GENERALIZED THEORY AND SOME SPECIALIZATIONS OF THE REGION CONTRACTION ALGORITHM I - BALL OPERATION

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Abstract.

We describe a new algorithm named Region Contraction Algorithm for solving certain nonlinear equations, and establish the convergence of the algorithm and give an error estimation. It is shown that this general theory includes all of present existing ball iterations as special cases.

To find a zero of a quasi-strongly monotone mapping, which arises often from the field of differential equations, variational calculus and optimization etc., the authors [2] recently proposed a new algorithm called Region Contraction Algorithm (abbreviated RCA henceforth) in real Hilbert spaces. Stemming from T.E. Williamson's geometric estimation for fixed points of contractive mappings [3], the algorithm establishes a convergent iterative process which keeps well defined and automatically covers the errors by constructing a sequence of closed balls containing the zero set. Later on, proceeding in a completely different view from the authors, Wu Yujiang and Wang Deren [4] rewrited our algorithm in the language of interval analysis, and also suggested a new globally convergent scheme in the case that F is strongly monotone. It showed the authors that the RCA is almost Nickel's Ball Newton Method [1] (abbreviated BNM henceforth) except for the difference of the class of mappings to which it applies.

In this paper we develop a more general algorithm called stationary region contracting algorithm (abbreviated SRCA) with RCA, BNM and some other methods as its specializations.

In Section 1 we present the algorithm and give some basic properties in Section 2. In Section 3 we prove convergence of the algorithm and discuss some specializations in the last section.

In what follows, we always let H be a real Hilbert space with inner product $(.,.)$, and use $B(x,r)$ to denote the closed ball with center x and radius r.

1. Algorithm.

Let D be a subset of H, $B(b,d) \subset D$ a given closed ball with $d > 0$, and $F: D \subset H \to H$ a given nonlinear mapping. We want to find a zero of the mapping F in the ball $B(b,d)$. Let us suppose that there exists a non-linear mapping $g: B(b,d) \to H$ and a nonnegative functional $r: B(b,d) \to R^{+}$ such that for all $x \in B(b,d)$

$$r(x) \leqq \|g(x)\| \tag{1.1}$$

and

$$N(F) \subset Gx, \tag{1.2}$$

where $N(F)$ is the set of zeros of F in $B(b,d)$ and Gx is defined as

$$Gx = B(x - g(x),\ r(x)).$$

For any two closed balls B' and B", let $\langle B' \cap B'' \rangle$ stand for the minimum-volume closed ball which contains their intersection if $B' \cap B'' \neq \emptyset$, and $\langle B' \cap B'' \rangle = \emptyset$ if $B' \cap B'' = \emptyset$.

We develop our general algorithm SRCA as follows:

I. Initial Step
Set $B_0 = B(x_0, r_0) = B(b,d)$.

II. Continuation Step
Suppose that $B(x_k, r_k)$ has been constructed; we then continue to construct the next ball B_{k+1} in the following way:

II.1. Starting Step. If $r_k = 0$, then stop the algorithm at (*) when $Fx_k \neq 0$, otherwise (**) when $Fx_k = 0$. If $r_k \neq 0$, then calculate Gx_k.

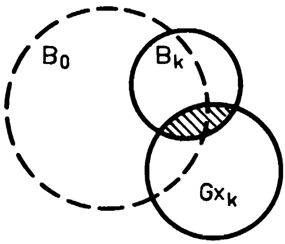
II.2. Contraction Step. Stop the algorithm at (*) if $Gx_k \cap B_0 = \emptyset$, otherwise set $\bar{B}_{k+1} = B(\bar{x}_{k+1}, \bar{r}_{k+1}) = \langle Gx_k \cap B_k \rangle$.

II.3. Modification Step. Stop the algorithm at (*) if $\bar{B}_{k+1} \cap B_0 = \emptyset$, otherwise set

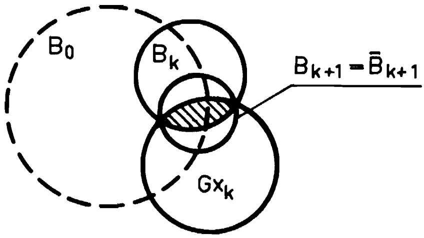
$$B_{k+1} = B(x_{k+1}, r_{k+1}) = \begin{cases} \bar{B}_{k+1} & \text{if } \bar{x}_{k+1} \in B_0, \\ \langle \bar{B}_{k+1} \cap B_0 \rangle & \text{if } \bar{x}_{k+1} \bar{\in} B_0 . \end{cases}$$

III. Return to II. with k := k+1.

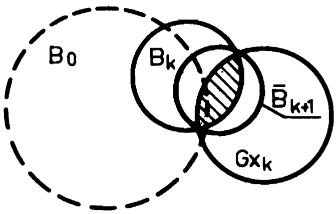
A continuation step from B_k to B_{k+1} for SRCA is shown in the following figures (where $H = R^2$).



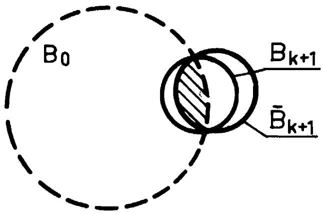
II.1.



II.2. and II.3. $\bar{x}_{k+1} \in B_0$



II.1. and II.2.



II.2. and II.3. $\bar{x}_{k+1} \bar{\in} B_0$

Remark. (1) It is easily seen (see Theorem 1 in next section) that the SRCA generates successively a sequence of closed balls which contain $N(F)$, whose centers are located in B_0, and whose radii decrease step by step. Thus, the algorithm is always well defined, the error of computation is automatically covered, and we may expect to find $N(F)$.

(2) If $B' = B(c_1,s_1)$ and $B'' = B(c_2,s_2)$ are any two closed balls, and $s_1 \geqq s_2 > 0$, then $\langle B' \cap B'' \rangle$ can be represented by the following formula (see Lemma 2 in [3] and [1]):

$$\langle B' \cap B'' \rangle = \begin{cases} \emptyset & \text{if } \|c_1 - c_2\| > s_1 + s_2, \\ B'' & \text{if } \|c_1 - c_2\| \leqq (s_1^2 - s_2^2)^{\frac{1}{2}}, \\ B(c,s) & \text{otherwise}, \end{cases} \tag{1.3}$$

where

$$\begin{aligned} c &= w_1 c_1 + (1 - w_1) c_2 = (1 - w_2) c_1 + w_2 c_2 \in B' \cap B'', \\ s &= (s_2^2 - w_1^2 \|c_1 - c_2\|^2)^{\frac{1}{2}} = (s_1^2 - w_2^2 \|c_1 - c_2\|^2)^{\frac{1}{2}}, \\ w_1 &= \tfrac{1}{2}(1 - (s_1^2 - s_2^2)/\|c_1 - c_2\|^2) = 1 - w_2. \end{aligned} \tag{1.4}$$

Particularly, if $\langle B' \cap B'' \rangle = B(\xi,\eta) \neq \emptyset$, then $\xi \in B' \cap B''$ and $\eta \leqq \min(s_1,s_2)$.

(3) A natural idea is that one always takes $B_{k+1} = \langle \bar{B}_{k+1} \cap B_0 \rangle$ in the Modification Step of II.3., but a simple analysis shows that this modification affects only weakly the convergence of the algorithm, and hence, there is no need to do so for saving time of computation.

2. Basic Properties.

Let $\{B_k\}$ be the sequence of closed balls generated by the algorithm, and for convenience, we define $B_j = B_k$ for all $j > k$ if the algorithm stops at (**) in the (k+1)-th step.

Theorem 1.

The following statements are all valid:

(1) $x_k \in B_0$ for each k .

(2) $N(F) \subset B_k$ for each k .

(3) If the algorithm stops in the (k_0+1)-th step, then either $N(F) = \emptyset$ if (*) appears or $N(F) = \{x^*\}$ if (**) appears.

(4) If $N(F) \neq \emptyset$ and $r^* = \lim\limits_{(k)} r_k = 0$, then x_k converges to x^*, the unique element of $N(F)$, and

$$\|x_k - x^*\| \leqq r_k \qquad \forall\ k \geqq 0.$$

Proof. (1) and (2): It is obvious for $k = 0$. Suppose that the conclusions hold for some $k > 0$ and B_{k+1} can be produced, then by (1.2) $N(F) \subset Gx_k \cap B_k \subset \langle Gx_k \cap B_k\rangle = \bar{B}_{k+1}$. If $\bar{x}_{k+1} \in B_0$, then $B_{k+1} = \bar{B}_{k+1}$ and $x_{k+1} = \bar{x}_{k+1}$ by the definition of the algorithm; otherwise, $B_{k+1} = \langle\bar{B}_{k+1} \cap B_0\rangle$, and hence $N(F) \subset \bar{B}_{k+1} \cap B_0 \subset B_{k+1}$ and $x_{k+1} \in \bar{B}_{k+1} \cap B_0 \subset B_0$. Therefore, the conclusion also holds for the case k+1. By induction, (1) and (2) are valid.

(3): If the algorithm stops at (*), then $N(F) = \emptyset$ because it comes about if and only if one of the following cases occurs: either $B_{k_0} = \{x_{k_0}\} \subset B_0$ and $Fx_{k_0} \neq 0$, or $Gx_{k_0} \cap B_0 = \emptyset$ or $\bar{B}_{k_0+1} \cap B_0 = \emptyset$. If the algorithm stops at (**), then $B_{k_0} = \{x_{k_0}\}$ and $Fx_{k_0} = 0$, i.e., $N(F) = \{x^*\}$ with $x^* = x_{k_0}$.

(4): If $N(F) \neq \emptyset$, the algorithm is never stopped at (*), so $\{r_k\}$ is infinite. Therefore, the result is direct consequence of (2) and (3).

Theorem 2.

The sequence $\{r_k\}$ decreases in the following way

$$\bar{r}_{k+1} \leqq \begin{cases} r_k / 2^{\frac{1}{2}} & \text{if } \|g(x_k)\|^2 + r(x_k)^2 \leqq r_k^2, \\ (1 - r_k^2/(4\|g(x_k)\|^2))^{\frac{1}{2}} r_k & \text{otherwise} \end{cases} \tag{2.1}$$

and

$$r_{k+1} \leqq \begin{cases} \bar{r}_{k+1} & \text{if } \bar{x}_{k+1} \in B_0, \\ (1 - \bar{r}_{k+1}^2/(4d^2))^{\frac{1}{2}}\, \bar{r}_{k+1} & \text{if } \bar{x}_{k+1} \bar{\in} B_0. \end{cases} \tag{2.2}$$

Proof. If $\|g(x_k)\|^2 + r(x_k)^2 \leqq r_k^2$, then $\bar{B}_{k+1} = \langle Gx_k \cap B_k \rangle = Gx_k$ by (1.3) and hence (1.1) yields that

$$2\bar{r}_{k+1}^2 = 2r(x_k)^2 \leqq \|g(x_k)\|^2 + r(x_k)^2 \leqq r_k^2,$$

that is, $\bar{r}_{k+1} \leqq r_k / 2^{\frac{1}{2}}$. If $\|g(x_k)\|^2 + r(x_k)^2 > r_k^2$, then it follows from (1.1) that $\|g(x_k)\|^2 \geqq |r_k^2 - r(x_k)^2|$, so (1.1) and (1.3)-(1.4) give that

$$\bar{r}_{k+1} = [r_k^2 - (\|g(x_k)\|^2 + r_k^2 - r(x_k)^2)^2/(2\|g(x_k)\|)^2]^{\frac{1}{2}}$$

$$\leqq [r_k^2 - (r_k^2/(2\|g(x_k)\|^2)]^{\frac{1}{2}} = r_k(1 - r_k^2/(4\|g(x_k)\|^2))^{\frac{1}{2}};$$

therefore, (2.1) follows.

Furthermore, if $\bar{x}_{k+1} \in B_0$, then $r_{k+1} = \bar{r}_{k+1}$ because $B_{k+1} = \bar{B}_{k+1}$. If $\bar{x}_{k+1} \bar{\in} B_0$, then $\|\bar{x}_{k+1} - b\| > d$ and $B_{k+1} = \langle \bar{B}_{k+1} \cap B_0 \rangle$, and hence one gets

$$r_{k+1} \leqq (1 - \bar{r}_{k+1}^2/(4d^2))^{\frac{1}{2}} \bar{r}_{k+1}$$

from Lemma 3 in [2]. Thus, the proof of (2.2) is completed.

A simple consequence of Theorem 2 is that $r^* = \lim_{(k)} r_k$ always exists if the algorithm is never stopped at (*).

Theorem 3.
For each pair $m \geqq n > 0$, $\{B_k\}$ satisfies that

$$r_m^2 \leqq r_n^2 - \sum_{k=n}^{m-1} \|x_{k+1} - x_k\|^2 .$$

Proof. Obviously, it is only necessary to prove the following inequality

$$\|x_{k+1} - x_k\|^2 \leqq r_k^2 - r_{k+1}^2 \tag{2.3}$$

for each $k \geqq 0$ in the case that B_{k+1} can be produced. We complete the proof considering two cases:

(1) $\bar{x}_{k+1} \in B_0$, then $B_{k+1} = \bar{B}_{k+1} = \langle Gx_k \cap B_k \rangle$. As done in the proof of Theorem 2, $\|g(x_k)\|^2 + r(x_k)^2 \leqq r_k^2$ implies that $B_{k+1} = \langle Gx_k \cap B_k \rangle = Gx_k$ and hence

$$\|x_{k+1} - x_k\|^2 = \|g(x_k)\|^2 \leqq r_k^2 - r(x_k)^2 = r_k^2 - r_{k+1}^2. \tag{2.4}$$

Similarly, from $\|g(x_k)\|^2 + r(x_k)^2 > r_k^2$, we get that $\|g(x_k)\|^2 \geqq$ $\geqq |r_k^2 - r(x_k)^2|$, so (1.3)-(1.4) gives directly that

$$\|x_{k+1} - x_k\|^2 = r_k^2 - r_{k+1}^2 .$$

Therefore, (2.3) holds in this case.

(2) $\bar{x}_{k+1} \bar{\in} B_0$, then $B_{k+1} = \langle \bar{B}_{k+1} \cap B_0 \rangle$. Since $\bar{r}_{k+1} \leqq r_k \leqq r_0 = d$ by Theorem 2, it follows that

$$\|\bar{x}_{k+1} - b\| > d \geqq (d^2 - \bar{r}_{k+1}^2)^{\frac{1}{2}}$$

and hence we conclude from (1.3)-(1.4) that $x_{k+1} = w_k b + (1-w_k)\bar{x}_{k+1}$ and

$$r_{k+1}^2 = \bar{r}_{k+1}^2 - w_k^2 \|\bar{x}_{k+1} - b\|^2 , \tag{2.5}$$

where

$$w_k = \tfrac{1}{2}(1-(d^2 - \bar{r}_{k+1}^2)/ \|\bar{x}_{k+1} - b\|^2). \tag{2.6}$$

Noting the following identity

$$\begin{aligned}\|x_{k+1} - x_k\|^2 &= \|w_k(b-x_k) + (1-w_k)(\bar{x}_{k+1} - x_k)\|^2 \\ &= w_k\|b-x_k\|^2 + (1-w_k)\|\bar{x}_{k+1} - x_k\|^2 - w_k(1-w_k)\|\bar{x}_{k+1} - b\|^2\end{aligned}$$

and the fact that (2.4) indicates that $\|\bar{x}_{k+1} - x_k\|^2 \leqq r_k^2 - \bar{r}_{k+1}^2$, we obtain

$$\begin{aligned}\|x_{k+1} - x_k\|^2 &\leqq w_k d^2 + (1-w_k)(r_k^2 - \bar{r}_{k+1}^2) - w_k(1-w_k)\|\bar{x}_{k+1} - b\|^2 \\ &= (1-w_k)(r_k^2 - \bar{r}_{k+1}^2) + w_k\bar{r}_{k+1}^2 + w_k((d^2 - \bar{r}_{k+1}^2)/\|\bar{x}_{k+1} - b\|^2 \\ &\quad -1 + w_k)\ \|\bar{x}_{k+1} - b\|^2 \\ &= (1-w_k)r_k^2 + (2w_k - 1)\bar{r}_{k+1}^2 - w_k^2\ \|\bar{x}_{k+1} - b\|^2. \end{aligned} \tag{2.7}$$

Combining (2.5) with (2.7), we therefore know a sufficient condition for (2.3) holds is the following

$$(1-w_k)r_k^2 + (2w_k - 1)\bar{r}_{k+1}^2 - w_k^2 \|\bar{x}_{k+1}-b\|^2 \leqq r_k^2 - \bar{r}_{k+1}^2 + w_k^2 \|\bar{x}_{k+1} - b\|^2$$

or equivalently,

$$2\bar{r}_{k+1}^2 - r_k^2 \leqq 2w_k \|\bar{x}_{k+1} - b\|^2. \tag{2.8}$$

But from (2.6),

$$2w_k \|\bar{x}_{k+1} - b\|^2 = \|\bar{x}_{k+1} - b\|^2 - d^2 + \bar{r}_{k+1}^2$$

so (2.8) is equivalent to

$$\bar{r}_{k+1}^2 + d^2 \leqq r_k^2 + \|\bar{x}_{k+1} - b\|^2 .$$

Consequently, the validity of (2.3) immediately follows from $\bar{r}_{k+1} \leqq r_k$ and $d \leqq \|\bar{x}_{k+1} - b\|$.

Remark. Theorems 1-3 generalize Theorems 1-3 of [2]; for a nonexpansive mapping T.E. Williamson, Jr. has proved a similar estimation as in Theorem 3 (see Theorem 6 in [3]).

Using the parameter $\lambda \in [0,1]$ defined by

$$\lambda = \operatorname{Sup} \{ r(x)/\|g(x)\| \ ; \ x \in B_0 \text{ and } g(x) \neq 0 \}$$

we characterice now the algorithm in another way.

Theorem 4.
The sequence $\{r_k\}$ decreases in the following form

$$r_k \leqq \lambda r_{k-1} \leqq \lambda^k d.$$

Proof. For any possible $k \geqq 0$, if $\|g(x_k)\|^2 + r(x_k)^2 \leqq r_k^2$, then $\|g(x_k)\| \leqq r_k$, and hence $\bar{r}_{k+1} \leqq r(x_k) = (r(x_k)/\|g(x_k)\|)\|g(x_k)\| \leqq \lambda r_k$; if $\|g(x_k)\|^2 + r(x_k)^2 > r_k^2$, then $\|g(x_k)\| \geqq |r_k^2 - r(x_k)^2|^{\frac{1}{2}}$ and by (1.3)-(1.4)

$$\bar{r}_{k+1}^2 = r_k^2 - [(\|g(x_k)\|^2 + r_k^2 - r(x_k)^2)/(2\|g(x_k)\|)]^2. \qquad (2.9)$$

Hence, from the simple inequality $2ab \leqq a+b$, it follows that

$$2r_k(\|g(x_k)\|^2 - r(x_k)^2)^{\frac{1}{2}} \leqq r_k^2 + (\|g(x_k)\|^2 - r(x_k)^2);$$

therefore we conclude from (2.9) that

$$\bar{r}_{k+1}^2 \leqq r_k^2 - [2r_k(\|g(x_k)\|^2 - r(x_k)^2)^{\frac{1}{2}} / (2\|g(x_k)\|)]^2$$

$$= (r(x_k)/\|g(x_k)\|)^2 r_k^2 \leqq \lambda^2 r_k^2,$$

i.e., $\bar{r}_{k+1} \leqq \lambda r_k$. Thus, the conclusion immediately follows from the fact $r_{k+1} \leqq \bar{r}_{k+1}$.

3. Convergence

We now establish the convergence of the SRCA. In the first result, a generalization to Theorem 2 of [2], we only presuppose the boundedness of g.

Theorem 5 (Bounded Convergence).
If g is bounded on B_0 and $N(F) \neq \emptyset$, then x_k converges to x^*, the unique element of $N(F)$, in the following way

$$\|x_k - x^*\| \leqq r_k \to 0 \text{ as } k \to \infty.$$

Proof. By (4) of the Theorem 1, it is sufficient to show $r^* = \lim r_k = 0$. We assume the contrary, namely that $r^* > 0$ and the algorithm never stops. Since g is bounded on B_0, there is a constant $M > r^*$ such that

$$\operatorname{Sup}\{\|g(x)\| ;\ x \in B_0\} \leqq \frac{1}{2}M.$$

It follows from $x_k \in B_0$ that $2\|g(x_k)\| \leqq M$ for all $k \geqq 0$. Thus, we have for all $k \geqq 0$ that

$$r_k^2/(4\|g(x)\|^2) \geqq r^{*2}/M^2$$

and hence from Theorem 2 that

$$r_{k+1} \leqq \bar{r}_{k+1} \leqq q r_k, \tag{3.1}$$

where

$$q = \mathrm{Max}\,\{\,2^{-\frac{1}{2}}\,,\ (1 - r^*/M^2)^{\frac{1}{2}}\,\} < 1.$$

Taking the limit as $k \to \infty$, we get the obvious contradiction: $r^* \leqq qr^* < r^*$ which shows that r^* must be zero.

Generally speaking, some additional conditions are necessary for guaranteeing the sequence $\{B_k\}$ shrink to $N(F)$. We below consider a important particular case in which g and r are of the following forms

$$g(x) = u(x)P(Fx) \tag{3.2}$$

$$r(x) = v(x)\ \|P(Fx)\|, \tag{3.3}$$

where u and v are nonnegative functionals on B_O, and $P\colon H \to H$ is a mapping with O as its unique zero.

Recall that a mapping $T\colon C \subset H \to H$ is said to be closed if its graph $\{(x,Tx) \in H\times H;\ x \in C\}$ is closed in the product space $H\times H$.

<u>Theorem 6</u> (Global convergence).
If $\lambda < 1$, then $\{B_k\}$ shrinks as $k \to \infty$ to a singleton containing $N(F)$ provided the algorithm never stops at $(*)$. In addition, if the composed mapping PF restricted to B_O is closed (especially, continuous), and

$$\varepsilon = \mathrm{Inf}\,\{\,v(x);\ x \in B_O \text{ and } u(x)P(Fx) \neq O\} > O \tag{3.4}$$

then

(1) $N(F) = \emptyset$ iff the algorithm terminates at $(*)$;

(2) $N(F) \neq \emptyset$ iff the algorithm never terminates at $(*)$, and in this case x_k converges to $N(F) = \{x^*\}$ with the following estimation

$$\|x_k - x^*\| \leqq r_k \leqq \lambda^k d.$$

Proof. If $\lambda < 1$ and the algorithm never stops at (*), then $\{B_k\}$ is infinite, and $r_k \to 0$ as $k \to \infty$ by Theorem 4. Also, since $\|x_{k+1} - x_k\| \leqq r_k \leqq d\lambda^k$ by (2.3) and Theorem 4, $\{x_k\}$ is a Cauchy sequence so that it is convergent, and hence the first part follows.

For the last part, by the definition of the algorithm and $\{B_k\}$, we only need to prove the sufficiency of (2). Suppose that the algorithm never stops at (*), then $\{B_k\}$ must be infinite. If the algorithm stops at (**), the conclusion follows directly from (4) of Theorem 1, so it remains to discuss the case that the algorithm never stops.

For any $k \geqq 0$, $\overline{B}_{k+1} = \langle Gx_k \cap B_k \rangle \neq \emptyset$ implies that $\|g(x_k)\| \leqq r(x_k)+r_k$. It follows from (3.2)-(3.3) that

$$r(x_k) = v(x_k)\|P(Fx_k)\| = \|u(x_k)P(Fx_k)\| v(x_k)/u(x_k)$$

$$= \|g(x_k)v(x_k)/u(x_k)\| \leqq (r(x_k)+r_k)\lambda$$

and hence $r(x_k) \leqq \lambda(1-\lambda)^{-1}r_k$. Thus, we conclude from (3.4) that

$$\|P(Fx_k)\| = r(x_k)/v(x_k) \leqq \lambda r_k/(1-\lambda)\varepsilon$$

which shows $P(Fx_k) \to 0$ as $k \to \infty$. On the other hand, by the first part of the Theorem, there exists an $x^* \in B_0$ such that $x_k \to x^*$ as $k \to \infty$ and we have that $P(Fx^*) = 0$ since PF is closed, hence $x^* \in N(F)$ because $0 \in H$ is the unique zero of P. The uniqueness of x^* is obvious.

Remark. Theorem 6 is a generalization of a main result established by K. Nickel for his BNM (see Theorem 1 in [1]).

It should be noted that in application the functionals u and v in (3.2)-(3.3) can both often taken to be positive and constant. In this case, the assumption (3.4) is naturally satisfied.

4. Specializations

In this section we specify the SRCA and its convergent properties to some concrete classes of nonlinear mappings.

4.1 Nickel's Class of Functions $\mathcal{F}$ and His BNM.

The class of functions $\mathcal{F}$, introduced by K.L. Nickel [1], is the set of mappings $f\colon B_0 \to R^n$ which satisfy the following property: For each set C, of the form $C = B(\bar{x},r) \cap B_0$ with $\bar{x} \in B_0$ and $r \geqq 0$, there exists a regular $n \times n$ matrix $\Lambda = \Lambda(C)$ and a real number $\lambda = \lambda(C)$ such that $0 \leqq \lambda < 1$ and for all $x,y \in C$

$$\|x - y - \Lambda(f(x) - f(y))\| \leqq \lambda \; \|\Lambda(f(x) - f(y))\| . \tag{4.1}$$

It is known that $\mathcal{F}$ is a subset of the Lipschitz bicontinuous mappings and for such a class Nickel established his BNM. We observe that every f in $\mathcal{F}$ obviously satisfies the hypotheses (1.1)-(1.2) of Section 1 for the choice

$$u(x) = 1, \quad v(x) = \lambda, \quad \text{and} \quad P = \Lambda \tag{4.2}$$

in (3.2)-(3.3), and hence Nickel's BNM and his Theorem 1 on global convergence are proper specializations of the SRCA and Theorem 6 of this paper.

Based on the approach here, however, Nickel's class $\mathcal{F}$ can now clearly be amplified so that the BNM is applicable and convergence still holds. E.g., suppose that $N(f) \neq \emptyset$ and allow $\lambda \leqq 1$ and (4.1) holds just for all $y \in N(f)$, then (1.1)-(1.2) are also satisfied and hence the Theorems of Section 2-3 are all valid for the BNM.

4.2 Quasi-contractive Mappings and Williamson's Geometric Estimation Method.

A mapping $T\colon D \to H$ is said to be contractive if, there exists a positive constant $\alpha < 1$ such that for all $x,y \in D$

$$\|Tx - Ty\| \leq \alpha \|x - y\| \tag{4.3}$$

if the inequality holds just for all $y \in F(T)$, the fixed point set of T, we call such a T quasi-contractive mapping (in what follows, we always use "quasi" in the same way to indicate this restriction of y).

Let $\varepsilon = \alpha(1-\alpha^2)^{-1}$, $\delta = (1-\alpha^2)^{-1}$. T.E. Williamson, Jr. [3] has established the global estimation

$$F(T) \subset B(x - \delta(x - Tx), \varepsilon \|x - Tx\|) \quad \forall\, x \in D \tag{4.4}$$

for a contractive mapping T, and in virtue of the estimation, designed a geometric estimation algorithm (abbreviate GEA) to construct a fixed point of T. Except for the difference of the choice of the initial point, his algorithm corresponds basically to the SRCA, namely when $u(x) = \delta$, $v(x) = \varepsilon$ in (3.2)-(3.3). But, his algorithm is not globally convergent. From the discussion here, cf. Theorem 6, apparently, the defect has been completely removed.

All conclusions for a contractive mapping can naturally extend to a quasi-contractive one, for the estimation (3.4) is really true for the latter. However, it is easily shown that the latter class of mappings is much larger than the first.

4.3 Quasi-strongly Monotone Mappings and the Authors' RCA.

A mapping $F: D \to H$ is said to be strongly monotone if, there exists a constant $\alpha > 0$ such that for all $x, y \in D$

$$(Fx - Fy, x - y) \geq \alpha \|x - y\|^2 \tag{4.5}$$

holds. We call the mapping F monotone if the inequality holds for $\alpha = 0$. For the equation $Fx = 0$, with F a quasi-strongly monotone mapping, we have really shown all of the convergence properties (except for Theorem 6) of the SRCA with the choice $u(x) = v(x) = (2\alpha)^{-1}$ and $P = I$ in (3.2)-(3.3). Especially, it is emphasized that, by the boundedness of a monotone mapping, the Bounded Convergence Theorem indicates that the SRCA is unconditional and always locally convergent for a finite and infinite dimension space, respectively.

In order to get the global convergence, we assume that F is also Lipschitzian, i.e., for some constant $L > 0$, $\|Fx - Fy\| \leq L\|x - y\|$, and specify the algorithm by

$$u(x) = \alpha^{-1} \quad \text{and} \quad v(x) = (\alpha^{-2} - L^{-2})^{\frac{1}{2}}.$$

Then the SRCA is globally convergent. To see this, notice that, $\lambda = (1-\alpha^2 L^{-2})^{\frac{1}{2}} < 1$, by the definition of λ, and every $x \in D$ and $y \in N(F)$, the quasi-strongly monotonicity and L-continuity of F gives that

$$\|x - \alpha L^{-2} Fx - y\|^2 = \|x - y\|^2 - 2\alpha L^2 (Fx, x-y) + (\alpha L^{-2} \|Fx - Fy\|)^2$$

$$\leqq (1 - 2\alpha^2 L^{-2} + (\alpha L^{-1})^2)\|x-y\|^2 = (1-\alpha^2 L^{-2}) \quad \|x-y\|^2$$

which shows that the mapping T defined by $Tx = x - \alpha L^{-2} Fx$ is really a contractive mapping with modulus of contractivity λ, so it immediately follows from 4.3 that

$$N(F) \subset B(x - g(x), r(x)),$$

i.e., the hypotheses (1.1)-(1.2) and assumptions of Theorem 6 are all satisfied.

4.4 Strictly Pseudo-contractive Mappings.

A mapping $G: D \to H$ is said to be strictly pseudo-contractive if, there is a positive constant $\beta < 1$ such that for all $x, y \in D$

$$\|Gx - Gy\|^2 \leqq \|x - y\|^2 + \beta \|(x - Gx) - (y - Gy)\|^2. \tag{4.6}$$

It is known [6] that G is strictly pseudo-contractive iff $F = I - G$ is monotone with the following property

$$(Fx - Fy, x - y) \geqq \frac{1}{2}(1 - \beta) \|Fx - Fy\|^2. \tag{4.7}$$

So, we consider the latter class here, where $\frac{1}{2}(1-\beta)$ is replaced by $\alpha > 0$.

Let F be a monotone mapping with the property (4.7) and assume that F satisfies the following quasi-expansive condition

$$\|x-y\| \leq L\|Fx\| \qquad \forall\, x \in B_0, \quad y \in N(F). \tag{4.8}$$

We then easily show that the SRCA with the specialization

$$g(x) = \alpha^{-1}L^2Fx \text{ and } \quad R(x) = L^2((\alpha^2 - L^{-2}))^{\frac{1}{2}}\,\|Fx\|$$

is globally convergent (the reasoning is almost similar to the previous one).

Remark. Under the assumption that $\alpha < L < (\frac{1}{2}(1+5))^{\frac{1}{2}}\alpha$ in Subsection 4.3 and $\alpha < L \leq 2^{\frac{1}{2}}\alpha$ in Subsection 4.4, Wu and Wang [4] specify the SRCA by setting $u(x) = \alpha/L^2$, $v(x) = (\alpha^{-2} - L^{-2})^{\frac{1}{2}}$, $P = I$ and $u(x) = \alpha$, $v(x) = ((L^2 - \alpha^2)^{\frac{1}{2}}$, $P = I$, respectively. Clearly, our specializations here not only abstain from their restrictions on L and α, but also increase the speed of convergence greatly.

Some more sophisticated specializations can also be done, for example, see [7].

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