is a cone which induces the prescribed partial order.

Let V be a cone. Then V is closed and convex. Applying Lemma 68.1 and Lemma 68.2 one concludes that the following lemmas hold true:

LEMMA 68.3. *Let E be a real linear topological space. Suppose that $V \subset E$ ($V \neq E$) is a cone with the non-empty interior. This being assumed, if for every $\xi \in V'$, $0 \leqslant \xi x$, then $x \in V$. Moreover, if $0 < \xi x$ for every $\xi \in V'$, then x belongs to the interior of V.*

LEMMA 68.4. *Let E be a locally convex real linear topological space. Suppose that $V \subset E$ ($V \neq E$) is a cone. Then $x \in V$ if and only if for every $\xi \in V'$ the inequality $\xi x \geqslant 0$ holds.*

Notice that the above discussion applies to spaces with complex field of scalars, provided we consider the real parts of complex-valued functionals.

§ 69. Mean value theorems. It is of some interest to consider functions of a real variable t with values in a linear space. Let $x(t)$ be defined on the interval Δ and suppose that $x(t) \in E$ for $t \in \Delta$. For the sake of simplicity we assume now that E is a Banach space. The symbol $\|x\|$ stands for the norm of the element $x \in E$. Let $x(t)$ be strongly differentiable to $x'(t)$ on Δ, i.e.

$$\lim_{h\to 0}\left\|\frac{x(t+h)-x(t)}{h}-x'(t)\right\| = 0$$

for each $t \in \Delta$. The analogue of a classical theorem of advanced calculus is the following conjecture: if $x'(t) = \theta$ (zero vector) for $t \in \Delta$, then $x(t) = \text{const}$ on Δ.

We can attack the problem as follows: let $t_0 \in \Delta$ and notice that

$$\|x(t+h)-x(t_0)\| - \|x(t)-x(t_0)\| \leqslant \|x(t+h)-x(t)\|\,.$$

Hence, for $\varphi(t) = \|x(t)-x(t_0)\|$ the inequality $D_+\varphi(t) \leqslant 0$ holds all over Δ. This implies, by Theorem 2.1, that $\varphi(t)$ decreases and consequently $x(t) = x(t_0)$ for $t > t_0$. But t_0 was an arbitrary point of Δ. This shows that $x(t) = \text{const}$. The above statement can be proved by using much more sophisticated arguments. It is a classical theorem in the theory of Banach spaces that for every $z \in E$ there exists a $\xi \in E'$ such that $|\xi| \leqslant 1$ and $\xi z = \|z\|$. Take now $z = x(t_1)-x(t_0)$ ($t_1 \in \Delta$) and ξ such that $\xi[x(t_1)-x(t_0)] = \|x(t_1)-x(t_0)\|$. Consider the real-valued function $\psi(t) = \xi[x(t)-x(t_0)]$. The assumption $x'(t) = \theta$ implies $\psi'(t) = 0$ on Δ. Hence $\psi(t) = \text{const}$ on Δ. This implies that $\|x(t_1)-x(t_0)\| = \psi(t_0) = 0$, which completes the proof.

Notice now that we have used essentially the fact that the real-valued function $\psi(t)$ has the derivative equal to zero. We can replace the assumption $x'(t) = \theta$ by the requirement that the weak derivative $x_w'(t)$ be zero on Δ. The weak derivative is defined by

$$\lim_{h\to 0} \xi\left\{\frac{x(t+h)-x(t)}{h}\right\} = \xi x_w'(t)\,, \qquad \xi \in E'\,.$$

If we assume that $x_w'(t) = \theta$ on Δ, then the previous arguments apply and we thus obtain a stronger result: if $x_w'(t) = \theta$ on Δ, then $x(t) = \text{const}$ on Δ.

The further generalization runs in three directions. First of all we require that the function $x(t)$ be merely weakly continuous, that is for every $\xi \in E'$ the real-valued function $\xi x(t)$ is continuous. In the second step we replace the very particular set, consisting of the zero vector, by a closed and convex one. The last move is to replace the derivative by a certain analogue of a derivative with respect to this convex, closed set. All these three points are mentioned in the subsequent theorem.

THEOREM 69.1. *Let E be a real linear topological space and let $V \subset E$ be a convex body. Suppose that:*

(69.1) *For every $\xi \in E'$ the function $\xi x(t)$ is continuous on Δ.*

(69.2) *For every $\xi \in E'$ the set $\Delta - Z_\xi$ is at most countable, where $Z_\xi \subset \Delta$ is the set of those t for which there is a sequence of reals $\tau_n \to 0+$ and a sequence $y_n \in V$ (both sequences depending possibly on t) such that*

$$\lim_{n\to\infty} \xi\left\{\frac{x(t+\tau_n)-x(t)}{\tau_n} - y_n\right\} = 0\,.$$

Then

$$\frac{x(t_1)-x(t_2)}{t_1-t_2} \in V \qquad \textit{for} \qquad t_1 \neq t_2;\ t_1, t_2 \in \Delta\,.$$

Proof. We will use Lemma 68.1. It suffices to prove that if

$$V \subset K(\xi, a)\,, \tag{69.3}$$

then

$$\frac{x(t_1)-x(t_2)}{t_2-t_2} \in K(\xi, a)\,, \qquad t_1 \neq t_2\,.$$

Consider the function $\psi(t) = \xi x(t)$. By (69.1) $\psi(t)$ is continuous on Δ. For the fixed ξ we take the set Z_ξ. Let $t \in Z_\xi$. We take the sequence $y_n \in V$ corresponding to t. Hence

$$\xi y_n \leqslant a\,. \tag{69.4}$$

It follows now from (69.2) that

$$\frac{\psi(t+\tau_n)-\psi(t)}{\tau_n} - \xi y_n \to 0\,. \tag{69.5}$$

Relations (69.4) and (69.5) show that

$$D_+\psi(t) \leqslant a . \tag{69.6}$$

Inequality (69.6) holds for every $t \in Z_\xi$. The set $\Delta - Z_\xi$ being at most countable we get by Theorem 2.2

$$\frac{\psi(t_1)-\psi(t_2)}{t_1-t_2} \leqslant a , \qquad t_1 \neq t_2 ,$$

that is

$$\xi\left\{\frac{x(t_1)-x(t_2)}{t_1-t_2}\right\} \leqslant a .$$

The last relation means that

$$\frac{x(t_1)-x(t_2)}{t_1-t_2} \in K(\xi, a)$$

what was to be proved.

Remark. Theorem 2.2 is obviously a very particular case of the above theorem.

COROLLARY. *It follows from Lemma* 68.2 *and from the above proof that Theorem* 69.1 *remains true if E is locally convex and V is a closed, convex subset of E, not necessarily possessing interior points.*

For the sake of completeness we will prove the following theorem:

THEOREM 69.2. *Let V be an open, convex subset of the real linear topological space E. Suppose that the function $x(t)$ is weakly continuous on the interval Δ. We assume that for every $\xi \in E'$ the set $\Delta - Z_\xi$ is at most countable, where $Z_\xi \subset \Delta$ is the set of those t for which there exists a sequence $\tau_n \to 0+$ and an element $z_t \in V$ so that*

$$\lim_{n\to\infty} \xi\left\{\frac{x(t+\tau_n)-x(t)}{\tau_n}\right\} = \xi z_t .$$

Then

$$x(t_1)-x(t_2) \in (t_1-t_2)V , \qquad t_1, t_2 \in \Delta .$$

Proof. Suppose that $V \subset K^0(\xi, \xi x)$, where $x \in \partial V$. We take the function $\psi(t) = \xi x(t)$. This function is continuous and for $t \in Z_\xi$

$$D_+\psi(t) \leqslant \lim_{n\to\infty} \xi\left\{\frac{x(t+\tau_n)-x(t)}{\tau_n}\right\} = \xi z_t .$$

But $z_t \in V$; hence $\xi z_t < \xi x$ and consequently $D_+\psi(t) < \xi x$ for $t \in Z_\xi$. It follows then, by Theorem 2.2, that for $t_1 \neq t_2$

$$\xi\left\{\frac{x(t_1)-x(t_2)}{t_1-t_2}\right\} < \xi x .$$

This means that $x(t_1)-x(t_2) \in (t_1-t_2)K^0(\xi, \xi x)$. We conclude that

$$\frac{x(t_1)-x(t_2)}{t_1-t_2}$$

belongs to every $K^0(\xi, \xi x)$ such that $V \subset K^0(\xi, \xi x)$. By Lemma 68.1 this implies our assertion.

It is easy to verify that the above mean value theorems remain true if we assume that $\xi x(t)$ is absolutely continuous for every $\xi \in E'$ and $\Delta - Z_\xi$ are of Lebesgue measure zero. To do this we have to use Theorem 3.1. We formulate right now one of the possible theorems.

THEOREM 69.3. *Let E be a locally convex real linear topological space and let V be its closed and convex subset. Suppose that $x(t)$ is weakly absolutely continuous on the interval Δ, i.e. for every $\xi \in E'$ the function $\xi x(t)$ is absolutely continuous on Δ. Assume that for every $\xi \in E'$ there is a set $Z_\xi \subset \Delta$, $\Delta - Z_\xi$ being of Lebesgue measure zero, such that for each $t \in Z_\xi$ there is a sequence $\tau_n \to 0$ and a sequence $y_n \in V$ so that*

$$\lim_{n\to\infty} \xi\left\{\frac{x(t+\tau_n)-x(t)}{\tau_n} - y_n\right\} = 0 .$$

Then

$$\frac{x(t_1)-x(t_2)}{t_1-t_2} \in V , \qquad t_1 \neq t_2 ,\ t_1, t_2 \in \Delta .$$

Next we introduce the following definition: the weak right-hand derivative of $x(\tau)$ at t equals y, $D_+^w x(t) = y$, if for every $\xi \in E'$

$$\lim_{h\to 0+} \xi\left\{\frac{x(t+h)-x(t)}{h}\right\} = \xi y .$$

We will say that a certain property holds *nearly everywhere* if it holds except an at most countable set of points.

The following theorem is an immediate consequence of what we have proved already:

THEOREM 69.4. *Let E be a real linear topological space and let V be a convex body (or merely closed and convex in case E is locally convex). Let $x(t)$ be weakly continuous. If $D_+^w x(t) \in V$ nearly everywhere on Δ, then $x(t_1)-x(t_2) \in (t_1-t_2)V$ for $t_1, t_2 \in \Delta$. If $D_+^w x(t)$ belongs to the interior of V nearly everywhere on Δ, then $x(t_1)-x(t_2) \in (t_1-t_2)$* int *$V$ for $t_1, t_2 \in \Delta$.*

Let V be a cone. We write $\theta < x$ if $x \in \operatorname{int} V$ and $x < y$ if $y - x > \theta$. The above theorem implies the following one:

THEOREM 69.5. *Let V be a cone with a non-empty interior. If $\theta \leqslant D_+^w x(t)$ nearly everywhere on Δ and $x(t)$ is weakly continuous, then $x(t_1) \leqslant x(t_2)$ for $t_1 \leqslant t_2$. If $\theta < D_+^w x(t)$ nearly everywhere on Δ and $x(t)$ weakly continuous on Δ, then $x(t_1) < x(t_2)$ for $t_1 < t_2$. If E is locally convex, then the inequality*

$D_+^w x(t) \geqslant 0$ satisfied nearly everywhere implies $x(t_1) \leqslant x(t_2)$, $t_1 \leqslant t_2$, V being an arbitrary cone, not necessarily possessing interior points.

The following theorem is an immediate consequence of Theorem 69.2:

THEOREM 69.6. *Let E be a real linear topological space and let $R(x)$ be a continuous functional defined on E. Assume that $R(x)$ is convex in this sense that*

$$R\big(\lambda x + (1-\lambda)y\big) \leqslant \lambda R(x) + (1-\lambda)R(y) \quad \text{for} \quad 0 \leqslant \lambda \leqslant 1 .$$

Suppose that the function $x(t)$ is weakly continuous on the interval Δ. We assume that the weak derivative $D_+^w x(t)$ exists nearly everywhere on Δ. Then for any $t_1, t_2 \in \Delta$ there is a $\tau \in [t_1, t_2]$ such that $D_+^w x(\tau)$ exists and

$$R\left(\frac{x(t_1)-x(t_2)}{t_1-t_2}\right) \leqslant R(D_+^w x(\tau)) .$$

Proof. Suppose our assertion is not true. Define

$$V = \left\{x\colon\ R(x) < R\left(\frac{x(t_1)-x(t_2)}{t_1-t_2}\right)\right\} .$$

The negation of the assertion means that $D_+^w x(t) \in V$ nearly everywhere on $[t_1, t_2]$. It follows from the properties of R that V is open and convex. Hence, by Theorem 69.2,

$$\frac{x(t_1)-x(t_2)}{t_1-t_2} \in V ,$$

which is a contradiction with the definition of V. This completes the proof.

COROLLARY. *If E is a Banach space and $R(x)$ stands for the norm of x, then Theorem 69.6 states that*

$$\|x(t_1)-x(t_2)\| \leqslant \|D_+^w x(\tau)\|\,|t_1-t_2|$$

for some τ, provided that $x(t)$ be weakly continuous and right-hand weakly differentiable nearly everywhere (see [2]).

We will now present some simple examples. More advanced applications of mean value theorems are given in subsequent sections.

EXAMPLE. Let E be a Banach space and let $x(t)$ be weakly continuous on the interval Δ. Suppose that the weak right-hand derivative $D_+^w x(t)$ exists nearly everywhere on Δ and satisfies nearly everywhere the inequality

$$\|D_+^w x(t)\| \leqslant M , \qquad M = \text{const} .$$

Theorem 69.4 then applies with $V = \{x\colon \|x\| \leqslant M\}$ and consequently

$$\|x(t_1)-x(t_2)\| \leqslant M\,|t_1-t_2| \qquad (t_1, t_2 \in \Delta) .$$

EXAMPLE. Let E be a Banach space and suppose that $x(t)$ is strongly differentiable for $0 < t < \eta$ $(\eta > 0)$. Assume that

$$\lim_{t\to 0+} x(t) = \theta, \qquad \lim_{t\to 0+} x'(t) = z_0$$

(both limits in the strong sense). We generalize the l'Hôpital rule by showing that

$$\lim_{t\to 0+} \frac{x(t)}{t} = z_0 .$$

Let $\varepsilon > 0$. Then there is a $\delta > 0$ such that $\|z_0 - x'(t)\| \leqslant \varepsilon$ for $0 < t < \delta$. We put in Theorem 69.4

$$V = \{x\colon \|z_0 - x\| \leqslant \varepsilon\}$$

and thus obtain

$$\left\| z_0 - \frac{x(t_1) - x(t_2)}{t_1 - t_2} \right\| \leqslant \varepsilon$$

if $0 < t_1, t_2 < \delta$, $t_1 \neq t_2$. The limit passage in the above inequality with $t_2 \to 0+$ shows that

$$\left\| z_0 - \frac{x(t_1)}{t_1} \right\| \leqslant \varepsilon$$

for $0 < t_1 < \delta$ which completes the proof.

Let us mention that our main assumption that E' contains nontrivial functionals cannot be omitted in the presented generalization of mean values theorems. We consider the following example of [44].

Let S be the metric space of Lebesgue measurable functions on $[0, 1]$, the distance function being defined by

$$\varrho(f, g) = \int_0^1 \frac{|f(v) - g(v)|}{1 + |f(v) - g(v)|}\, dv .$$

It is known that S' reduces to the zero functional. Consider the function $x(t)$ of the variable $t \in [0, 1]$ with values in S defined by

$$x(t) = \begin{cases} 1 & \text{if} \quad 0 \leqslant v \leqslant t, \\ 0 & \text{if} \quad t < v \leqslant 1 . \end{cases}$$

We have

$$x(t+h) - x(t) = \begin{cases} 1 & \text{if} \quad t < v \leqslant t+h, \\ 0 & \text{if} \quad v \notin (t, t+h], \end{cases}$$

and consequently

$$\varrho\left(\frac{x(t+h) - x(t)}{h}, \theta\right) \leqslant \left|\frac{h}{1+h}\right|$$

which implies that $x'(t) = \theta$. But $x(t)$ is not identically constant in $\Delta = [0, 1]$.

§ 70. Strong differential inequalities. Let V be a cone in the linear topological space E and suppose that V has a non-empty interior. We have introduced the definition

$$x < y \equiv y - x \in \operatorname{int} V .$$

In what follows we apply notation of § 69. It is easy to check that the following conditions hold true:

(70.1) If $x \leqslant y$ and $y < z$, then $x < z$.

(70.2) If $\lambda > 0$ and $x < y$, then $\lambda x < \lambda y$.

(70.3) If $x < y$ and $z \in E$, then $x + z < y + z$.

Let the space E satisfy the separation axiom of Hausdorff; then, we are able to introduce the definition (of Cauchy type) of the right-hand limit $\lim\limits_{t \to t_0+} x(t)$ for functions of the real variable t with values in E. It is a simple matter to verify that this limit has the following property:

(70.4) If $\lim\limits_{t \to t_0+} x(t) > x_0$, then $x(t) > x_0$ for $t > t_0$ and t sufficiently close to t_0.

We define the strong right-hand derivative by the formula

$$D_+^s x(t) = \lim_{h \to 0+} \frac{x(t+h) - x(t)}{h} .$$

THEOREM 70.1. *Let the function $f(t, x)$ be defined on the product $[t_0, t_0 + a) \times E$. Suppose that $f(t, x) \in E$ and*

(70.5) $$\bar{y} \leqslant \bar{\bar{y}} \quad \textit{implies} \quad f(t, \bar{y}) \leqslant f(t, \bar{\bar{y}}) .$$

Let the functions $x(t)$, $y(t)$ be continuous on $[t_0, t_0 + a)$ and suppose that:

(70.6) $$x(t_0) \leqslant y(t_0) ,$$

(70.7) $$D_+^s x(t) < f\big(t, x(t)\big) \quad \textit{on} \quad [t_0, t_0 + a) ,$$

(70.8) $$D_+^s y(t) \geqslant f\big(t, y(t)\big) \quad \textit{on} \quad [t_0, t_0 + a) .$$

Then $x(t) < y(t)$ on $(t_0, t_0 + a)$.

Proof. It follows from (70.5) and (70.6) that

$$f\big(t_0, x(t_0)\big) \leqslant f\big(t_0, y(t_0)\big)$$

and consequently by (70.7) and (70.8)

$$D_+^s x(t_0) < D_+^s y(t_0) .$$

Hence, by (70.4),

$$\frac{x(t) - x(t_0)}{t - t_0} < \frac{y(t) - y(t_0)}{t - t_0}$$

for $t_0 < t < t_0 + \delta$ with a suitable $\delta > 0$.

The above inequality and (70.2) imply

$$x(t)-x(t_0) < y(t)-y(t_0) .$$

But $x(t_0) \leqslant y(t_0)$, and by (70.1) and (70.3) we infer therefore $x(t) < y(t)$ for $t_0 < t < t_0+\delta$. Suppose now that the set

$$Z = \{t \in (t_0, t_0+a):\ y(t)-x(t) \notin \operatorname{int} V\}$$

is non-empty and write $\tau = \inf Z$. Obviously $\tau \geqslant t_0+\delta$ and $x(t) < y(t)$ for $t_0 < t < \tau$. The functions $x(t)$, $y(t)$ are continuous and V is closed. Hence $x(\tau) \leqslant y(\tau)$ and consequently $D^s_+ x(\tau) < D^s_+ y(\tau)$. This implies that there is an $\eta > 0$ such that $x(t) < y(t)$ for $t \in (\tau, \tau+\eta)$. We see that $x(t) \leqslant y(t)$ in the interval $[t_0, \tau+\eta)$. It follows then by (70.5), (70.7) and (70.8) that $D^s_+ x(t) < D^s_+ y(t)$ on $[t_0, \tau+\eta)$. Applying Theorem 69.5 to the difference $y(t)-x(t)$ we get $x(t)-y(t) < x(t_0)-y(t_0)$ and consequently $x(t) < y(t)$ on $(t_0, \tau+\eta)$, which is a contradiction with the definition of τ. Hence, Z is empty as was to be proved.

Using the above theorem one can easily imitate the classical procedure of § 8 in order to construct the maximum solution for the equation $x' = f(t, x)$. It is necessary to have some existence theorems which combined with Theorem 70.1 give the desired result. This is the case when for example E is a Banach space and $f(t, x)$ is completely continuous. For other details in this matter, see [25].

§ 71. Bendixson equation and differential inequalities. Let $u(t)$ be a real-valued function and suppose that

$$u'(t) \leqslant Ku(t)\,, \qquad a \leqslant t \leqslant b\,,$$

with $K = \text{const}$. Multiplying this inequality by e^{-Kt} we get

$$\frac{d}{dt}\big(u(t)e^{-Kt}\big) = u'(t)e^{-Kt} - Ku(t)e^{-Kt} \leqslant 0\,.$$

Hence, $u(t)e^{-Kt}$ decreases in $[a, b]$, and consequently

$$u(t) \leqslant u(a)e^{K(t-a)}\,, \qquad a \leqslant t \leqslant b\,.$$

This classical approach admits some generalization. Notice that the function $\varphi(t, \xi, \eta) = \eta e^{K(t-\xi)}$ satisfies the equation

$$\frac{\partial\varphi(t, \xi, \eta)}{\partial\xi} + f(\xi, \eta)\frac{\partial\varphi(t, \xi, \eta)}{\partial\eta} = 0 \tag{71.1}$$

with $f(\xi, \eta) = K\eta$ and $\bar{\eta} \leqslant \bar{\bar{\eta}}$ implies $\varphi(t, \xi, \bar{\eta}) \leqslant \varphi(t, \xi, \bar{\bar{\eta}})$. Hence

$$\frac{\partial\varphi}{\partial\eta} \geqslant 0\,. \tag{71.2}$$

It is a classical result that (71.1), (71.2) hold for $f(t, u)$ of class C^1. In this case $\varphi(t, \xi, \eta)$ stands for the value of solution of $u' = f(t, u)$ which at $t = \xi$ takes on the value η.

Suppose now that the function $u(t)$ satisfies

$$u'(t) \leqslant f\big(t, u(t)\big) \tag{71.3}$$

on the interval $[a, b]$. By analogy with the linear case we form the function $v(\xi) = \varphi\big(t, \xi, u(\xi)\big)$. We have

$$v'(\xi) = \frac{\partial\varphi(t, \xi, \eta)}{\partial\xi}\bigg|_{\eta=u(\xi)} + u'(\xi)\frac{\partial\varphi(t, \xi, \eta)}{\partial\eta}\bigg|_{\eta=u(\xi)}. \tag{71.4}$$

Multiplying (71.3) by $\partial\varphi/\partial\eta$ ($\geqslant 0$) and using (71.2) we get

$$u'(\xi)\frac{\partial\varphi(t, \xi, \eta)}{\partial\eta}\bigg|_{\eta=u(\xi)} \leqslant f(\xi, \eta)\frac{\partial\varphi(t, \xi, \eta)}{\partial\eta}\bigg|_{\eta=u(\xi)} \tag{71.5}$$

which by (71.1) and by (71.4) shows that $v'(\xi) \leqslant 0$. Hence $v(t) = \varphi\big(t, t, u(t)\big) = u(t) \leqslant \varphi\big(t, \xi, u(\xi)\big)$ if $\xi \leqslant t$.

We will now try to extend the above method. First of all the cone $S = (-\infty, 0]$ will be replaced by the closed and convex set V in a linear real topological space. Inequality (71.2) expresses the fact that S is invariant under the operator of multiplication by $\partial\varphi/\partial\eta$. It is then natural to require that the analogue of this operation leave invariant the set V. Inequality (71.3) should be replaced by the inclusion $x'(t) - f\big(t, x(t)\big) \in V$. The most difficult part concerns the proper interpretation of formula (71.1). On the other hand, we need the formula for differentation of composite functions. All these properties can be stated formally as assumptions. We will get then a formal theorem. Anyhow, it will be worthwhile for its assumptions admit a great deal of interesting interpretations.

We start with some notation and definitions. First we assume that the space E in question is locally convex. Let $\Delta = [0, a)$ ($a \leqslant +\infty$) and suppose that the function $f(t, x)$ is defined on $\Delta \times Z$, where Z is a subset of E. It is supposed that the values of $f(t, x)$ belong to E. Let the function $x(t)$ be defined on Δ and let $t_0 \in \Delta$. The equality $\widetilde{D}_+ x(t_0) = y$ means that there is a sequence $\tau_n \to 0+$ such that for each $\xi \in E'$

$$\lim_{n\to\infty} \xi\left\{\frac{x(t_0 + \tau_n) - x(t_0)}{\tau_n}\right\} = \xi y .$$

We say then that $x(t)$ is *quasidifferentiable* to $\widetilde{D}_+ x(t_0)$ at the point $t = t_0$.

THEOREM 71.1. *Let $V \subset E$ be closed and convex. Suppose that $x(s) \in Z$ for $s \in \Delta$ and let $\widetilde{D}_+ x(t)$ exist nearly everywhere on Δ. Assume that*

$$\widetilde{D}_+ x(s) - f\big(s, x(s)\big) \in V \tag{71.6}$$

nearly everywhere on Δ.

We assume that there exists a vector-valued function $\varphi(t, s, x)$ defined on $\Delta \times \Delta \times Z$ such that:

(71.7) $\varphi(t, s, x(s))$ *is weakly continuous in s on Δ, for $t \in \Delta$.*

(71.8) $\varphi(t, s, x)$ *is weakly right-hand differentiable in s on Δ to* $\dfrac{\partial_+\varphi(t, s, x)}{\partial s}$.

It is supposed that there exists a linear mapping $\varphi_x(t, s, x)$ from E into E, depending on parameters (t, s, x), for which the following conditions hold true:

(71.9) $$\varphi_x(t, s, x)V \subset V .$$

(71.10) *For every* $(t, s, x) \in \Delta \times \Delta \times Z$

$$\frac{\partial_+\varphi(t, s, x)}{\partial s} + \varphi_x(t, s, x)f(t, x) = \theta .$$

We assume that for each fixed $t \in \Delta$ the function $\varphi(t, s, x(s))$ is quasi-differentiable in s to $\tilde{D}_+[\varphi(t, s, x(s))]$ for those s for which $\tilde{D}_+x(s)$ exists and, moreover,

(71.11) $$\tilde{D}_+[\varphi(t, s, x(s))] = \varphi_x(t, s, x(s))\tilde{D}_+x(s) + \frac{\partial_+\varphi(t, s, x)}{\partial s}\bigg|_{x=x(s)}$$

for such s.

Under these assumptions

$$\varphi(t, s_1, x(s_1)) - \varphi(t, s_2, x(s_2)) \in (s_1 - s_2)V$$

for $s_1, s_2, t \in \Delta$.

Proof. Using (71.6) and (71.9) we get

(71.12) $$\varphi_x(t, s, x(s))\tilde{D}_+x(s) - \varphi_x(t, s, x(s))f(s, x(s)) \in V$$

nearly everywhere on the interval Δ. It follows from (71.10) and from (71.11)

(71.13) $$\begin{aligned}\tilde{D}_+[\varphi(t, s, x(s))] &= \varphi_x(t, s, x(s))\tilde{D}_+x(s) + \frac{\partial_+\varphi(t, s, x)}{\partial s}\bigg|_{x=x(s)} \\ &= \varphi_x(t, s, x(s))\tilde{D}_+x(s) - \varphi_x(t, s, x(s))f(s, x(s)) .\end{aligned}$$

(71.12) and (71.13) imply that

(71.14) $$\tilde{D}_+[\varphi(t, s, x(s))] \in V$$

nearly everywhere on Δ. Assumption (71.7) and (71.14) and Theorem 69.1 imply the assertion of the theorem.

A few comments are now necessary. We are not precise and omit the analytical details.

Formally equation (71.10) means that $\varphi(t, s, x)$ is for every fixed t the so-called *first integral* for the equation $x' = f(t, x)$. In our case the first integral is a vector-valued function. The linear operator $\varphi_x(t, s, x)$ is nothing else but the analogue of the Fréchet differential of $\varphi(t, s, x)$ in x. Given the function $f(t, x)$, we look for the first integrals of equation $x' = f(t, x)$. Then we form the operators $\varphi_x(t, s, x)$ and just try to characterize some closed and convex subsets invariant under the mappings $\varphi_x(t, s, x)$. In that way the question when and how our theorem can be used is reduced to the question of invariant sets of some linear operators.

Suppose that the right-hand Cauchy problem for equation

$$x' = f(t, x) \tag{71.15}$$

has the unique solution within a sufficiently large class of initial conditions. Let us take the solution $x(t)$ of (71.15) such that $x(s) = x$. Denote by $\varphi(t, s, x)$ the vector $x(t)$ $(s < t,\, s, t \in \Delta)$, i.e. $\varphi(t, s, x) = x(t)$. It follows then that $\varphi\big(t, s, x(s)\big)$ $\big(= x(t)\big)$ does not depend on s. If $\varphi(t, s, x)$ is a sufficiently regular function, then it satisfies (71.10). This is in general the case when $f(t, x)$ is a regular function. In a scalar case, if f is of class C^1, (71.10) holds. In what follows (71.10) will be called the *Bendixson equation*. If E is a Banach space and $f(t, x)$ is Fréchet differentiable in (t, x) in a continuous way, then (71.10) holds for the above defined $\varphi(t, s, x)$. Notice yet that the function φ just discussed is, in general, linear in x provided that $f(t, x)$ be linear in x. Formally this can be written as $\varphi(t, s, z) - \varphi(t, s, y) = \varphi_x(t, s, x)(z-y)$ for each x.

§ 72. Linear differential inequalities in Banach spaces I. We noticed in the previous section that if E is a Banach space, then the Bendixson equation holds for sufficiently regular functions $f(t, x)$ with a natural choice of the function φ. The method of Bendixson equation for integration of finite systems of ordinary differential inequalities was used in [50]. This is the case when E is finite dimensional.

Different choices of the space E give the interpretations of Theorem 71.1. Here are included infinite systems of first order ordinary differential inequalities of the form

$$\xi_i'(t) \leqslant f_i\big(t, \xi_1(t), \ldots\big) \quad (i = 1, 2, \ldots),$$
$$\eta_i'(t) \geqslant f_i\big(t, \eta_1(t), \ldots\big) \quad (i = 1, 2, \ldots).$$

E will stand for a space of sequences with suitable restrictions on their behaviour as $i \to \infty$. Also we need that $f = \{f_i\} \in E$. The abstract regularity assumptions about f will be translated onto corresponding classical properties of functions and sequences in question.

There are, however, some other interesting interpretations of Theorem 71.1 in which the regularity assumptions about $f(t, x)$ are of different and more delicate character. What we have in mind is the case when $f(t, x)$ is linear in x. There is a great variety of methods of establishing the Bendixson equation in the linear case. We mention here two methods. The first one is the method based on the theory of distributions. The Bendixson equation can be obtained in this case by using the Holmgren method of adjoint systems. This approach was developed in [29], where are discussed first order partial linear differential inequalities for distribution-valued function. The second method is the method of the Hille-Yosida theory of one-parameter semi-groups of operators in Banach spaces. This theory will be used in the present and subsequent sections.

We will give now a brief outline of basic facts on one-parameter semi-groups of operators. We follow here the monograph [12]. From now on E will stand for a Banach space.

First we introduce some notational conventions. For definitions and other details, we refer to [12].

The Bochner integral of the function $x(t)$ is denoted by $\int_\alpha^\beta x(\tau)d\tau$, the Pettis integral by $(\mathrm{P})\int_\alpha^\beta x(\tau)d\tau$. The symbol w-lim denotes the weak limit, s-lim the strong one. Let the function $x(t)$ be defined in the neighborhood of t_0. We define

$$D_+^{\mathrm{w}} x(t_0) = \operatorname*{w-lim}_{h\to 0+} \frac{x(t_0+h)-x(t_0)}{h},$$

$$D_+^{\mathrm{s}} x(t_0) = \operatorname*{s-lim}_{h\to 0+} \frac{x(t_0+h)-x(t_0)}{h},$$

$$x'(t_0) = \operatorname*{s-lim}_{h\to 0} \frac{x(t_0+h)-x(t_0)}{h}.$$

The right-hand weak (strong) partial differentiation is denoted by $\partial_+^{\mathrm{w}}/\partial s$ $(\partial_+^{\mathrm{s}}/\partial s)$. The bilateral strong partial derivative is denoted by $\partial^{\mathrm{s}}/\partial s$. Given the operator U its domain is denoted by $D[U]$, the range by $R[U]$.

Let $\{T(t)\}$ be a one-parameter family of bounded, linear operators in E defined for $t>0$. We say that $\{T(t)\}$ is a *semi-group* if the following condition holds true:

$$T(t_1+t_2) = T(t_1)T(t_2), \qquad t_1, t_2 > 0. \tag{72.1}$$

We always assume that for each $x \in E$ the function $T(t)x$ is strongly continuous on the half-line $(0, \infty)$. Next we define

$$\operatorname*{s-lim}_{h\to 0+} \frac{T(h)-I}{h} x = A_0 x \tag{72.2}$$

whenever the limit exists. Notice that if $0<\alpha<\beta$ and x is an arbitrary element of E, then

$$\int_\alpha^\beta T(\tau)x\,d\tau = x_{\alpha,\beta} \in D[A_0].$$

The linear manifold $D[A_0]$ is dense in $\bigcup_{a>0} E_a = E_0$ where $E_a = \{y: y = T(a)x, x \in E\}$. In what follows we assume that E_0 is dense in E and consequently $D[A_0]$ is dense in E.

Assume now that for each $x \in E$ the function $\|T(t)x\|$ is summable over the interval $[0, 1]$, i.e.

$$\int_0^1 \|T(\tau)x\| d\tau < +\infty . \tag{72.3}$$

One can prove that there are two finite constants $M > 0$, ω such that

$$\|T(t)\| \leqslant Me^{\omega t} \tag{72.4}$$

for sufficiently large t. (72.3) and (72.4) imply that the integral

$$R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x\,dt$$

converges strongly for λ with $\operatorname{Re}\lambda > \omega$ and arbitrary x. The operator $R(\lambda)$ is linear and bounded.

We say that the *semi-group is of class* $(0, A)$ if it satisfies (72.3) and additionally the following equality holds true:

$$\operatorname*{s-lim}_{\lambda\to\infty} \lambda R(\lambda)x = x \quad \text{for} \quad x \in E . \tag{72.5}$$

One can prove that if $\{T(t)\}$ is of class $(0, A)$, then the corresponding operator A_0 defined by (72.2) has the smallest closed extension $A - A_0 \subset A$. A is called the *infinitesimal generator* of $\{T(t)\}$. The resolvent $R(\lambda, A)$ exists for $\operatorname{Re}\lambda > \omega$ and $R(\lambda) = R(\lambda, A)$. In case when

$$\operatorname*{s-lim}_{h\to 0+} \frac{1}{h} \int_0^h T(\tau)x\,d\tau = x \quad \text{for} \quad x \in E$$

we have exactly $A = A_0$. This obviously happens when $T(t)$ is of class (C_0), that is when $\operatorname*{s-lim}_{h\to 0+} T(h)x = x$ for $x \in E$.

V being a cone in E we say that the operator U is *positive*, in symbols $U \geqslant \theta$, if $U(D[U] \cap V) \subset V$. The basic property we need is the following one:

The semi-group $\{T(t)\}$ *of class* $(0, A)$ *consists of positive operators (simply—is positive) if and only if* $R(\lambda, A) \geqslant \theta$ *for sufficiently large real* λ.

Let us consider the one-parameter family $A(t)$ of linear operators with domains and ranges in E. We are interested in the abstract differential inequality

$$x'(t) - A(t)x(t) \in V$$

where V is a closed and convex subset of E. Following the general ideas of § 71 we introduce the following conditions:

(72.6) There exists a family $U(t, s)$ $(0 < s < t < a)$ of linear bounded operators which leave invariant the set V, i.e. $U(t, s)V \subset V$.

(72.7) The strong derivative

$$\frac{\partial_+^s U(t, s)x}{\partial s} \quad (s < t)$$

exists for $x \in D[A(s)]$ and

$$\frac{\partial_+^s U(t, s)x}{\partial s} + U(t, s)A(s)x = \theta$$

for those x.

(72.8) For each $x \in E$ the function $U(t, s)x$ is strongly continuous in s.

We will prove the following theorem:

THEOREM 72.1. *Let* (72.6), (72.7) *and* (72.8) *be satisfied and suppose that the function* $x(t) \in D[A(t)]$, $0 < t < a$, *is strongly continuous on* $(0, a)$. *We assume that*

$$D_+^s x(s) - A(s)x(s) \in V \tag{72.9}$$

nearly everywhere on $(0, a)$.

Then

$$U(t, s_1)x(s_1) - U(t, s_2)x(s_2) \in (s_1 - s_2)V \tag{72.10}$$

for $s_1 < t$, $s_2 < t$, $s_1, s_2, t \in (0, a)$.

Proof. We will verify that the function

$$\varphi(t, s, x) = U(t, s)x \tag{72.11}$$

satisfies the assumptions of Theorem 71.1. Obviously we put $f(t, x) = A(t)x$. It follows from (72.8) that $\varphi(t, s, x(s))$ is weakly continuous in s. Moreover, $\varphi_x(t, s, x) = U(t, s)x$. By (72.7) we see that φ, φ_x satisfy the Bendixson equation. Also $\varphi_x(t, s, y) = \varphi(t, s, y) \in V$ if $y \in V$.

Suppose that $D_+^s x(s_0)$ exists. It follows from the formula $(h > 0)$

$$\begin{aligned}&\frac{1}{h}[U(t, s_0+h)x(s_0+h) - U(t, s_0)x(s_0)]\\ &= U(t, s_0+h)D_+^s x(s_0) + \frac{1}{h}[U(t, s_0+h) - U(t, s_0)]x(s_0) + U(t, s_0+h)\frac{o(h)}{h}\end{aligned}$$

and from the equiboundedness of $U(t, s)$ on compact subsets of $(0, a)$ that

$$\left.\frac{\partial_+^s\big(U(t, s)x(s)\big)}{\partial s}\right|_{s=s_0} = \left.\frac{\partial_+^s\big(U(t, s)x(s_0)\big)}{\partial s}\right|_{s=s_0} + U(t, s_0)D_+^s x(s_0)\,. \tag{72.12}$$

Hence $U(t,s)x(s)$ satisfies (71.11) of Theorem 71.1. The assertion (72.10) follows now from Theorem 71.1.

Suppose now that V is a cone. It induces the semi-order $\leqslant$. We need the following lemma:

LEMMA 72.1. *Suppose that the functions $x(t)$, $y(t)$ satisfy either one of the following conditions:*

(a) *The functions $x(t)$, $y(t)$ are weakly continuous on $(0, a) = \Delta$ and for every $\xi \in V'$ there exists an at most countable subset $\Delta - Z_\xi$ of Δ such that $D_+ \xi x(t) \leqslant \xi y(t)$ for $t \in Z_\xi$. The function $y(t)$ is Pettis integrable.*

(b) *The function $x(t)$ is weakly absolutely continuous and $\frac{d}{dt}\xi x(t) \leqslant \xi y(t)$ for $t \in Z_\xi$,* $\operatorname{mes}(\Delta - Z_\xi) = 0$, *$\xi \in V'$. The function $y(t)$ is Pettis integrable.*

Then

$$x(t_2) - x(t_1) \leqslant (\mathrm{P}) \int_{t_1}^{t_2} y(\tau)\, d\tau\,, \qquad t_1 < t_2\,.$$

The above lemma can be easily proved by using methods developed in § 69.

Now we are able to prove the following

THEOREM 72.2. *Let V be a cone and let conditions* (72.6), (72.7) *and* (72.8) *be satisfied. Assume that*

$$U(\tau, \tau) = I \quad \text{for} \quad \tau \in (0, a)\,, \tag{72.13}$$

I denoting the identity operator. Let the strongly continuous function $x(t) \in D[A(t)]$, $0 < t < a$, satisfy nearly everywhere on $(0, a)$ the inequality

$$D_+^{\mathrm{s}} x(t) \leqslant A(t)x(t) + y(t)\,. \tag{72.14}$$

We suppose that $y(t)$ is strongly continuous on $(0, a)$.
Then

$$x(t) \leqslant U(t,s)x(s) + \int_s^t U(t,\tau)y(\tau)\, d\tau\,, \qquad s < t\,.$$

Proof. It follows from (72.14) that

$$U(t,s)D_+^{\mathrm{s}} x(s) - U(t,s)A(s)x(s) \leqslant U(t,s)y(s)\,.$$

Using (72.12) of the previous proof and (72.7) we get that

$$\frac{\partial_+^{\mathrm{s}}\big(U(t,s)x(s)\big)}{\partial s} \leqslant U(t,s)y(s) \tag{72.15}$$

holds nearly everywhere on $(0, t)$. Applying Lemma 72.1 and (72.13) we get the assertion.

Suppose that $U(t,s) = T(t-s)$, where $\{T(t)\}$ is a semi-group of class $(0, A)$. In that case (72.6) is satisfied if V is invariant under $T(t)$ for $t > 0$. It is obvious that $A(t) = \text{const} = A$, where A stands for the infinitesimal generator of $\{T(t)\}$. Notice that

$$\frac{\partial}{\partial s}\big(T(t-s)x\big) = -\frac{d}{d\tau}T(\tau)x\big|_{\tau=t-s} = -AT(t-s)x = -T(t-s)Ax\,, \qquad x \in D[A]$$

(see Theorem 11.5.3 of [12]). If $\{T(t)\}$ is of class (C_0) then $T(\tau-\tau) = T(0) = I$ and (72.13) holds. Observe that the assumptions of the above theorem are true if V is a cone and $\{T(t)\}$ is positive. The corresponding theorem, which is a generalization of the classical theorem about linear differential inequalities, is the following one:

THEOREM 72.3. *Let V be a cone and let $\{T(t)\}$ be a positive semi-group of class (C_0). Assume that the function $x(t)$ is strongly continuous on $[0, a)$ and*

(72.16) $\qquad D_+^s x(t) \leqslant Ax(t)$ *nearly everywhere on* $[0, a)$,

(72.17) $\qquad x(0) \leqslant \theta$.

Then $x(t) \leqslant \theta$ on $[0, a)$.

Proof. We put $y(t) \equiv \theta$ and $U(t,s) = T(t-s)$ in Theorem 72.2 and thus get

$$x(t) \leqslant T(t-s)x(s)$$

for $s < t$. For $s = 0$ we have $x(t) \leqslant T(t)x(0) \leqslant T(t)\theta \leqslant \theta$, q.e.d.

Going back to Theorem 72.1 we point out that usually the construction of the function $U(t,s)$ is achieved by using semi-groups generated by $A(t)$. One assumes that, for a fixed t, $A(t)$ is an infinitesimal generator of a semi-group $\{T(\tau; t)\}$ of class (C_0). One can prove that the required operator function U may be obtained by the formula

$$\operatorname*{s-lim}_{\max|t_{i+1}-t_i|\to 0} \Pi T(t_{i+1}-t_i; t_i) = U(t,s)\,, \qquad s = t_0 < t_1 < \ldots < t_n = t\,,$$

provided that $A(t)$ satisfy some regularity assumptions. This way of integration of equation

(72.18) $$x'(t) = A(t)x(t)$$

was initiated by T. Kato in [15]. For an extensive review of related topics, see [17].

In general the Bendixson equation is a consequence of the integration procedure of (72.18). The assumptions about $A(t)$ are of that type that for x-es belonging to a dense subset of $\bigcap D[A(t)]$

(72.19) $$\frac{\partial}{\partial t}\big(U(t,s)x\big) = A(t)U(t,s)x \qquad (s \leqslant t)$$

and

(72.20) $$U(t, t)x = x \quad \text{for all} \quad x \in E\,.$$

Moreover, $U(t, s)$ are bounded and strongly continuous in (t, s). On the other hand, the Cauchy problem for (72.18) has the uniqueness property. It follows then that $U(t, s)$ satisfies the functional equation

(72.21) $$U(t, s)\,U(s, u) = U(t, u)\,, \qquad u \leqslant s \leqslant t\,.$$

If the above properties hold, then the Bendixson equation may be proved as follows. Let $x \in \bigcap D[A(t)]$ and consider $(h > 0)$

$$\zeta(h) = \frac{U(t, s+h)x - U(t, s)x}{h}\,.$$

By (72.21)

$$\zeta(h) = U(t, s+h)\frac{x - U(s+h, s)x}{h}\,.$$

By (72.19)

$$\frac{x - U(s+h, s)x}{h} \underset{h\to 0+}{\to} -A(s)x\,.$$

Hence $\zeta(h) \to -U(t, s)A(s)x$ as was to be proved.

The function $U(t, s)$ being in general the multiplicative integral of $T(\tau; t)$, the inclusion $U(t, s)V \subset V$ holds whenever $T(\tau; t)V \subset V$. For example, if V is a cone and every semi-group $\{T(\tau; t)\}$ is positive, then $U(t, s)$ is positive for $0 \leqslant s \leqslant t \leqslant a$.

Now we will present an application of Theorem 72.3 to integration of countable systems of linear ordinary differential inequalities.

The temporally homogeneous Markoff process with a countable number of possible states is described by the infinite matrix $\{p_{jk}(t)\}$, $0 \leqslant t < +\infty$, of transition probabilities. Inequalities $p_{jk} \geqslant 0$ and the semi-group property

$$p_{ik}(t+s) = \sum_{j|1}^{\infty} p_{ij}(t)p_{jk}(s)$$

are satisfied. Under some general conditions $p_{jk}(t)$ satisfy the Kolmogoroff equations

(72.22) $$p'_{ik}(t) = \sum_{j|1}^{\infty} p_{ij}a_{jk}\,.$$

(72.23) $$p'_{ik}(t) = \sum_{j|1}^{\infty} a_{ij}p_{jk}\,,$$

with the initial conditions

(72.24) $$p_{jk}(0) = \delta_{jk}\,.$$

The constants a_{jk} satisfy

$$-a_{jj} = a_j \geqslant 0\,, \quad a_{jk} \geqslant 0 \quad \text{for} \quad j \neq k\,,$$
(72.25)
$$\sum_{j\,1}^{\infty} a_{jk} = 0\,, \quad k = 1, 2, 3, \ldots$$

Conversely, given the matrix $\{a_{jk}\}$ which satisfies (72.25) we can ask about the integration of (72.22), (72.23).

T. Kato in [16] constructed the solution of the Kolmogoroff equations by using the semi-group theory. We will give here a brief summary of his results.

Let l^1 be the space of one-sided summable real valued sequences $x = \{\xi_i\}$ with the usual norm

$$\|x\| = \sum_{1}^{\infty} |\xi_i|\,.$$

The matrix $\{a_{jk}\}$ defines the operator A with domain and range in l^1 as follows: the domain $D[A]$ consists of those $x = \{\xi_i\} \in l^1$ for which the series $\eta_k = \sum_{j|1}^{\infty} a_{jk}\xi_j$ are absolutely convergent and $\sum_{k|1}^{\infty} |\eta_k| < +\infty$. We define then $Ax = \{\eta_k\}$. Let D_0 be a linear manifold spanned by vectors $y_i = \{\delta_{ik}\}$ (δ_{ik} — Kronecker symbol). It is easy to see that $D_0 \subset D[A]$. The restriction of A to D_0 is denoted by A^0. Observe that D_0 is dense in l^1. In l^1 we define a natural cone V by

$$V = \{x\colon\ x = \{\xi_i\} \in l^1\,,\ \xi_i \geqslant 0\,,\ i = 1, 2, 3, \ldots\}\,.$$

In what follows the term "positive" is used in the sense of that cone. The main result of [16] is the following statement: *There is at least one positive semi-group of class* (C_0) *in* l^1 *with a generator being an extension of* A^0. *Among these semi-groups there is the unique minimal one* (¹) $\{T(t)\}$ *such that its generator* G *satisfies* $A \supset G \supset A^0$.

The domain $D[G]$ is not characterized explicitly. Anyhow, $D[A^0] \subset D[G]$. For $x \in D[A^0]$ we have

$$Gx = A^0x = \Big\{\sum_{j|1}^{\infty} a_{jk}\xi_j\Big\}\,.$$

Notice that D_0 contains pretty regular, non-trivial curves of type $x = x(t)$. Indeed, let the real-valued function $e(t)$ be of class C^2 on $[0, \infty)$ and $e(t) = 0$ for $t \geqslant 1$ and $t \leqslant 0$. Define $\{\xi_k(t)\} = x(t)$ with

$$\xi_k(t) = \frac{1}{2^{3k}}\, e(2^k t)$$

(¹) *Minimal* in the sense of semi-order relation.

and assume that $e(t) \neq 0$ in $(0, 1)$. It is a simple matter to verify that $\{\xi_k(t)\} \in l^1$; $x(t)$ is strongly continuously differentiable. Moreover, in every neighborhood of zero the curve $x = x(t)$ does not run in a finite dimensional space.

Suppose we are given the function $x(t) = \{\xi_k(t)\} \in D_0$ strongly right-hand differentiable to $D^s_+ x(t)$. Notice that the strong convergence in l^1 implies the convergence in coordinates. Hence $D^s_+ x(t) = \{D_+ \xi_k(t)\}$. By definition of A^0 and by theorem of Kato, $Gx(t) = \{\sum_{j|1}^{\infty} a_{jk}\xi_j(t)\}$. The abstract inequality $D^s_+ x(t) \leqslant Gx(t)$ is equivalent with the countable system of ordinary inequalities.

THEOREM 72.4. *Suppose that the matrix $\{a_{jk}\}$ satisfies (72.25). Let the continuous function $x(t) = \{\xi_k(t)\}$ be strongly right-hand differentiable on $(0, a)$, in l^1, to $D^s_+ x(t)$. Assume that for every $t \in (0, a)$ there is a finite number of $\xi_k(t)$ different from zero.*

Suppose that for $0 < t < a$

$$D_+ \xi_k(t) \leqslant \sum_{j|1}^{\infty} a_{jk}\xi_j(t) \quad (k = 1, 2, 3, \ldots), \tag{72.26}$$

and

$$\xi_k(0) \leqslant 0 \quad (k = 1, 2, 3, \ldots). \tag{72.27}$$

Then $\xi_k(t) \leqslant 0$ $(k = 1, 2, \ldots)$, $0 \leqslant t < a$.

Proof. We see that $x(t) \in D_0$. By (72.26) and (72.27)

$$D^s_+ x(t) \leqslant Gx(t), \quad x(0) \leqslant \theta.$$

The semi-group generated by G is positive. By Theorem 72.3 we get therefore $x(t) \leqslant \theta$, i.e. $\xi_k(t) \leqslant 0$, q.e.d.

COROLLARY. *Notice that the infinitesimal generators of positive semi-groups in some functional spaces of continuous or merely summable functions are in a certain sense necessarily second order elliptic operators satisfying a version of maximum principle* (see [9] and [68]).

Thus the theorems of the present section give the operator-theoretical treatment of linear parabolic inequalities of second order.

The final result of the present section is the following theorem:

THEOREM 72.5. *Let A be an infinitesimal generator of a positive semi-group $\{T(t)\}$ of class (C_0). Let $B(t)$ be a strongly continuous operator-valued function. Assume that there is a real number β such that $B(t) + \beta I \geqslant \theta$ for $t \in [0, a)$. If $x(t)$ is strongly differentiable to $x'(t)$ on $[0, a)$ and*

$$x'(t) \leqslant [A + B(t)]x(t) \quad \text{on} \quad [0, a), \quad x(0) \leqslant \theta,$$

then $x(t) \leqslant \theta$ on $[0, a)$.

Proof. Write $z(t) = e^{\beta t}x(t)$. Then

$$z'(t) \leqslant Az(t) + [B(t) + \beta I]z(t) .$$

We put $U(t, s) = T(t-s)$ and $y(t) = [B(t) + \beta I]z(t)$ in Theorem 72.2 and thus obtain

$$z(t) \leqslant \int_0^t T(t-\tau)[B(\tau) + \beta I]z(\tau)\,d\tau .$$

The operator $T(t-\tau)[B(\tau) + \beta I]$ is positive. It follows that the sequence

$$z_0(t) = z(t) ,$$

$$z_{n+1}(t) = \int_0^t T(t-\tau)[B(\tau) + \beta I]z_n(\tau)\,d\tau$$

is an increasing one: $z_n(t) \leqslant z_{n+1}(t)$. Obviously

$$z_n(t) \to \theta \quad \text{on} \quad [0, a) ,$$

which completes the proof.

§ 73. Linear differential inequalities in Banach spaces II. So far the functions in inequalities have been strongly differentiable. In what follows we will assume less, namely that the functions are weakly differentiable. For the sake of clarity we restrict ourselves (not essentially of course) to the case when V is a cone. We assume that A is an infinitesimal generator of a positive semi-group $\{T(t)\}$ of class $(0, A)$.

LEMMA 73.1. *Let the function $x(t)$ be right-hand weakly differentiable to $D_+^w x(t_0)$ at t_0. Write*

$$x_\lambda(t) = \lambda R(\lambda, A)x(t)$$

for sufficiently large λ.

Then the function $T(t-s)x_\lambda(s)$ is right-hand weakly differentiable in s at $s = t_0$ and

$$\frac{\partial_+^w}{\partial s}\big(T(t-s)x_\lambda(s)\big)_{s=t_0} = T(t-t_0)D_+^w x_\lambda(t_0) - T(t-t_0)Ax_\lambda(t_0) .$$

Proof. Let $\xi \in E'$. We have

$$\xi\frac{T(t-t_0-h)x_\lambda(t_0+h) - T(t-t_0)x_\lambda(t_0)}{h}$$

$$= -\xi\frac{T(t-t_0-h) - T(t-t_0)}{-h}x_\lambda(t_0) + \xi T(t-t_0-h)\frac{x_\lambda(t_0+h) - x_\lambda(t_0)}{h} .$$

The first member tends to $-\xi AT(t-t_0)x_\lambda(t_0)$. The other one equals to

$$\xi[T(t-t_0-h)-T(t-t_0)]\lambda R(\lambda, A)\frac{x(t_0+h)-x(t_0)}{h}+$$
$$+\xi T(t-t_0)\frac{x_\lambda(t_0+h)-x_\lambda(t_0)}{h}.$$

The second member of that sum tends to $\xi T(t-t_0)D_+^{\mathrm{w}}x_\lambda(t_0)$. On the other hand, the formula

$$[T(\tau_2)-T(\tau_1)]\lambda R(\lambda, A)x = \int_{\tau_1}^{\tau_2} T(\tau)\lambda AR(\lambda, A)x\,d\tau\,, \qquad x \in E\,,$$

implies

$$\|[T(\tau_2)-T(\tau_1)]\lambda R(\lambda, A)x\| \leqslant \sup_\tau\|\lambda T(\tau)\|\,\|AR(\lambda, A)x\|\,|\tau_2-\tau_1| \tag{73.1}$$

and consequently

$$\left|\xi[T(t-t_0-h)-T(t-t_0)]\lambda R(\lambda, A)\frac{x(t_0+h)-x(t_0)}{h}\right| \leqslant MNh\,,$$

where

$$\left\|\frac{x(t_0+h)-x(t_0)}{h}\right\| \leqslant M$$

and N is a suitable constant derived from (73.1).

Summing up the above relations we get the assertion of the lemma.

THEOREM 73.1. *Let $x(t)$ be weakly continuous in $(0, a)$ and let it satisfy nearly everywhere the inequality*

$$D_+^{w}x(t) \leqslant Ax(t)\,. \tag{73.2}$$

Then $x(t) \leqslant T(t-s)x(s)$ for $0 < s < t < a$.

Proof. Write $x_\lambda(t) = \lambda R(\lambda, A)x(t)$. We have by Lemma 73.1

$$\frac{\partial_+^{\mathrm{w}}}{\partial s}\big(T(t-s)x_\lambda(s)\big) = T(t-s)D_+^{\mathrm{w}}x_\lambda(s)-T(t-s)Ax_\lambda(s)$$

whenever $D_+^{\mathrm{w}}x(s)$ exists. The resolvent $R(\lambda, A)$ is positive for large λ and commutes with A. Hence, by (73.2),

$$D_+^{\mathrm{w}}x_\lambda(s) \leqslant Ax_\lambda(s)\,. \tag{73.3}$$

Using the arguments similar to those used in the proof of Lemma 73.1 one shows that $T(t-s)x_\lambda(s)$ is weakly continuous in s. By the same lemma and by (73.3) we get that $x_\lambda(s)$ satisfies the assumptions of Theorem 71.1 with suitable φ, φ_x and f. Thus

$$T(t-s_2)x_\lambda(s_2) \leqslant T(t-s_1)x_\lambda(s_1)\,, \qquad s_1 < s_2\,.$$

But $x_\lambda(s) \to x(s)$. Hence $T(t-s_2)x(s_2) \leqslant T(t-s_1)x(s_1)$. We put $\tau = t-s_2$ and get $T(\tau)x(s_2) \leqslant T(\tau)T(s_2-s_1)x(s_1)$. Hence

$$\text{(73.4)} \qquad \lambda R(\lambda, A)x(s_2) = \lambda \int_0^\infty e^{-\lambda\tau}T(\tau)x(s_2)\,d\tau$$
$$\leqslant \lambda \int_0^\infty e^{-\lambda\tau}T(\tau)T(s_2-s_1)x(s_1)\,d\tau = \lambda R(\lambda, A)T(s_2-s_1)x(s_1)\,.$$

But $\operatorname*{s-lim}_{\lambda\to\infty} \lambda R(\lambda, A)x = x$, $x \in E$. The assertion follows from (73.4) by a limit passage.

COROLLARY. *Assume additionally in the above theorem that*

$$\operatorname*{w-lim}_{t\to 0+} x(t) = x(0) \leqslant \theta\,.$$

By (73.4) *and by the theorem*

$$x_\lambda(t) \leqslant T(t-s)x_\lambda(s)\,, \qquad s < t\,,$$

and consequently

$$\operatorname*{w-lim}_{s\to 0+} T(t-s)x_\lambda(s) = T(t)x_\lambda(0) \leqslant \theta\,.$$

Hence

$$x(t) = \operatorname*{s-lim}_{\lambda\to\infty} x_\lambda(t) \leqslant \theta\,.$$

Using the same technique as in the proof of the above theorem and applying Lemma 72.1 and Lemma 73.1 one proves the following theorem:

THEOREM 73.2. *Let the function $x(t)$ have the Bochner summable (over every compact in $(0, a)$) derivative $x'(t)$ and suppose that*

$$x(\tau_2)-x(\tau_1) = \int_{\tau_1}^{\tau_2} x'(\tau)\,d\tau \qquad \textit{for} \qquad \tau_1, \tau_2 \in (0, a)\,.$$

Suppose that $x(t)$ satisfies almost everywhere the inequality

$$x'(t) \leqslant Ax(t)\,.$$

Then $x(t) \leqslant T(t-s)x(s)$ for $0 < s < t$.

The assumptions concerning the differentiability of $x(t)$ can be weakened at the cost of some additional assumptions. That possibility is included in the following theorem:

THEOREM 73.3. *Let the function $x(t)$ be weakly absolutely continuous and Bochner integrable in any compact subinterval of $(0, a)$. Assume that*

$$\int_{\tau_1}^{\tau_2} x(\tau)\,d\tau \in D[A] \qquad \textit{for} \qquad \tau_1, \tau_2 \in (0, a)$$

and let $Ax(t)$ be Bochner integrable in every compact subinterval of $(0, a)$. *It is supposed that for every positive functional $\xi \in V'$ the inequality*

$$\frac{d}{dt}\xi x(t) \leqslant \xi A x(t) \tag{73.5}$$

holds for $t \in Z_\xi$ where $\operatorname{mes}\big((0, a) - Z_\xi\big) = 0$. *Then*

$$x(t) \leqslant T(t-s)x(s)\,, \qquad s < t\,.$$

Proof. We take the function

$$x_h(t) = \int_t^{t+h} x(\tau)\,d\tau$$

and verify by integration of (73.5) that

$$\xi\big(x(t+h) - x(t)\big) \leqslant \xi\int_t^{t+h} Ax(\tau)\,d\tau\,, \qquad \xi \in V'\,. \tag{73.6}$$

The summability of $Ax(t)$ and the fact that A is closed imply

$$\xi\int_t^{t+h} Ax(\tau)\,d\tau = \xi A x_h(t)\,. \tag{73.7}$$

Notice that

$$\frac{d}{dt}x_h(t) = x(t+h) - x(t)$$

for almost all t. By (73.6) and (73.7) we get therefore

$$\frac{d}{dt}x_h(t) \leqslant Ax_h(t) \tag{73.8}$$

almost everywhere on $(0, a)$. It is easy to see that $x_h(t)$ satisfies the regularity assumptions required in theorem 73.2. Hence (73.8) implies

$$x_h(t) \leqslant T(t-s)x_h(s)\,.$$

The weak continuity of $x(t)$ and the limit passage $h \to 0+$ in the inequality

$$\frac{1}{h}\xi x_h(t) \leqslant \frac{1}{h}\xi T(t-s)x_h(s)$$

complete the proof.

Previously we assumed always that A generates the semi-group. Suppose now that A is merely closed and $(\lambda I - A)^{-1}$ exists for $\lambda > 0$. A is defined and linear on a manifold included in E.

Let $\varphi(\lambda)$ be a real-valued function of class C^∞ on $(0, \infty)$. It is called *completely monotone* if

$$(-1)^n\frac{d^n\varphi}{d\lambda^n} \geqslant 0\,, \qquad \lambda > 0 \ \ (n = 0, 1, 2, \ldots)\,.$$

The classical Berstein-Widder theorem [67] states that a necessary and sufficient condition for φ to be completely monotone is that it be of the form

$$\varphi(\lambda) = \int_0^\infty e^{-\lambda t} d\alpha(t)$$

with an increasing $\alpha(t)$, the Stieltjes integral being convergent for $\lambda > 0$. It follows then that if φ is completely monotone and

$$\varphi(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt$$

with a continuous $f(t)$, then $f(t) \geqslant 0$.

THEOREM 73.4. *Let the function $x(t)$ be bounded, strongly measurable and weakly absolutely continuous over the interval $[0, \infty)$. Suppose that $x(0) = \theta$ and let for every $\xi \in V'$ the inequality*

$$\frac{d}{dt}\xi x(t) \leqslant \xi A x(t) \tag{73.9}$$

be satisfied almost everywhere on $(0, \infty)$. We assume that $(\lambda I - A)^{-1} \geqslant \theta$.

It is supposed that $Ax(t)$ is strongly measurable and

$$\int_0^\infty \|Ax(\tau)\|\, d\tau < +\infty .$$

Then $x(t) \leqslant \theta$ for $t \geqslant 0$.

Proof. Let us multiply (73.9) by $e^{-\lambda t}$ and integrate over $[0, R]$. We get

$$e^{-\lambda t}\xi x(t)\Big|_0^R + \lambda \int_0^R e^{-\lambda t}\xi x(t)\, dt \leqslant \xi \int_0^R e^{-\lambda t} A x(t)\, dt . \tag{73.10}$$

Write

$$L(\lambda) = \int_0^\infty e^{-\lambda t} x(t)\, dt$$

and let $R \to +\infty$. It follows then from (73.10) and from the closedness of A that

$$\lambda \xi L(\lambda) \leqslant \xi A L(\lambda) , \qquad \xi \in V' .$$

Hence $-(\lambda I - A)L(\lambda) \in V$ and consequently, by positivity of $(\lambda I - A)^{-1}$

$$L(\lambda) \leqslant \theta , \qquad \lambda > 0 .$$

The multiplication of (73.9) by $t^n e^{-\lambda t}$ and integration from 0 to $+\infty$ gives us

$$\xi(\lambda I - A) \int_0^\infty t^n e^{-\lambda t} x(t)\, dt \leqslant n\xi \int_0^\infty t^{n-1} e^{-\lambda t} x(t)\, dt \quad \text{for} \quad \xi \in V' .$$

Hence

$$(\lambda I - A)\int_0^\infty t^n e^{-\lambda t}x(t)\,dt \leqslant n\int_0^\infty t^{n-1}e^{-\lambda t}x(t)\,dt\,.$$

By induction

$$\int_0^\infty t^n e^{-\lambda t}x(t)\,dt \leqslant \theta \qquad (n = 0, 1, 2, \ldots)\,,$$

and consequently the function $\varphi_\xi(\lambda) = -\xi L(\lambda)$ $(\xi \in V')$ satisfies

$$(-1)^n\frac{d^n\varphi_\xi}{d\lambda^n} = -\xi\int_0^\infty t^n e^{-\lambda t}x(t)\,dt \geqslant 0\,.$$

We infer by the previous discussion that $-\xi x(t)$ is non-negative, which completes the proof.

§ 74. Almost linear differential inequalities in Banach spaces. This section concerns inequalities of the form

$$x'(t) \leqslant Ax(t) + f\big(t, x(t)\big)\,.$$

The operator A is the generator of a positive semi-group $\{T(t)\}$ of class (C_0). As usually the relation of inequality is induced by a cone V. In what follows the symbol $C_E[0, a]$ denotes the space of vector-valued functions with values in E, continuous on $[0, a]$, with the sup norm.

THEOREM 74.1. *Let the function $f(t, x)$ be strongly continuous in (t, x) and bounded, $\|f(t, x)\| \leqslant M < +\infty$. Assume that $f(t, x)$ increases in x. Let the transformation*

$$F\colon\, z(t) \to \int_0^t T(t-\tau)f\big(\tau, z(\tau)\big)\,d\tau$$

be completely continuous when considered in the space $C_E[0, a]$. Suppose that the function $x(t)$ is strongly differentiable on $[0, a]$ and

$$x'(t) \leqslant Ax(t) + f\big(t, x(t)\big)\,, \qquad 0 \leqslant t \leqslant a\,. \tag{74.1}$$

Under our assumptions there is a solution $y(t)$ of

$$y(t) = T(t)x(0) + \int_0^t T(t-\tau)f\big(\tau, y(\tau)\big)\,d\tau$$

such that

$$x(t) \leqslant y(t)\,, \qquad 0 \leqslant t \leqslant a\,.$$

Proof. Inequality (74.1) implies

$$x(t) \leqslant T(t)x(0) + \int_0^t T(t-\tau)f\big(\tau, x(\tau)\big)\,d\tau\,. \tag{74.2}$$

Define now

$$N = \sup_{[0,a]} \|T(\tau)\| , \qquad K = \sup_{[0,a]} \|x(\tau)\|$$

and

$$Z = \{z(\cdot)\colon\ z(\cdot) \in C_E[0, a] ,\ z(0) = x(0) ,\ x(t) \leqslant z(t) \text{ on } [0, a]$$
$$\text{and } \|z(t)\| \leqslant \max(N\|x(0)\| + MNa, K)\} .$$

Notice that $x(\cdot) \in Z$.

Z is closed, bounded and convex in $C_E[0, a]$. The monotonicity of $f(t, x)$ and (74.2) imply that $F(Z) \subset Z$. By Schauder fixed point theorem there is y such that $y = Fy$. Obviously $x(t) \leqslant y(t)$, q.e.d.

COROLLARY. *Using the method of successive approximations one verifies easily that theorem remains true if the complete continuity of F is replaced by the Lipschitz condition in x, for $f(t, x)$. In that case $y(t)$ is unique.*

Now we will discuss the couple of inequalities

(74.3) $$x'(t) \leqslant Ax(t) + f\big(t, x(t)\big) ,$$

(74.4) $$y'(t) \geqslant Ay(t) + f\big(t, y(t)\big) .$$

The linearization procedure we apply requires the following condition:

(74.5) The function $f(t, x)$ is Fréchet differentiable in x to $f_x(t, x)$ and $f_x(t, x)$ is strongly continuous in (t, x).

THEOREM 74.2. *Let $x(t)$ and $y(t)$ satisfy on $[0, a]$ the inequalities* (74.3), (74.4) *and let $f(t, x)$ satisfy* (74.5). *Suppose that*

$$f_x\Big(t, y(t) + \tau\big(x(t) - y(t)\big)\Big) + \beta I \geqslant \theta , \qquad 0 \leqslant \tau \leqslant 1 ,$$

for $0 \leqslant t \leqslant a$ and some real β. Then, if $x(0) \leqslant y(0)$, then $x(t) \leqslant y(t)$ on $[0, a]$.

Proof. It follows from (74.3) and from (74.4) that

(74.6) $$[x(t) - y(t)]' \leqslant A[x(t) - y(t)] + \Big[f\big(t, x(t)\big) - f\big(t, y(t)\big)\Big] .$$

On the other hand,

(74.7) $$f\big(t, x(t)\big) - f\big(t, y(t)\big) = B(t)\big(x(t) - y(t)\big) ,$$

where

$$B(t) = \int_0^1 f_x\Big(t, y(t) + \tau\big(x(t) - y(t)\big)\Big) d\tau .$$

Moreover, $B(t) + \beta I \geqslant \theta$. We see that $z(t) = x(t) - y(t)$ satisfies the assumptions of Theorem 72.5. Hence $z(t) \leqslant \theta$, q.e.d.

If $A = \theta$, then (74.3), (74.4) reduce to

$$x'(t) \leqslant f\big(t, x(t)\big), \qquad y'(t) \geqslant f\big(t, y(t)\big).$$

Taking E as a space of sequences $\{\xi_i\}$ we can get interpretations of the above theorem and just apply it to integration of infinite systems of ordinary differential inequalities. The systems are of the form

$$\xi_i'(t) \leqslant f_i\big(t, \xi_1(t), \xi_2(t), \ldots\big),$$

$$\eta_i'(t) \geqslant f_i\big(t, \eta_1(t), \eta_2(t), \ldots\big),$$

where the properties of $\{\eta_i\}$, $\{\xi_i\}$, $\{f_i\}$ are restricted by the fact that all these sequences belong to E. The cone V is defined by

$$V = \{x\colon\ x = \{\xi_i\},\ \xi_i \geqslant 0\}.$$

The Fréchet differentiability of f implies usually the existence of classical derivatives and the abstract condition

$$f_x + \beta I \geqslant \theta \tag{74.8}$$

produces $\big(f_i = f_i(t, x_1, x_2, x_3, \ldots)\big)$

$$\frac{\partial f_i}{\partial x_k} \geqslant 0 \quad (i \neq k); \qquad \frac{\partial f_i}{\partial x_i} \geqslant -\beta. \tag{74.9}$$

In case E is finite dimensional the above theorem gives us some particular case of Theorems of § 9.

We will apply now the previous theory to some extension of the Chaplygin method (see § 31). Let (74.5) hold and suppose we are given a function $x(t)$ and write down the equation

$$y' = Ay + f\big(t, x(t)\big) + f_x\big(t, x(t)\big)\big(y - x(t)\big). \tag{74.10}$$

Following the general ideas of § 69 one proves easily the following lemma:

LEMMA 74.1. *Suppose that the Fréchet differential f_x of the function $f(x)$ satisfies the following condition*:

If $x_1 \leqslant x_2$, $\theta \leqslant z$, then $f_x(x_1)z \leqslant f_x(x_2)z$. Then

$$f_x(x)(y-x) + f(x) \leqslant f(y) \quad \text{for} \quad x \leqslant y.$$

Now we can prove the following theorem:

THEOREM 74.3. *Let the function $y(t)$ satisfy on $[0, a]$ equation* (74.10). *Assume that for every $t \in [0, a]$ if $x_1 \leqslant x_2$, $\theta \leqslant z$, then $f_x(t, x_1)z \leqslant f_x(t, x_2)z$. Suppose that*

$$f_x\big(t, x(t)\big) \geqslant \gamma I \quad \textit{for some real } \gamma, \tag{74.11}$$

$$x'(t) \leqslant Ax(t) + f\big(t, x(t)\big). \tag{74.12}$$

Then, if $x(0) = y(0)$, *then*

$$x(t) \leqslant y(t) \quad on \quad [0, a], \tag{74.13}$$

$$y'(t) \leqslant Ay(t) + f\big(t, y(t)\big) \quad on \quad [0, a], \tag{74.14}$$

$$f_x\big(t, y(t)\big) \geqslant \gamma I \quad on \quad [0, a]. \tag{74.15}$$

Proof. It follows from (74.10) and (74.12) that $z(t) = x(t) - y(t)$ satisfies

$$z'(t) \leqslant Az(t) + f_x\big(t, x(t)\big) z(t),$$
$$z(0) = \theta.$$

We can apply Theorem 72.5 and thus obtain $z(t) \leqslant \theta$. On the other hand, $f_x(t, x)$ increases in x. The inequality $x(t) \leqslant y(t)$ and Lemma 74.1 imply

$$f_x\big(t, x(t)\big)\big(y(t) - x(t)\big) + f\big(t, x(t)\big) \leqslant f\big(t, y(t)\big)$$

which together with (74.10) proves (74.14).

Notice that $f_x(t, x)$ increases in x. Hence

$$\gamma I \leqslant f_x\big(t, x(t)\big) \leqslant f_x\big(t, y(t)\big) \quad \text{for} \quad t \in [0, a],$$

which completes the proof.

The above theorem can be used in the abstract treatment of the Chaplygin method. For details in this field we refer to [27] and [28].

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LIST OF SPECIAL SYMBOLS

$\in$	is an element of
$\notin$	is not an element of
$\subset$	is a subset of
$\{x: P(x)\}$	set of x's for which the proposition P is true
$\bigcup$	union of sets
$\bigcap$	intersection of sets
$\overline{D}$	closure of D
∂D	boundary of D
$\|x\|$	norm of a vector
D^+	right-hand upper derivative
D_+	right-hand lower derivative
D^-	left-hand upper derivative
D_-	left-hand lower derivative
D_+^{S}	right-hand strong derivative
D_+^{W}	right-hand weak derivative
s-lim	strong limit
w-lim	weak limit
$A \leqslant B$	for two points $A = (a_1, \dots, a_n)$, $B = (b_1, \dots, b_n)$ such that $a_j \leqslant b_j$ $(j = 1, 2, \dots, n)$
$A < B$	for two points $A = (a_1, \dots, a_n)$, $B = (b_1, \dots, b_n)$ such that $a_j < b_j$ $(j = 1, 2, \dots, n)$
$A \overset{i}{\leqslant} B$	for two points $A = (a_1, \dots, a_n)$, $B = (b_1, \dots, b_n)$ such that $A \leqslant B$ and $a_i = b_i$, the index i being fixed
$-A$	for a point $A = (a_1, \dots, a_n)$ we denote $-A = (-a_1, \dots, -a)$
$\lvert A \rvert$	for a point $A = (a_1, \dots, a_n)$ we denote $\lvert A \rvert = (\lvert a_1\rvert, \dots, \lvert a_n\rvert)$
$p \Rightarrow q$	p implies q
$\varphi_n(X) \underset{D}{\rightrightarrows} \varphi(X)$	the sequence $\varphi_n(X)$ converges to $\varphi(X)$ uniformly on D
$\Omega(t; H)$	right-hand maximum solution of comparison system through the point $(0, H) = (0, \eta_1, \dots, \eta_n)$
u_X	for a function $u(X) = u(x_1, \dots, x_n)$ its gradient
u_{XX}	for a function $u(X) = u(x_1, \dots, u_n)$ the sequence of its second derivatives

(a, b)	open interval $a < t < b$
$[a, b]$	closed interval $a \leqslant t \leqslant b$
$[a, b)$	interval $a \leqslant t < b$
$(a, b]$	interval $a < t \leqslant b$
$G_1 \times G_2$	topological product
$\lvert A - B \rvert$	for two points $A = (a_1, \dots, a_n)$, $B = (b_1, \dots, b_n)$ their Euclidean distance
E'	adjoint space of E
I	identity operator
$\operatorname{sgn} x$	denotes 1 if $x \geqslant 0$, and -1 if $x < 0$

INDEX

CONTENTS

CHAPTER IV

Ordinary differential inequalities of higher order and some integral inequalities

CHAPTER V

Cauchy problem for ordinary differential equations

CHAPTER VI

Some auxiliary theorems

CHAPTER VII

Cauchy problem for first order partial differential equations

CHAPTER VIII

Mixed problems for second order partial differential equations of parabolic and hyperbolic type

CHAPTER IX

Partial differential inequalities of first order

CHAPTER X

Second order partial differential inequalities of parabolic type

CHAPTER XI

Differential inequalities in linear spaces