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Inclusion Methods for Elliptic Boundary Value Problems

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1. INTRODUCTION

This article is concerned with inclusion methods for nonlinear second-order elliptic boundary value problems of the form

$$\begin{aligned} -\Delta u + F(x, u, \nabla u) &= 0 \quad \text{on } \Omega, \\ B[u] &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. Its boundary $\partial\Omega$ is assumed to be Lipschitz-continuous (i. e., $\partial\Omega$ is everywhere locally representable as the graph of a Lipschitz-continuous function). Some additional smoothness properties of $\partial\Omega$ will be specified later.

B is a mixed type boundary operator: There exists a closed subset $\Gamma_0 \subset \partial\Omega$ such that $B[u] := u$ on Γ_0, and $B[u] := \partial u/\partial\nu := (\nabla u) \cdot \nu$ on $\Gamma_1 := \partial\Omega \setminus \Gamma_0$, with $\nu : \partial\Omega \to \mathbb{R}^n$ denoting the outer unit normal field at $\partial\Omega$, and with the dot indicating the canonical inner product in $\mathbb{R}^n$.

We assume that the triple $(\Omega, \Gamma_0, \Gamma_1)$ is *regular* in the following sense: Let

$$H_2^B(\Omega) := cl\{u \in C_2(\bar{\Omega}) : B[u] = 0 \quad \text{on } \partial\Omega\},$$

with "cl" indicating the closure in the Sobolev space $H_2(\Omega)$. The regularity condition requires that, for some $\sigma \in \mathbb{R}$, the boundary value problem

$$u \in H_2^B(\Omega), \quad -\Delta u + \sigma u = r \quad \text{on } \Omega$$

is (uniquely) solvable for each r in the space $L_2(\Omega)$ of square integrable functions. The complete class of regular triples $(\Omega, \Gamma_0, \Gamma_1)$ seems to be unknown. However, regularity can be shown for many relevant examples, several of which are listed in the appendix of this article.

The nonlinearity F in (1.1) is defined on $\bar{\Omega} \times I\!R \times I\!R^n$ with values $F(x,y,z) \in I\!R$. We assume that F and its derivatives $F_y := \partial F/\partial y$ and $F_z := (\partial F/\partial z_1, \ldots, \partial F/\partial z_n)^t$ are continuous. If $n \geq 2$, we assume in addition that F grows at most quadratically with respect to z, i. e., for each $\alpha \geq 0$, there exists some $C \geq 0$ such that

$$|F(x,y,z)| \leq C(1 + |z|^2) \quad \text{for } x \in \bar{\Omega},\ y \in I\!R,\ |y| \leq \alpha,\ z \in I\!R^n, \tag{1.2}$$

with $|\cdot|$ denoting both the modulus in $I\!R$ and the Euclidean norm in $I\!R^n$.

We will discuss *existence* and *inclusion* methods for problem (1.1). These are methods providing the existence of a solution of problem (1.1) within explicitly computable bounds.

The search for bounds (resp. inclusions) for solutions of PDE and ODE problems has a long history. In particular, methods based on *monotonicity* ideas and *differential inequalities* have been extensively studied and applied to many examples. We will report on these methods, particularly on L. Collatz' and J. Schröder's work, in Section 3.

However, there are drawbacks of the monotonicity methods with respect to application to large classes of problems of the type (1.1): The monotonicity methods require all real eigenvalues of the linearization L of the left-hand side of problem (1.1) (at some approximate solution) to be *positive*. We will discuss the background and the consequences of these restrictions in Section 2, in the frame of a general description of inclusion methods via fixed-point formulations.

In Section 4, the author's existence and inclusion method, which avoids all kinds of monotonicity assumptions, will be presented. Its *basis* is closely related to the Newton-Cantorovich Theorem. Due to the lack of sufficient differential inequalitites in non-monotone situations, we propose to use *normwise* bounds for the terms entering the fixed-point equation, in particular, for the inverse of the linearized operator L mentioned above. For this purpose, we use bounds for *eigenvalues* of L or of L^*L (which have to be computed numerically), in combination with explicit Sobolev space imbeddings (which have been determined by theoretical means).

Other existence and inclusion methods which are applicable to problem (1.1) and avoid monotonicity assumptions, have been proposed by M. T. Nakao and by S. Oishi. Their methods avoid the (direct) computation of bounds for the operator L^{-1}. Instead, they both have to deal with the inverse of some finite-dimensional *projection* of L, and bound the infinite-dimensional "remainder" by other means. In Section 5, we will discuss these methods, and try to make clear that this infinite-dimensional remainder causes significant disadvantages.

For boundary value problems with ordinary differential equations (which are, as long as they are scalar and of second order, of course contained in (1.1)), a large variety of existence and inclusion methods without monotonicity assumptions has been proposed. Urabe [56] and Kedem [21] compute bounds for the Green function via initial value problems. Lehmann [26] uses differential operators with piecewise constant coefficients to enclose the Green function. Kaucher and Miranker [20] simply invert the operator $-u''$ to obtain a fixed-point equation, which simplifies the numerics but implicitly requires a very restrictive contraction property (see the discussion in Section 2). Lohner [27] applies shooting methods, treating initial value problems by interval-versions of one-step methods based on local Taylor expansions. In [14], the given problem is transformed by introduction of certain *breakpoints* (see also the short description in Section 3). This approach by J. Schröder preserves the ideas of monotonicity methods, without however requiring their restrictive assumptions.

The particular situation of *ordinary* differential equations seems to be obligatory for all these methods (except for Kaucher's approach, which however has other severe drawbacks; see Section 2). Since the main emphasis of the present article is on *elliptic* boundary value problems, we will not comment further on these methods here.

In Section 6, we give a brief description of the numerical procedures we used to compute the terms needed for the practical application of our method; however, we will not describe the computation of the eigenvalue bounds needed to estimate L^{-1}, since such methods are extensively described in the article [7] in the present volume. Section 7 contains some examples which illustrate our method.

In the appendix-section 8, we present the most important solvability and regularity results for linear elliptic boundary value problems needed throughout the article. In particular, we formulate Sobolev's Imbedding Theorem (for bounded Lipschitz-domains), which will be cited in many places of the article, without explicit reference to Section 8 or to other sources.

2. GENERAL CONCEPTS

2.1. Fixed-point formulation

The usual general way of proceeding in order to derive the desired existence and inclusion statements is to transform problem (1.1) into some equivalent fixed-point equation

$$u \in X, \quad u = Tu, \tag{2.1}$$

and to apply some fixed-point theorem. Under appropriate conditions on the space X and the operator $T : X \to X$ (e. g., continuity, compactness), which usually have to be verified by theoretical means, the fixed-point theorem yields the existence of a solution u of problem (2.1) (and thus, of problem (1.1)) in some suitable set $U \subset X$, provided that

$$TU \subset U. \tag{2.2}$$

The statement "$u \in U$" constitutes the desired inclusion result. In order to compute an explicit inclusion, one must therefore construct U *explicitly*. Moreover, U should provide tight bounds, i. e., it should be *"small in diameter"* in an appropriate sense. These requirements can usually be satisfied only by numerical means. For the numerical verification of condition (2.2), one has to use *interval-analysis* on many levels between basic interval arithmetic and functional analysis, as we will discuss in detail in later sections.

The transformation of problem (1.1) into some fixed-point equation (2.1) can be carried out in various ways. In the following, we will describe the most "reasonable" transformations, which at the same time gives a brief overview of the most important existence and inclusion methods for elliptic problems of the type considered here. To avoid an overload of this brief description with too many technical details, we will not be very precise with respect to spaces and norms in this (and in the next) subsection.

1) The simplest transformation is possibly the following: Provided that the "Dirichlet-part" Γ_0 of $\partial\Omega$ has positive measure in $\partial\Omega$, so that the operator $-\Delta$ is invertible on $H_2^B(\Omega)$, one may apply $(-\Delta)^{-1}$ to the differential equation in (1.1) (regard Theorem A.2 of the Appendix) to obtain a fixed-point equation (2.1) with

$$Tu := (-\Delta)^{-1}[-F(\cdot, u, \nabla u)], \tag{2.3}$$

and with $X \supseteq H_2^B(\Omega)$ denoting some appropriate Banach space (for instance, $X = H_{1,p}(\Omega)$ for some suitable p, or $X = C_1(\bar{\Omega})$ in the case of an ODE problem ($n = 1$), or $X = C(\bar{\Omega})$ if $n \leq 3$ and F is independent of ∇u; all these spaces will be discussed later).

The choice (2.3) was proposed (in a slightly different, but equivalent formulation) by Kaucher and Miranker [20] for the ODE case ($n = 1$), and in Nakao's earlier papers [29] for the elliptic PDE case.

In the next subsection, we will show that the choice (2.3) for the fixed-point formulation is *very restrictive* in the sense that the condition (2.2), with a practically computable set U, restricts the class of problems (1.1) which can be treated to a rather small one.

2) Instead of using the linear operator $-\Delta$ and its inverse to define T, one may use a more general operator

$$\tilde{L}[u] := -\Delta u + \tilde{b} \cdot \nabla u + \tilde{c}u, \tag{2.4}$$

with suitable functions $\tilde{b} \in L_\infty(\Omega)^n$, $\tilde{c} \in L_p(\Omega)$ (for some $p > n$, $p \geq 2$) such that $\tilde{L}$ is invertible on $H_2^B(\Omega)$. The differential equation in (1.1) may be rewritten as $\tilde{L}[u] = \tilde{b} \cdot \nabla u + \tilde{c}u - F(\cdot, u, \nabla u)$, and application of $\tilde{L}^{-1}$ (regard Theorem A.2 of the Appendix) provides a fixed-point equation (2.1) with

$$Tu := \tilde{L}^{-1}[\tilde{b} \cdot \nabla u + \tilde{c}u - F(\cdot, u, \nabla u)], \tag{2.5}$$

and with the same possibilities of choosing the space X as in **1)**.

An important special choice in (2.4) is $\tilde{b} = b$, $\tilde{c} = c$, $\tilde{L} = L$, where

$$b := F_z(\cdot, \omega, \nabla\omega), \quad c := F_y(\cdot, \omega, \nabla\omega), \quad L[u] := -\Delta u + b \cdot \nabla u + cu, \tag{2.6}$$

with ω denoting some *approximate solution* for problem (1.1) obtained, for instance, from numerical (floating-point) computations. This choice provides that the linearization of the operator T at ω is zero, which has many pleasant consequences (see Section 4).

The choice (2.5), (2.6) was proposed in several of the author's papers [33 – 37], and in a recent paper by Oishi [32]. Oishi uses an interesting variant of the author's method which we will discuss in Section 5.

3) The oldest and most extensively studied method uses *monotone* operators $T : X \to X$ in (2.1). These are operators satisfying

$$u \leq v \Longrightarrow Tu \leq Tv \quad \text{for all } u, v \in X, \tag{2.7}$$

with "$\leq$" denoting some suitable partial order relation in X. To explain the basic ideas we assume, for simplicity of presentation, that the nonlinearity F in (1.1) depends only linearly on ∇u, i. e., that

$$F(x, y, z) = b(x) \cdot z + \tilde{F}(x, y), \tag{2.8}$$

and that $n \leq 3$. Then, with $\tilde{L}$ given by (2.4) and $\tilde{b} := b$, (2.5) takes the form

$$Tu := \tilde{L}^{-1}[\tilde{c}u - \tilde{F}(\cdot, u)], \tag{2.9}$$

and $\tilde{c}$ is chosen such that T is monotone, with respect to the canonical order relation $u \leq v :\Longleftrightarrow u(x) \leq v(x)$ $(x \in \bar{\Omega})$ in $X := C(\bar{\Omega})$, on some function interval

$$[v_0, w_0] := \{u \in X : v_0(x) \leq u(x) \leq w_0(x) \ (x \in \bar{\Omega})\}.$$

This is achieved by choosing (a constant) $\tilde{c} > 0$, which provides that $\tilde{L}$ is *inverse-positive* on $H_2^B(\Omega)$, i. e.,

$$\tilde{L}[u] \geq 0 \Longrightarrow u \geq 0 \quad \text{for all } u \in H_2^B(\Omega) \tag{2.10}$$

(see Corollary 3.2) and by requiring, moreover, that

$$\tilde{c} \geq \max\left\{\frac{\partial \tilde{F}}{\partial y}(x, y) : x \in \bar{\Omega},\ v_0(x) \leq y \leq w_0(x)\right\},$$

which obviously yields the monotonicity of the mapping $u \mapsto \tilde{c}u - \tilde{F}(\cdot, u)$ on $[v_0, w_0]$. Since (2.10) implies the monotonicity of $\tilde{L}^{-1}$, T given by (2.9) is therefore monotone on $[v_0, w_0]$.

The monotonicity of T implies $Tv_0 \leq Tu \leq Tw_0$ for all $u \in [v_0, w_0] =: U$, so that the crucial condition (2.2) takes the simple form

$$v_0 \leq Tv_0, \quad Tw_0 \leq w_0. \tag{2.11}$$

Due to the monotonicity of $\tilde{L}^{-1}$, the inequalities $\tilde{L}[v_0] \le \tilde{L}[Tv_0]$, $\tilde{L}[Tw_0] \le \tilde{L}[w_0]$ are *sufficient* for (2.11), provided that $v_0, w_0 \in H_2^B(\Omega)$. According to (2.9), (2.4) (with $\tilde{b} = b$) and (2.8), these sufficient *differential inequalities* read

$$-\Delta v_0 + F(\cdot, v_0, \nabla v_0) \le 0 \le -\Delta w_0 + F(\cdot, w_0, \nabla w_0). \tag{2.12}$$

The first who systematically investigated monotone operators with respect to applications to differential equations was L. Collatz [8]. J. Schröder continued and extended his work [48 – 55]. Inequalities of the type (2.11) or (2.12) (together with $v_0 \le w_0$) played a central role in their investigations; see Section 3 for more details. In Subsection 2.2, we will also show that the "monotonicity method" has drawbacks, compared with the choice (2.6) explained above, with respect to the class of problems it can be applied to.

4) Starting from some approximate solution ω of problem (1.1), we can choose the *simplified Newton* operator

$$Tu := u - \mathcal{F}'(\omega)^{-1}\mathcal{F}(u), \tag{2.13}$$

with $\mathcal{F}(u)$ denoting the left-hand side of (1.1). The (Fréchet-) derivative $\mathcal{F}'(\omega)$ therefore coincides with the operator L given by (2.6). Thus, (2.13) reads

$$Tu = u - L^{-1}[-\Delta u + F(\cdot, u, \nabla u)]. \tag{2.14}$$

Since $-\Delta u = L[u] - b \cdot \nabla u - cu$, we obtain

$$Tu = L^{-1}[b \cdot \nabla u + cu - F(\cdot, u, \nabla u)],$$

i. e., T coincides with the operator given by (2.5), (2.6) proposed by the author.

In his recent papers (e. g., [30, 31]), M. T. Nakao uses a modification of this operator T, in order to avoid the severe drawbacks of his earlier choice (2.3). Essentially, he replaces L^{-1} in (2.14) with the aid of some projection P onto some finite dimensional space S. More precisely, he replaces L^{-1} in (2.14) by

$$\left\{\varepsilon I + \left[P(-\Delta)^{-1}L\Big|_S\right]^{-1} P\right\}(-\Delta)^{-1}, \tag{2.15}$$

with $\varepsilon > 0$ denoting some small number, so that T in (2.14) is replaced by

$$Tu = u - \left\{\varepsilon I + \left[P(-\Delta)^{-1}L\Big|_S\right]^{-1} P\right\}[u + (-\Delta)^{-1}F(\cdot, u, \nabla u)]. \tag{2.16}$$

The term εI is needed to make T a *condensing* operator, a condition which is satisfied if T is the sum of a contractive and a compact mapping, which allows the application of Sadovskii's Fixed-Point-Theorem.

If ε is "small" and the finite dimensional space S is "large" in an appropriate sense, then the operator given by (2.15) is "close" to L^{-1}, so that some pleasant properties of the operator T in (2.14) are preserved in (2.16). A more detailed comparison of this new operator with the old choice (2.14) will be given in Section 5.

2.2. Construction of U, "almost" necessary conditions

In this subsection, we will describe general concepts for the numerical construction of some (closed, bounded) set $U \subset X$ satisfying (2.2). Besides the basic concept, our main emphasis will here be on conditions which are an "almost" necessary consequence of (2.2) in the sense that they cannot be avoided in practical computations if U shall be "small in diameter". These "almost" necessary conditions will provide a lot of information about the quality of the different choices of the operator T described in the previous subsection. *Sufficient* conditions for (2.2) will be discussed in the following sections.

For many examples where *monotone* operators T can be constructed, it was shown by Collatz (e. g., [8 – 10]), Schröder (e. g., [48 – 55]), Albrecht (e. g., [3, 4]), Walter (e. g., [57]) and others, that conditions (2.11) or (2.12) (which are equivalent to resp. sufficient for (2.2) with $U = [v_0, w_0]$) can often be fulfilled by an *ansatz* for v_0 and w_0. In this way, verified enclosure results for many PDE problems and a vast variety of ODE problems were obtained already several decades ago! Rounding errors were avoided by calculation "by hand" in rational arithmetic, or they were neglected, which of course violated the rigor of the results. However, this violation is not very severe since the needed calculations are, in most cases, very stable.

In non-monotone situations, however, where conditions of the type (2.11) or (2.12) are not available, it is difficult to construct a set U satisfying (2.2) by ansatz. Instead, it has become a usual way of proceeding (not only for differential equation problems) to start with an *approximate solution* (approximate fixed-point) ω, and then to transform the fixed-point equation (2.1) into "midpoint-form", which in our situation means to introduce the operator $\tilde{T} : X \to X$ defined by

$$\tilde{T}u := T(\omega + u) - \omega. \tag{2.17}$$

Obviously, each fixed-point u of $\tilde{T}$ provides $\omega + u$ as a fixed-point of T, and vice versa. Moreover, if one is interested in solutions of the given problem which are close to ω, one now looks for fixed-points of $\tilde{T}$ which are close to 0. Thus, one will try to construct some "small" (closed and bounded) neighborhood $V \subset X$ of 0 such that

$$\tilde{T}V \subset V, \tag{2.18}$$

which is often much easier, from the mathematical and from the computational point of view, than the direct construction of U satisfying (2.2). Once a V satisfying (2.18) has been found, (2.2) is fulfilled for $U := V + \omega$.

To construct V, we now assume that T is Fréchet-differentiable at ω, and define

$$\psi(u) := T(\omega + u) - T\omega - T'(\omega)[u] \quad (u \in X).$$

Then,

$$\tilde{T}u = (T\omega - \omega) + T'(\omega)[u] + \psi(u) \quad \text{for } u \in X. \tag{2.19}$$

Therefore, a sufficient condition for (2.18) is

$$(T\omega - \omega) + T'(\omega)[V] + \psi(V) \subset V. \tag{2.20}$$

In general, (2.20) is not a necessary consequence of (2.18). However, it is "almost" necessary if V is "small", since $\|\psi(u)\| = o(\|u\|)$ for $u \to 0$ (this is just the definition of the Fréchet-differentiability of T at ω) and $\psi(V)$ is therefore only a "very small" perturbation of the "small" term $(T\omega - \omega) + T'(\omega)[V]$. Moreover, it is almost unavoidable in practice to compute a *separate* enclosure for the higher order terms $\psi(V)$, i. e., to verify, in reinforcement of (2.18), condition (2.20) or even a stronger condition containing a practically computed *enclosure* $W \supset \psi(V)$ in place of $\psi(V)$ (for instance, a ball W centered at 0):

$$(T\omega - \omega) + T'(\omega)[V] + W \subset V. \tag{2.21}$$

Such an enclosure W of $\psi(V)$ will usually satisfy

$$0 \in \overset{\circ}{W}, \tag{2.22}$$

since $0 \in \overset{\circ}{V}$ and $\psi(0) = 0$ which shows that (2.22) can only be avoided if rather special properties of the higher order terms $\psi(V)$ can be exploited.

For the rest of this section, we will be concerned with consequences of the "almost" necessary conditions (2.21), (2.22). The sufficient condition (2.20) will be further investigated in the following sections, in particular in Section 4.

The following theorem was proved by S. M. Rump for matrices [46, Thm. 5, p. 38]. The proof can be carried over, without many changes, to bounded linear operators in Banach spaces. However, some changes are necessary due to the lack of compactness properties in our infinite dimensional situation.

Theorem 2.1. *Let V be a closed, bounded, nonempty subset of the real Banach space X, and let some $W \subset X$ exist such that (2.21) and (2.22) hold. Then, all real eigenvalues of $T'(\omega)$ lie within* $(-1, 1)$.

Remarks. a) In Rump's original theorem, it is shown a fortiori that all *complex* eigenvalues lie within the open complex unit disc. The same result can be proved here, based on the usual complex extension techniques for real Banach spaces and their linear operators. We omit this stronger result here since we do not need it for later purposes.

b) If $T'(\omega)[V]$ is *compact* (for instance, if X is finite-dimensional), the existence of some $W \subset X$ satisfying (2.21), (2.22) is equivalent to Rump's original condition

$$(T\omega - \omega) + T'(\omega)[V] \subset \overset{\circ}{V}.$$

Proof of Theorem 2.1. Let $Y := V - V = \{v_1 - v_2 : v_1, v_2 \in V\}$. Then,

$$T'(\omega)[Y] + W \subset Y \tag{2.23}$$

since, for $v_1, v_2 \in V$, $y := v_1 - v_2$ and $w \in W$, (2.21) and (2.22) yield $T'(\omega)[y] + w = [(T\omega - \omega) + T'(\omega)[v_1] + w] - [(T\omega - \omega) + T'(\omega)[v_2] + 0] \in V - V = Y$.

Now let $(\lambda, u) \in I\!R \times X$ denote an eigenpair of $T'(\omega)$, and define

$$\Gamma := \{\gamma \in I\!R : \gamma u \in Y\}.$$

Γ is nonempty because $0 \in \Gamma$ (which holds since V is nonempty), and Γ is bounded since $u \neq 0$ and V is bounded. Therefore, $\gamma^* := \sup \Gamma$ is finite and nonnegative. Due to (2.22), we can choose some $\gamma \in \Gamma$ and some $\varepsilon > 0$ such that

$$\lambda[(\gamma^* - \gamma) + \varepsilon]u \in W. \tag{2.24}$$

Since $\gamma u \in Y$ and $T'(\omega)[\gamma u] = \lambda \gamma u$, (2.24) and (2.23) imply

$$\lambda(\gamma^* + \varepsilon)u = \lambda\gamma u + \lambda[(\gamma^* - \gamma) + \varepsilon]u \in Y.$$

The symmetry $Y = -Y$ therefore yields $|\lambda|(\gamma^* + \varepsilon)u \in Y$ and thus, $|\lambda|(\gamma^* + \varepsilon) \in \Gamma$, so that the definition of γ^* provides $|\lambda|(\gamma^* + \varepsilon) \leq \gamma^*$. Consequently, $|\lambda| < 1$. ■

To derive further consequences of this theorem (resp., of (2.21), (2.22)), we now assume more concretely that T is given in the form (2.5), with functions $\tilde{b} \in L_\infty(\Omega)^n$, $\tilde{c} \in L_p(\Omega)$ (for some $p > n$, $p \geq 2$) such that $\tilde{L} : H_2^B(\Omega) \to L_2(\Omega)$ defined by (2.4) is one-to-one (and thus, due to Theorem A.2 of the Appendix, also onto). Moreover, we assume that the space X in (2.1) is chosen such that $H_2^B(\Omega) \subset X$ (with continuous imbedding) and that the operator $\hat{F}$ defined by $\hat{F}(u) := F(\cdot, u, \nabla u)$ is Fréchet-differentiable at ω as a mapping from X into $L_2(\Omega)$, which is true for the spaces X in the corresponding situations mentioned in **1)** in Subsection 2.1. Then, $T : X \to X$ is Fréchet-differentiable at ω.

Investigating T of this form we cover all choices discussed in the previous subsection except Nakao's new choice (see **4)** in Subsection 2.1) which will be analyzed separately in Section 5.

Furthermore, let $L : H_2^B(\Omega) \to L_2(\Omega)$ denote the linearization of the left-hand side of the given problem at $\omega \in H_2^B(\Omega)$, i. e., let L be given by (2.6). We assume that ω is such that $b \in L_\infty(\Omega)^n$, $c \in L_p(\Omega)$ for some $p > n$, $p \geq 2$ (which is satisfied, for instance, if $\omega \in L_\infty(\Omega)$ and $\nabla\omega \in L_\infty(\Omega)^n$ and thus, for all usual finite element approximations ω). For $\lambda \in [-1, 1]$, define

$$\begin{aligned} L_{(\lambda)}[u] :&= (1 - \lambda)\tilde{L}[u] + \lambda L[u] \\ &= -\Delta u + [(1 - \lambda)\tilde{b} + \lambda b] \cdot \nabla u + [(1 - \lambda)\tilde{c} + \lambda c]u. \end{aligned} \tag{2.25}$$

Theorem 2.2. *Let the assumptions of Theorem 2.1 hold, with T and X as described above. Then,*

a) $L_{(\lambda)} : H_2^B(\Omega) \to L_2(\Omega)$ is one-to-one for each $\lambda \in [-1,1]$,

b) if $L_{(\lambda)}$ is inverse-positive on $H_2^B(\Omega)$ for one $\lambda \in [-1,1]$ (i. e., (2.10) holds for $L_{(\lambda)}$, with the canonical order relation $u \geq v :\iff u(x) \geq v(x)$ for almost all $x \in \Omega$), then $L_{(\lambda)}$ is inverse-positive on $H_2^B(\Omega)$ for all $\lambda \in [-1,1]$.

Proof. **ad a):** Let $L_{(\lambda)}[u] = 0$ for some $\lambda \in [-1,1]$ and some $u \in H_2^B(\Omega)$. Then,

$$\tilde{L}[u] = \lambda[(\tilde{b} - b) \cdot \nabla u + (\tilde{c} - c)u].$$

Moreover, (2.5) and (2.6) provide

$$T'(\omega)[u] = \tilde{L}^{-1}[(\tilde{b} - b) \cdot \nabla u + (\tilde{c} - c)u],$$

so that we obtain

$$u = \lambda T'(\omega)[u]. \tag{2.26}$$

Now suppose for contradiction that $u \neq 0$. Then, (2.26) shows that $\lambda \neq 0$ and that λ^{-1} is an eigenvalue of $T'(\omega)$. Thus, Theorem 2.1 provides $|\lambda| > 1$ which contradicts our assumption.

ad b): The proof of this part of the assertion is postponed to Section 3 (Theorem 3.3). ∎

Before discussing consequences of Theorem 2.1 with respect to the special choices of $\tilde{L}$ (resp., of $\tilde{b}$ and $\tilde{c}$) described in the previous subsection, we formulate the following theorem, the proof of which is again postponed to Section 3 (Theorem 3.4).

Theorem 2.3. *For each $\lambda \in [-1,1]$, the inverse-positivity of $L_{(\lambda)}$ on $H_2^B(\Omega)$ implies that all its real eigenvalues are positive.*

The following conclusions, which follow from Theorem 2.2 and are illuminated by Theorem 2.3, constitute an important result of this subsection. The terms b, c, L are given by (2.6).

Conclusions:

i) For the choice (2.3) for T proposed by Kaucher and Miranker, and in Nakao's earlier papers, the "almost" necessary conditions (2.21), (2.22) imply:

$$\begin{array}{l} L_{(\lambda)} \text{ given by } L_{(\lambda)}[u] = -\Delta u + \lambda b \cdot \nabla u + \lambda c u \\ \text{is inverse-positive on } H_2^B(\Omega) \text{ for all } \lambda \in [-1,1]. \end{array} \tag{2.27}$$

ii) For the "monotonicity" choice (2.9) proposed by Collatz, Schröder et al., with $\tilde{c}$ as described after (2.9), the "almost" necessary conditions (2.21), (2.22) imply:

$$\begin{aligned}&L \text{ given by } L[u] = -\Delta u + b \cdot \nabla u + cu \\ &\text{is inverse-positive on } H_2^B(\Omega).\end{aligned} \tag{2.28}$$

iii) For the choice (2 4), (2.5), with $\tilde{b} = b$, $\tilde{c} = c$, $\tilde{L} = L$ from (2.6), which was proposed by the author, the "almost" necessary conditions (2.21), (2.22) have no restrictive consequences.

Proof. Conclusions i) and ii) follow from Theorem 2.2 if we show that $L_{(0)} = \tilde{L}$ is inverse-positive on $H_2^B(\Omega)$. For ii), this follows immediately from the choice $\tilde{c} > 0$ implying (2 10) (see Corollary 3.2). For i), where $\tilde{L} = -\Delta$, it follows from Corollary 3.3.

In the situation considered in iii), we have $T'(\omega) \equiv 0$ so that (2.21), (2.22) have no consequences concerning T, without further conditions on V and W (such as the condition $\psi(V) \subset W$, which makes these conditions *sufficient* for (2.18) and occurs, of course, for all choices of T). ■

In addition to conclusion ii), it should be noted that, for the choice (2.9), Theorem 2.2 *does not imply more* than (2.28) if v_0 and w_0 (see (2.11), (2.12)) are constructed such that $\omega \in U = [v_0, w_0]$ (or equivalently, that $0 \in V = U - \omega$; compare the remarks before and after (2.18)). This can be seen as follows. Since $\tilde{b} = b$ here (see (2.8), (2.9)), (2.25) shows that

$$L_{(\lambda)}[u] = -\Delta u + b \cdot \nabla u + [(1-\lambda)\tilde{c} + \lambda c] = L[u] + (1-\lambda)(\tilde{c} - c)u$$

for $\lambda \in [-1, 1]$, and $\tilde{c} - c$ is nonnegative due to the condition required for $\tilde{c}$ after (2.10). Theorem 3.2 therefore implies that (2.28) is equivalent to the inverse-positivity of *all* operators $L_{(\lambda)}$ ($\lambda \in [-1, 1]$) asserted in Theorem 2.2.

Condition (2.27), however, is much more restrictive than (2 28). It requires, for instance, not only the negative part, but also the positive part of c to be "not too large" (compare Theorem 2.3), while large positive parts of c are no problem for condition (2.28), and are even *pleasant* for the monotonicity method.

Consequently, the class of problems (1.1) which can be treated by the choice (2 3) is *much smaller* than the class of problems tractable by the monotonicity method.

However, also condition (2.28) is rather restrictive. Via Theorem 2.3, it requires all real eigenvalues of L to be positive.

The choice (2.4), (2.5), with $\tilde{b} = b$ and $\tilde{c} = c$ from (2.6), does not pose restrictions on the sign of the (real) eigenvalues of L. It only requires all (real) eigenvalues to be *nonzero,* so that L is one-to-one. To show that this choice is in fact very powerful we consider the following example resulting from *semiconductor physics*:

$$-u'' - (u \cdot \sin x)' + \lambda(u^2 + u - 1) = 0 \quad (0 < x < 2\pi), \quad u'(0) = u'(2\pi) = 0. \tag{2.29}$$

The following bifurcation diagram of approximate solutions ω was obtained by a Newton-collocation procedure. An additional branch is formed by the vertical line at $\lambda = 0$ where $u(x) := \mu \exp(\cos x)$ is a solution for each $\mu \in I\!R$.

On the branches (which were plotted after interpolating a computed "grid" of approximate solutions), we noted the respective number of *negative eigenvalues* of the operator L. Only two branches show a "0"; so only on these two branches all eigenvalues of L are positive, as required by condition (2.28) for the "monotonicity" method.

The author's choice (2.4), (2.5), with $\tilde{b} = b$ and $\tilde{c} = c$ from (2.6) (with all details explained in Sections 4 and 6), was successful on *all* branches, except in immediate neighborhoods of the turning- and bifurcation points, where one eigenvalue of L is zero so that L is not one-to-one.

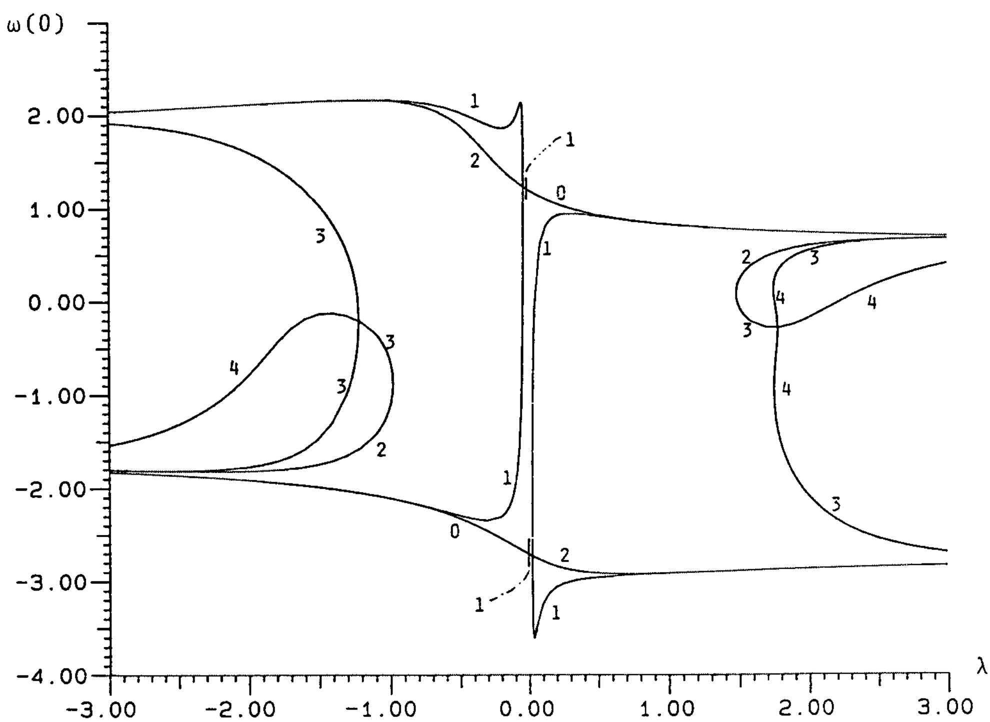


Figure 1: Bifurcation diagram for problem (2.29)

3. MONOTONICITY METHODS

In this section, we report on methods using certain inequalities in function spaces (operator inequalities) as sufficient conditions for inclusion results for problems of the form (1.1). Problems which allow application of such methods were entitled "problems of monotone kind" ("Aufgaben monotoner Art") by L. Collatz (see [8]). For differential equation problems, sufficient inequalities of the required form are typically *differential* inequalities (compare (2.12)).

For simplicity of presentation, we assume here, as partly in the previous section, that the nonlinearity F in (1.1) depends only linearly on ∇u, i. e., that

$$F(x,y,z) = b(x) \cdot z + \tilde{F}(x,y). \tag{3.1}$$

3.1. A brief survey

From the author's point of view, the most significant and extensive research work in the field of monotonicity methods and their application to boundary value problems has been done by L. Collatz and especially by J. Schröder (see the descriptions below). Since the corresponding theory is closely related to the maximum principle, important contributions to the subject have also been made by Hopf [17], and by Protter and Weinberger [42]. A lot of guiding work in the field of differential inequalities has been done by W. Walter [57], also together with R. Redheffer [43, 44]. Albrecht [3, 4] in particular investigated iterative procedures based on monotonicity methods. For a much more detailed discussion of the related literature, see [53].

In his famous paper [8], Collatz presented the first systematic formulation of "problems of monotone kind". These are problems of the form

$$Mu = r, \qquad u \in D, \tag{3.2}$$

where $D \subset R$ and $r \in S$, with R and S denoting two ordered linear spaces, and $M : D \to S$ is a (possibly nonlinear) operator satisfying

$$(Mu \leq Mv \Longrightarrow u \leq v) \qquad \text{for all } u,v \in D. \tag{3.3}$$

Such operators M were entitled "operators of monotone kind" by Collatz. Later, Schröder called them "inverse-monotone operators" which is perhaps more appropriate since (3.3) obviously means that the inverse operator $M^{-1} : M(D) \to R$ exists and is monotone.

For problems (3.2) of monotone kind, a very simple inclusion method is at hand: Suppose that elements $v_0, w_0 \in D$ have been found such that

$$Mv_0 \leq r \leq Mw_0. \tag{3.4}$$

Then, (3.3) immediately shows that the solution u of problem (3.2) satisfies

$$v_0 \le u \le w_0, \tag{3.5}$$

provided that this solution exists in D. The existence problem is *not* addressed in [8].

Collatz applies this general result, in particular, to problem (1.1), with F from (3.1), with Mu denoting the left-hand side of (1.1), $r \equiv 0$, and with

$$\begin{aligned} R &:= \left\{ u \in C_2(\Omega) \cap C(\bar{\Omega}) : \frac{\partial u}{\partial \nu} \quad \text{exists on } \Gamma_1 \right\}, \\ S &:= C(\Omega), \quad D := \{u \in R : \big(x, u(x)\big) \in K \quad \text{for } x \in \Omega,\ B[u] = 0 \text{ on } \partial\Omega\}, \end{aligned} \tag{3.6}$$

where $K \subset \Omega \times I\!R$ is a set which is convex with respect to its last variable, and which contains the graph of the solution. The order relations in R and S are chosen canonically, i. e.,

$$u \le v :\Longleftrightarrow u(x) \le v(x) \quad \text{for all } x \in \Omega. \tag{3.7}$$

He proves that this problem is of monotone kind if

$$\frac{\partial \tilde{F}}{\partial y}(x, y) \ge 0 \quad \text{for } (x, y) \in K, \tag{3.8}$$

essentially by showing (with the aid of the strong maximum principle) that, for *non-negative* functions $\tilde{c}$, the operator $\tilde{L}$ given by

$$\tilde{L}[u] := -\Delta u + b \cdot \nabla u + \tilde{c}u, \tag{3.9}$$

is inverse-positive on $R^B := \{u \in R : B[u] = 0 \text{ on } \partial\Omega\}$, i. e., that (2.10) holds with the "classical" function space R^B in place of $H_2^B(\Omega)$.

In order to combine the enclosure result (3.5) with an *existence* statement, Collatz proposes to write the given problem (if possible) as a fixed point equation (2.1) with a *monotone* operator T (i. e., an operator satisfying (2.7)), and to apply Schauder's Fixed-Point Theorem (see, e. g., [10, Chapter III]). In Subsection 2.1, we showed how such a transformation may be carried out. Observe that the conditions (2.12) obtained there coincide with (3.4)!

Of course, the application of Schauder's Fixed-Point Theorem requires a precise setting of spaces and a detailed analysis of mapping properties. In particular, one needs a solvability and regularity theory for linear elliptic boundary value problems, in order to deal with *inverses* of linear differential operators, such as $\tilde{L}^{-1}$ in (2.9). Thus, the use of Sobolev spaces and/or Hölder spaces is at least very helpful, if not mandatory. Collatz did not carry out these details in a systematic way. He restricted himself — as far as elliptic boundary value problems are concerned — to more general remarks on the application of Schauder's Fixed-Point Theorem, and put his main research emphasis on applications of the inclusion result (3.4) $\Longrightarrow$ (3.5) to a large variety of (monotone-kind-) problems taken from science and technology.

J. Schröder extended and generalized Collatz' work into many directions. In particular, he *systematized* many aspects of "monotonicity": On one hand, he introduced abstract functional analytical and operator theoretical settings, in order to point out the main structures and to recognize further possible applications; on the other hand, he always aimed at methods inducing *automatic*, computer-implementable procedures for various classes of problems. Here, we give only a brief description of some selected results.

In [48] already, he proposes inclusion methods (for differential equation problems) which are based on the computation of an approximate solution ω and the investigation of the equation for the error $u - \omega$; this equation involves the defect of ω. Today, such a way of proceeding has become usual in almost all existing inclusion methods (not only for differential equation problems).

Schröder treats the error equation by differential inequalities, in order to obtain error *bounds* (and thus, inclusions for the exact solution). Using such inequalities he presents, in [50], an "automatic" procedure for computing inclusions for solutions of scalar *initial value problems* with ordinary differential equations. Of course, rounding errors could not be regarded automatically since the corresponding computer facilities were not available 30 years ago. In [28], Marcowitz carries Schröder's method and procedure over to initial value problems with *systems* of equations.

Later, Schröder put his main emphasis on boundary value problems. [51] contains a programmed procedure for inclusions of the type (3.4) $\Longrightarrow$ (3.5) for weakly nonlinear second-order elliptic boundary value problems of monotone kind. [14, 54] contain a programmed algorithm producing inclusion *and existence* results for scalar nonlinear second-order two-point boundary value problems which need *not* be of monotone kind. To obtain such results, certain *breakpoints* $\xi_1, \ldots, \xi_m$ are introduced within the interval of definition such that, very roughly speaking, the problem *is* of monotone kind on each subinterval between two consecutive breakpoints. In this way, a coupled system of differential inequalities on the subintervals and additional inequalities in the breakpoints is obtained which is suitable for numerical solution and provides the desired existence and inclusion result. Related work is also due to Küpper [22] and to Adams and Spreuer [1]. However, no practicable generalization of the breakpoint approach to elliptic problems, which are the main topic of the present article, has been found yet. For this reason, we will not discuss the breakpoint method in more detail here.

Concerning Schröder's various theoretical results (which, however, were always in interaction with applications), such as his general concepts of inverse-positive and inverse-monotone operators and his useful "Monotonicity Theorem", we refer to his book [53] which also contains a lot of references to related work. In this book, he applies his theory of monotonicity to *ordinary* but not to partial differential equation problems. However, the presented operator theoretical concepts are suited for ordinary as well as for parabolic and elliptic problems (of monotone kind), which unfortunately has not been widely recognized, but which is shown in detail in [55]. Of course, one major difference between ODE and PDE problems occurs when the inclusion results shall be combined with *existence* statements, since the needed existence theories for linear problems are much more complicated in the PDE case.

Schröder proposes the method of the "modified problem" to obtain such existence and inclusion results for various kinds of differential equation problems. For elliptic boundary value problems, the method is presented, e. g., in [52, 55]. Here, we give a brief description of its application to problem (1.1) (with F from (3.1)) which is slightly more general than Schröder's original and allows weaker assumptions on the smoothness of the data of the problem.

Let $R \subset C(\bar{\Omega})$ denote some function space and let $v_0, w_0 \in R$ satisfying $v_0 \leq w_0$ (in the pointwise sense) be given. For $u \in R$, define the cutoff function $u^\# \in C(\bar{\Omega})$ by

$$u^\#(x) := \sup\{v_0(x),\ \inf\{u(x), w_0(x)\}\} \quad (x \in \bar{\Omega}),$$

and consider, for some arbitrarily chosen (constant) $\tilde{c} > 0$, the *modified problem*

$$\left.\begin{array}{c} -\Delta u + b \cdot \nabla u + \tilde{c}u = \tilde{c}u^\# - \tilde{F}(\cdot, u^\#) \quad \text{on } \Omega \\ B[u] = 0 \quad \text{on } \partial\Omega \end{array}\right\} \tag{3.10}$$

Theorem 3.1. *Suppose that*

a) the modified problem has a solution $u \in R$,

b) each solution $u \in R$ of the modified problem satisfies $v_0 \leq u \leq w_0$.

Then, problem (1.1) (with F from (3.1)) has a solution $u \in R$ satisfying $v_0 \leq u \leq w_0$.

Proof. The solution $u \in R$ of problem (3.10) provided by assumption a) satisfies $v_0 \leq u \leq w_0$ according to assumption b), so that $u = u^\#$ and u is therefore a solution of problem (1.1). ∎

In order to verify the assumptions a) and b) of Theorem 3.1, Schröder assumes the boundary conditions to be of pure *Dirichlet* type (i. e., $B[u] \equiv u$ on $\partial\Omega$), and chooses

$$R := C_2(\Omega) \cap C(\bar{\Omega}) \tag{3.11}$$

(or more precisely, $R := \{u \in C(\bar{\Omega}) : u\big|_\Omega \in C_2(\Omega)\}$). Moreover, he makes the following smoothness assumptions for some $\alpha \in (0, 1)$:

($S1$) $\partial\Omega$ is a *global* $C_{2+\alpha}$-manifold

($S2$) $b \in C_\alpha(\bar{\Omega})^n$, $\tilde{F}(\cdot, u) \in C_\alpha(\bar{\Omega})$ for each $u \in C_\alpha(\bar{\Omega})$

$(S3)$ $v_0, w_0 \in C_\alpha(\bar{\Omega})$

Lemma 3.1. *With R from (3.11), and under the assumptions $(S1)$ to $(S3)$, assumption a) of Theorem 3.1 holds true.*

Proof (see [52, 55]). Let $p > n$, $p \geq 2$ be fixed. The linear operator $\tilde{L}$ occurring on the left-hand side of (3.10) is invertible on $H_{2,p}(\Omega) \cap \overset{0}{H}_{1,p}(\Omega)$ since $\tilde{c} > 0$ (see Corollary 3.2). Due to known existence results for linear elliptic boundary value problems (see, e. g.,[12, 13, 24]), the inverse operator $\tilde{L}^{-1} : L_p(\Omega) \to H_{2,p}(\Omega)$ exists (on the whole of $L_p(\Omega)$) and is bounded. Moreover, the mapping $\varphi : C(\bar{\Omega}) \to C(\bar{\Omega})$, $\varphi(u) := \tilde{c}u^{\#} - \tilde{F}(\cdot, u^{\#})$ is continuous, and the image set $\varphi\big(C(\bar{\Omega})\big)$ is bounded. Finally, the imbedding E_1 : $C(\bar{\Omega}) \to L_p(\Omega)$ is bounded, and the imbedding $E_2 : H_{2,p}(\Omega) \to C(\bar{\Omega})$ is compact due to Sobolev's Imbedding Theorem. Consequently, $T := E_2 \circ \tilde{L}^{-1} \circ E_1 \circ \varphi : C(\bar{\Omega}) \to C(\bar{\Omega})$ is continuous, and the image set $TC(\bar{\Omega})$ is relatively compact. Thus, according to Schauder's Fixed-Point Theorem, T has a fixed-point $u \in C(\bar{\Omega})$.

The fixed-point equation implies $u \in H_{2,p}(\Omega)$, so that Sobolev's Imbedding Theorem provides $u \in C_\alpha(\bar{\Omega})$. Using $(S3)$ we obtain $u^{\#} \in C_\alpha(\bar{\Omega})$ and thus, $\varphi(u) \in C_\alpha(\bar{\Omega})$ due to $(S2)$. Together with $(S1)$ and $(S2)$, known regularity results (see, e. g., [13, 24]) imply $\tilde{L}^{-1}\varphi(u) \in C_{2+\alpha}(\bar{\Omega})$. Since $u = \tilde{L}^{-1}\varphi(u)$, we obtain in particular that $u \in R$ and that (3.10) holds for u. ∎

Lemma 3.2. *With R from (3.11), assumption b) of Theorem 3.1 holds true if $v_0, w_0 \in R$ and if the following inequalities are satisfied (compare (2.12), (3.4)):*

$$\left.\begin{array}{r} -\Delta v_0 + b \cdot \nabla v_0 + \tilde{F}(\cdot, v_0) \leq 0 \leq -\Delta w_0 + b \cdot \nabla w_0 + \tilde{F}(\cdot, w_0) \quad \text{on } \Omega, \\ v_0 \leq 0 \leq w_0 \quad \text{on } \partial\Omega. \qquad\qquad \end{array}\right\} \tag{3.12}$$

Proof (see [52, 55]). Let $u \in R$ denote a solution of problem (3.10) and assume that $u \not\leq w_0$. Then, $u \leq w_0 + \lambda$ for some $\lambda > 0$ which may be assumed to be chosen minimal. Thus, for $u_\lambda := w_0 + \lambda - u$, we have $u_\lambda \geq 0$, $u_\lambda(\bar{x}) = 0$ for some $\bar{x} \in \bar{\Omega}$, and moreover, $u_\lambda(x) \geq \lambda > 0$ for $x \in \partial\Omega$. Consequently, $\bar{x} \in \Omega$, and therefore, $(\nabla u_\lambda)(\bar{x}) = 0$ and $(\Delta u_\lambda)(\bar{x}) \geq 0$ which implies $\tilde{L}[u_\lambda](\bar{x}) \leq 0$, with $\tilde{L}$ again denoting the differential operator occurring on the left-hand side of (3.10). On the other hand, $u(\bar{x}) = w_0(\bar{x}) + \lambda$ and thus, $u^{\#}(\bar{x}) = w_0(\bar{x})$, so that (3.10) and (3.12) imply $\tilde{L}[w_0](\bar{x}) \geq \tilde{L}[u](\bar{x})$ which provides $\tilde{L}[u_\lambda](\bar{x}) \geq \lambda\tilde{c} > 0$. Due to this contradiction, we have $u \leq w_0$. The inequality $v_0 \leq u$ follows analogously. ∎

The strong smoothness assumptions $(S1)$, $(S2)$, and the assumption that the boundary condition is of pure Dirichlet type, are not quite satisfactory. In particular, assumption $(S1)$ excludes all domains Ω with corners, such as rectangles. The necessity of those assumptions in Schröder's approach is generated by the "classical" choice (3.11) for the space R which makes it necessary to prove the existence of a *classical* solution u of problem (3.10). This is achieved by application of the result $\tilde{L}^{-1}\big(C_\alpha(\bar{\Omega})\big) \subset C_{2+\alpha}(\bar{\Omega})$ (requiring $(S1)$, $(S2)$) in the proof of Lemma 3.1.

Instead of (3.11) we propose, in the case $n \leq 3$, the choice

$$R := H_2(\Omega). \tag{3.13}$$

Now, assumption a) of Theorem 3.1 can be verified under much weaker assumptions, which do in fact not exceed our general assumptions made in the introduction.

Lemma 3.3. *With R from (3.13), assumption a) of Theorem 3.1 holds true.*

Proof. Let $\tilde{L}$ and φ as in the proof of Lemma 3.1. Due to Theorem A.2 of the appendix and to the Open Mapping Theorem, $\tilde{L}^{-1} : L_2(\Omega) \to H_2^B(\Omega)$ ($\subset H_2(\Omega)$) exists and is bounded, with $H_2^B(\Omega)$ defined in the introduction. Since the imbedding $E_1 : C(\bar{\Omega}) \to L_2(\Omega)$ is bounded, and the imbedding $E_2 : H_2(\Omega) \to C(\bar{\Omega})$ is compact due to Sobolev's Imbedding Theorem (regard that now $n \leq 3$), we can conclude as in the proof of Lemma 3.1 that the operator T defined there has a fixed-point $u \in C(\bar{\Omega})$. The fixed-point equation provides $u \in H_2^B(\Omega) \subset R$, and that u is a solution of problem (3.10). ■

While assumption a) of Theorem 3.1 can be proved, with R from (3.13), under weaker assumptions than before, the verification of assumption b) is now harder. In fact, the proof of Lemma 3.2 cannot be carried over to the present situation since the inequalities for the *single value* $\tilde{L}[u_\lambda](\bar{x})$ used there do not make sense for H_2-functions u_λ. Nevertheless, the *statement* of Lemma 3.2 remains true:

Lemma 3.4. *With R from (3.13), assumption b) of Theorem 3.1 holds true if $v_0, w_0 \in R$ and if*

$$\left.\begin{aligned} -\Delta v_0 + b \cdot \nabla v_0 + \tilde{F}(\cdot, v_0) \leq 0 \leq -\Delta w_0 + b \cdot \nabla w_0 + \tilde{F}(\cdot, w_0) \quad & a.\ e.\ on\ \Omega, \\ B[v_0] \leq 0 \leq B[w_0] \quad a.\ e.\ on\ \partial\Omega, & \end{aligned}\right\} \tag{3.14}$$

where "a. e." means "almost everywhere" with respect to the canonical measures on Ω and on $\partial\Omega$, respectively.

The proof of this Lemma is postponed to the next subsection (Corollary 3.1).

We wish to remark that the improvement obtained by the choice (3.13) only affects the *smoothness* assumptions on the problem data. The disadvantages of the monotonicity method concerning the required inverse-positivity of the linearized operator L, which we discussed and illustrated at the end of Section 2, remain unchanged! (In fact, the setting used in Section 2 was already based on the choice (3.13).)

3.2. Inverse-positivity on $H_2(\Omega)$

In this subsection, we develop a brief theory of inverse-positivity on the Sobolev space $H_2(\Omega)$. In the context of the present article, this is mainly to fill some gaps left in

Subsections 2.2 and 3.1. However, we believe that the results are also of interest in their own. A restriction on the dimension n is not needed here.

We wish to remark that many of the main ideas used here are already contained in Schröder's work (see, for instance, [53, Chapter III, Section 3]) but have never been carried out in a way which is needed in our context.

Throughout this subsection, let L denote an operator of the form

$$L[u] = -\Delta u + b \cdot \nabla u + cu,$$

with given coefficient functions $b \in L_\infty(\Omega)^n$, $c \in L_p(\Omega)$ (for some $p > n$, $p \geq 2$), which here may be seen free of the definition (2.6).

We say that the pair (L, B) is *inverse-positive on* $H_2(\Omega)$ if, for each $u \in H_2(\Omega)$, the following implication holds:

$$\left.\begin{array}{ll} L[u] \geq 0 & \text{a. e. on } \Omega \\ B[u] \geq 0 & \text{a. e. on } \partial\Omega \end{array}\right\} \Longrightarrow u \geq 0 \quad \text{a. e. on } \Omega,$$

where again "a. e." means "almost everywhere" with respect to the canonical measures on Ω and on $\partial\Omega$, respectively, and the expression $B[u]$ is to be understood in the trace sense. Moreover, we say that L is *inverse-positive on* $H_2^B(\Omega)$ if, for each $u \in H_2^B(\Omega)$,

$$L[u] \geq 0 \quad \text{a. e. on } \Omega \Longrightarrow u \geq 0 \quad \text{a. e. on } \Omega,$$

i. e., (2.10) holds for L. Since $B[u] = 0$ a. e. on $\partial\Omega$ for $u \in H_2^B(\Omega)$, we have the following

Lemma 3.5. *If (L, B) is inverse-positive on $H_2(\Omega)$, then L is inverse-positive on $H_2^B(\Omega)$.*

In the following, we will omit the expressions "a. e. on Ω" and "a. e. on $\partial\Omega$" if no misinterpretation of the corresponding inequalities is possible.

The following lemma constitutes the basis of all results in this subsection.

Lemma 3.6. *Let $\beta > 0$. Then, there exists a constant $C = C(\beta)$ such that, for all functions $\tilde{b} \in L_{2p}(\Omega)^n$, $\tilde{c} \in L_p(\Omega)$ (for some fixed $p > n$) satisfying $\frac{1}{2}\|\tilde{b}\|_{2p}^2 + \|\tilde{c}\|_p \leq \beta$, the following implication holds for each $u \in H_2(\Omega)$:*

$$B[u] \geq 0, \quad u \ngeq 0 \Longrightarrow \operatorname*{ess\,inf}_{\{x \in \Omega\, u(x) < 0\}} \tilde{L}[u] \leq C\|u^-\|_1,$$

where $\tilde{L}[u] := -\Delta u + \tilde{b} \cdot \nabla u + \tilde{c}u$, $u^- := \max\{0, -u\}$ (i. e., $u^- := -u$ on the measurable set $\{x \in \bar{\Omega} : u(x) < 0\}$, and $u^- := 0$ on the complement), and $u \ngeq 0$ means that the statement "$u \geq 0$ a. e. on Ω" does not hold.

To prove Lemma 3.6 we first need

Lemma 3.7. *Let $q \in (1,\infty)$ if $n \in \{1,2\}$, and $q \in \left(1, \frac{2n}{n-2}\right)$ if $n \geq 3$. Then, for each $\delta > 0$, there exists a constant $C_1 = C_1(\delta) > 0$ such that, for each $u \in H_1(\Omega)$,*

$$\|u\|_q \leq \delta \|\nabla u\|_2 + C_1 \|u\|_1.$$

Proof. Let $r := \max\{q, 2\}$, and choose $s \in (r, \infty)$ if $n \in \{1,2\}$, $s \in \left(r, \frac{2n}{n-2}\right)$ if $n \geq 3$. Due to Sobolev's Imbedding Theorem, there exists a constant $C_2 > 0$ such that, for each $u \in H_1(\Omega)$,

$$\begin{aligned} \|u\|_s &\leq C_2 \left[\|\nabla u\|_2^2 + \|u\|_2^2\right]^{\frac{1}{2}} \leq C_2 \left[\|\nabla u\|_2 + \|u\|_2\right] \\ &\leq C_2 \|\nabla u\|_2 + C_2 \operatorname{meas}(\Omega)^{\frac{1}{2}-\frac{1}{r}} \|u\|_r. \end{aligned} \tag{3.15}$$

Furthermore, Hölder's inequality provides, with $\lambda := (r^{-1} - s^{-1})/(1 - s^{-1})$,

$$\|u\|_r \leq \|u\|_s^{1-\lambda} \|u\|_1^{\lambda} \quad \big(u \in L_s(\Omega)\big).$$

Using here, for some arbitrary $\varepsilon > 0$, the inequality

$$ab \leq \varepsilon \frac{a^\mu}{\mu} + \varepsilon^{-\frac{\nu}{\mu}} \frac{b^\nu}{\nu} \quad \big(a, b \geq 0;\ \mu, \nu \in (1,\infty),\ \mu^{-1} + \nu^{-1} = 1\big),$$

we obtain

$$\|u\|_r \leq \varepsilon(1-\lambda)\|u\|_s + \varepsilon^{-\frac{1}{\lambda}+1} \cdot \lambda \|u\|_1 \quad \big(u \in L_s(\Omega)\big).$$

Using here (3.15) on the right-hand side we obtain, for $u \in H_1(\Omega)$,

$$\left[1 - \varepsilon(1-\lambda) C_2 \operatorname{meas}(\Omega)^{\frac{1}{2}-\frac{1}{r}}\right] \cdot \|u\|_r \leq \varepsilon(1-\lambda) C_2 \|\nabla u\|_2 + \varepsilon^{-\frac{1}{\lambda}+1} \cdot \lambda \|u\|_1.$$

Since $\|u\|_q \leq \operatorname{meas}(\Omega)^{\frac{1}{q}-\frac{1}{r}} \|u\|_r$ the assertion follows if ε is chosen sufficiently small, depending on the given $\delta > 0$. ■

Proof of Lemma 3.6. $q := 2p/(p-1)$ satisfies the assumptions of Lemma 3.7, so that this Lemma provides some constant $C_1 > 0$ such that

$$\|u\|_q \leq \frac{1}{2\sqrt{\beta}} \|\nabla u\|_2 + C_1 \|u\|_1 \quad \text{for} \quad u \in H_1(\Omega). \tag{3.16}$$

Choose $C := 2\beta C_1^2$, and let $\tilde{b}, \tilde{c}$ satisfying $\frac{1}{2}\|\tilde{b}\|_{2p}^2 + \|\tilde{c}\|_p \leq \beta$ and $u \in H_2(\Omega)$ satisfying $B[u] \geq 0$, $u \not\geq 0$ be given. We define $\eta := u^-$ and obtain from [13, Lemma 7.6] that $\eta \in H_1(\Omega)$ and

$$(\nabla \eta)(x) = -(\nabla u)(x) \quad \text{if } u(x) < 0, \qquad (\nabla \eta)(x) = 0 \quad \text{if } u(x) \geq 0.$$

Therefore, by partial integration,

$$\int_\Omega \tilde{L}[u]\cdot\eta\,dx = -\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\cdot\eta\,d\sigma - \int_\Omega |\nabla\eta|^2\,dx - \int_\Omega [(\tilde{b}\cdot\nabla\eta)\eta + \tilde{c}\eta^2]\,dx.$$

Due to the assumption $B[u]\geq 0$, the boundary integral is nonnegative since $\eta = u^-$ vanishes on Γ_0 and both η and $\partial u/\partial\nu$ are nonnegative on Γ_1. Furthermore, $|(\tilde{b}\cdot\nabla\eta)\eta| \leq |\tilde{b}||\nabla\eta||\eta| \leq \frac{1}{2}|\nabla\eta|^2 + \frac{1}{2}|\tilde{b}|^2\eta^2$. Consequently,

$$\int_\Omega \tilde{L}[u]\cdot\eta\,dx \leq -\frac{1}{2}\int_\Omega |\nabla\eta|^2\,dx + \int_\Omega \left[\frac{1}{2}|\tilde{b}|^2 + |\tilde{c}|\right]\eta^2\,dx. \tag{3.17}$$

Hölder's inequality and (3.16) provide

$$\begin{aligned}\int_\Omega \left[\frac{1}{2}|\tilde{b}|^2 + |\tilde{c}|\right]\eta^2\,dx &\leq \left\{\frac{1}{2}\left[\int_\Omega |\tilde{b}|^{2p}\,dx\right]^{\frac{1}{p}} + \left[\int_\Omega |\tilde{c}|^p\,dx\right]^{\frac{1}{p}}\right\}\cdot\left[\int_\Omega |\eta|^q\,dx\right]^{\frac{2}{q}} \\ &\leq \beta\|\eta\|_q^2 \leq 2\beta\left[\frac{1}{4\beta}\|\nabla\eta\|_2^2 + C_1^2\|\eta\|_1^2\right] = \frac{1}{2}\|\nabla\eta\|_2^2 + C\|\eta\|_1^2,\end{aligned}$$

so that we obtain with (3.17):

$$\int_\Omega \tilde{L}[u]\cdot\eta\,dx \leq C\|\eta\|_1^2.$$

On the other hand,

$$\int_\Omega \tilde{L}[u]\cdot\eta\,dx = \int_{\{x\in\Omega\ u(x)<0\}} \tilde{L}[u]\cdot\eta\,dx \geq \operatorname*{ess\,inf}_{\{x\in\Omega\ u(x)<0\}} \tilde{L}[u]\cdot\|\eta\|_1$$

which provides the assertion since $u \not\geq 0$ and thus, $\|\eta\|_1 \neq 0$. ■

Corollary 3.1. *The assertion of Lemma 3.4 holds true.*

Proof. Let $u\in H_2(\Omega)$ denote a solution of problem (3.10). To prove that $u\leq w_0$ we define, for $\lambda>0$,

$$u_\lambda := w_0 + \lambda - u,$$

and apply Lemma 3.6 to u_λ, with $\tilde{b} := b$ and $\tilde{c}$ from (3.10). For all $x\in\Omega$ such that $u_\lambda(x)<0$, we have $u(x) > w_0(x)+\lambda$ and thus, $u^\#(x) = w_0(x)$. Consequently, (3.10) and (3.14) provide, for these x,

$$\tilde{L}[u_\lambda](x) = \tilde{L}[w_0](x) + \lambda\tilde{c} - \tilde{L}[u](x) \geq \lambda\tilde{c}.$$

Since $B[u_\lambda] \geq B[w_0] - B[u] \geq 0$ due to (3.10) and (3.14), Lemma 3.6 yields, for each $\lambda > 0$ such that $u_\lambda \not\geq 0$,

$$\lambda \tilde{c} \leq C \|u_\lambda^-\|_1. \tag{3.18}$$

Since $0 \leq u_\lambda^- \leq (w_0 - u)^-$ and $\tilde{c} > 0$, (3.18) cannot hold for all $\lambda > 0$. Consequently, $u_\lambda \geq 0$ for some $\lambda > 0$. Let λ^* denote the infimum of these λ. Using elementary measure theory we obtain $u_{\lambda^*} \geq 0$.

Assume that $\lambda^* > 0$. Since $u_\lambda \not\geq 0$ for $\lambda \in (0, \lambda^*)$ and, for these λ, $u_\lambda^- = \max\{0, -u_\lambda\} = \max\{0, -u_{\lambda^*} + (\lambda^* - \lambda)\} \leq \lambda^* - \lambda$, we obtain from (3.18) that

$$\lambda \tilde{c} \leq C \cdot \mathrm{meas}(\Omega) \cdot (\lambda^* - \lambda) \quad \text{for } \lambda \in (0, \lambda^*)$$

which provides a contradiction for λ sufficiently close to λ^*. Consequently, $\lambda^* = 0$ and thus, $w_0 - u = u_0 \geq 0$. The inequality $v_0 \leq u$ follows analogously. ∎

The following theorem characterizes inverse-positive operators (L, B) on $H_2(\Omega)$ in the same way which is proposed by Schröder for "classical" function spaces.

Theorem 3.2. *(L, B) is inverse-positive on $H_2(\Omega)$ if and only if there exists some $z \in H_2(\Omega)$ satisfying $z \geq 0$, $B[z] \geq 0$, $\operatorname*{ess\,inf}_\Omega L[z] > 0$.*

Proof. If (L, B) is inverse-positive on $H_2(\Omega)$, L is invertible on $H_2^B(\Omega)$. Due to Theorem A.2 of the appendix, the problem $z \in H_2^B(\Omega)$, $L[z] \equiv 1$ on Ω, has therefore a solution. The inverse-positivity of (L, B) yields $z \geq 0$, so that z has all required properties.

Now let z with these properties be given, as well as some $u \in H_2(\Omega)$ satisfying $L[u] \geq 0$, $B[u] \geq 0$. Let $\varepsilon := \operatorname*{ess\,inf}_\Omega L[z]$.

Since, for all $\lambda \geq 0$, $B[u + \lambda z] \geq 0$ and $L[u + \lambda z] \geq \lambda\varepsilon$, Lemma 3.6 provides, with $\tilde{L} := L$ and with $u_\lambda := u + \lambda z$ in place of u,

$$\lambda \varepsilon \leq C \cdot \|u_\lambda^-\|_1 \tag{3.19}$$

for all $\lambda \geq 0$ such that $u_\lambda \not\geq 0$.

The assumption $z \geq 0$ provides $(u + \lambda z)^- \leq u^-$ for $\lambda \geq 0$, so that (3.19) cannot hold for all $\lambda \geq 0$. The assertion $u = u_0 \geq 0$ now follows analogously to the proof of Corollary 3.1. ∎

Corollary 3.2. *Let $\operatorname*{ess\,inf}_\Omega c > 0$. Then, (L, B) is inverse-positive on $H_2(\Omega)$.*

Proof. Apply Theorem 3.2 with $z \equiv 1$. ∎

The following theorem is concerned with connected sets of inverse-positive operators which were also considered by Schröder (see, e. g.,[53]). We remark that this theorem also provides the proof of Theorem 2.2 b) which was postponed to this section.

Theorem 3.3. *For λ within some real interval I, let $b_\lambda \in L_\infty(\Omega)^n$, $c_\lambda \in L_p(\Omega)$ for some $p > n$, $p \geq 2$. Suppose that b_λ and c_λ depend continuously on λ, with respect to the norms $\|\cdot\|_\infty$ and $\|\cdot\|_p$, respectively. For $\lambda \in I$, define*

$$L_{(\lambda)}[u] := -\Delta u + b_\lambda \cdot \nabla u + c_\lambda u.$$

Suppose that $L_{(\lambda)}$ is invertible on $H_2^B(\Omega)$ for all $\lambda \in I$, and that $(L_{(\lambda)}, B)$ is inverse-positive on $H_2(\Omega)$ for at least one $\lambda \in I$. Then $(L_{(\lambda)}, B)$ is inverse-positive on $H_2(\Omega)$ for all $\lambda \in I$.

Proof. Due to the invertibility of the operators $L_{(\lambda)}$ on $H_2^B(\Omega)$ and to Theorem A.2 of the appendix, the problems

$$z_\lambda \in H_2^B(\Omega), \quad L_{(\lambda)}[z_\lambda] \equiv 1 \quad \text{on } \Omega \tag{3.20}$$

have solutions for all $\lambda \in I$, and the Open Mapping Theorem provides the boundedness of the operators $L_{(\lambda)}^{-1}$. According to Theorem 3.2, it suffices to show that $z_\lambda \geq 0$ for all $\lambda \in I$ in order to prove the assertion. Define

$$\Lambda := \{\lambda \in I : z_\lambda \geq 0\}.$$

Since, due to our assumption, $(L_{(\lambda)}, B)$ is inverse-positive on $H_2(\Omega)$ for at least one $\lambda \in I$, Λ is nonempty. Consequently, it suffices to prove that $\Lambda \subset I$ is *closed and open* (with respect to the relative topology) in order to obtain the assertion $\Lambda = I$.

First we show that z_λ depends continuously on $\lambda \in I$, with respect to the norm $\|\cdot\|_{H_2}$ in $H_2(\Omega)$. (In fact, the norm $\|\cdot\|_1$ would be sufficient). Thus, let $\lambda_0 \in I$ be fixed, and let $\varepsilon_\lambda := \|b_{\lambda_0} - b_\lambda\|_\infty + \|c_{\lambda_0} - c_\lambda\|_p$ for $\lambda \in I$. (3.20) yields

$$z_\lambda - z_{\lambda_0} = L_{(\lambda_0)}^{-1}\left[(b_{\lambda_0} - b_\lambda) \cdot \nabla z_\lambda + (c_{\lambda_0} - c_\lambda) z_\lambda\right]$$

for $\lambda \in I$, so that we obtain from the boundedness of $L_{(\lambda_0)}^{-1} : L_2(\Omega) \to H_2(\Omega)$ and from Sobolev's Imbedding Theorem with $q := 2p/(p-2)$ ($:= \infty$ if $p = 2$),

$$\begin{aligned}
\|z_\lambda - z_{\lambda_0}\|_{H_2} &\leq C \left\|(b_{\lambda_0} - b_\lambda) \cdot \nabla z_\lambda + (c_{\lambda_0} - c_\lambda) z_\lambda\right\|_2 \\
&\leq C \left[\|b_{\lambda_0} - b_\lambda\|_\infty \|\nabla z_\lambda\|_2 + \|c_{\lambda_0} - c_\lambda\|_p \|z_\lambda\|_q\right] \\
&\leq \tilde{C}\varepsilon_\lambda \|z_\lambda\|_{H_1} \leq \tilde{C}\varepsilon_\lambda \|z_\lambda\|_{H_2} \leq \tilde{C}\varepsilon_\lambda \left[\|z_{\lambda_0}\|_{H_2} + \|z_\lambda - z_{\lambda_0}\|_{H_2}\right]
\end{aligned}$$

which provides the desired continuous dependence since $\varepsilon_\lambda \to 0$ for $\lambda \to \lambda_0$.

To prove that Λ is *closed* in I let (λ_j) denote some sequence in Λ converging to some $\lambda \in I$. Since $z_{\lambda_j} \to z_\lambda$ in $H_2(\Omega)$, there exists a subsequence $(z_{\lambda_{j_k}})$ converging to z_λ

almost everywhere in Ω. Therefore, since $z_{\lambda_j}(x) \geq 0$ for all $j \in I\!N$ and almost all $x \in \Omega$, it follows that $z_\lambda(x) \geq 0$ for almost all $x \in \Omega$, i. e., $\lambda \in \Lambda$.

To show that Λ is *open* in I let $\lambda_0 \in \Lambda$ be given. Since $\|b_\lambda\|_{2p}$ and $\|c_\lambda\|_p$ depend continuously on λ, we can choose some neighborhood U of λ_0 in I and some $\beta > 0$ such that $\frac{1}{2}\|b_\lambda\|_{2p}^2 + \|c_\lambda\|_p \leq \beta$ for $\lambda \in U$. Let $C = C(\beta)$ denote the constant provided by Lemma 3.6, and choose $\delta > 0$ such that

$$\|z_\lambda - z_{\lambda_0}\|_1 < \frac{1}{C} \quad \text{for } \lambda \in U, \ |\lambda - \lambda_0| < \delta \tag{3.21}$$

which is possible since $\|u\|_1 \leq \sqrt{\text{meas}(\Omega)} \cdot \|u\|_{H_2}$ for $u \in H_2(\Omega)$. Now assume for contradiction that $z_\lambda \not\geq 0$ for some $\lambda \in U$ satisfying $|\lambda - \lambda_0| < \delta$. Then, (3.20) and Lemma 3.6 imply $1 \leq C \cdot \|z_\lambda^-\|_1$, so that (3.21) provides

$$\|z_\lambda - z_{\lambda_0}\|_1 < \|z_\lambda^-\|_1. \tag{3.22}$$

On the other hand, $z_{\lambda_0} \geq 0$ since $\lambda_0 \in \Lambda$, and therefore,

$$0 \leq z_\lambda^- = \max\{0, -z_\lambda\} \leq \max\{0, z_{\lambda_0} - z_\lambda\} \leq |z_{\lambda_0} - z_\lambda|$$

which contradicts (3.22). ∎

Corollary 3.3. *Let $c \geq 0$, and suppose that L is invertible on $H_2^B(\Omega)$. Then, (L, B) is inverse-positive on $H_2(\Omega)$.*

Proof. Let $b_\lambda := b$, $c_\lambda := c + \lambda$ for $\lambda \in I := [0, 1]$, and define $L_{(\lambda)}$ as in Theorem 3.3. Due to Corollary 3.2, $(L_{(\lambda)}, B)$ is inverse-positive on $H_2(\Omega)$ for any $\lambda \in (0, 1]$. In particular, $L_{(\lambda)}$ is invertible on $H_2^B(\Omega)$ for all $\lambda \in (0, 1]$. Due to our assumption, this is also true for $\lambda = 0$. Thus, Theorem 3.3 provides the assertion. ∎

Theorem 3.4. *If (L, B) is inverse-positive on $H_2(\Omega)$, then all real eigenvalues of L on $H_2^B(\Omega)$ are positive. If $\operatorname*{ess\,inf}_\Omega c > -\infty$, the converse is also true.*

Proof. If (L, B) is inverse-positive on $H_2(\Omega)$, Theorem 3.2 provides some $z \in H_2(\Omega)$ satisfying $z \geq 0$, $B[z] \geq 0$, $\operatorname*{ess\,inf}_\Omega L[z] > 0$. Consequently, $\operatorname*{ess\,inf}_\Omega (L - \lambda I)[z] > 0$ for all $\lambda \leq 0$, so that Theorem 3.2 yields the inverse-positivity of $(L - \lambda I, B)$ on $H_2(\Omega)$ for $\lambda \leq 0$. In particular, $L - \lambda I$ is invertible on $H_2^B(\Omega)$ for $\lambda \leq 0$, so that no non-positive eigenvalue of L on $H_2^B(\Omega)$ exists.

Now suppose that $\operatorname*{ess\,inf}_\Omega c > -\infty$, and that all real eigenvalues of L on $H_2^B(\Omega)$ are positive. Let $b_\lambda := b$, $c_\lambda := c - \lambda$ for $\lambda \in I := (-\infty, 0]$, and define $L_{(\lambda)}$ as in Theorem 3.3, i. e., $L_{(\lambda)} = L - \lambda I$. Due to our assumption, $L_{(\lambda)}$ is invertible on $H_2^B(\Omega)$ for all $\lambda \in I$. Moreover, $(L_{(\lambda)}, B)$ is inverse-positive for $\lambda < \operatorname*{ess\,inf}_\Omega c$ due to Corollary 3.2.

Theorem 3.3 therefore provides the inverse-positivity of $(L_{(\lambda)}, B)$ on $H_2(\Omega)$ for all $\lambda \in I$. In particular, $(L_{(0)}, B) = (L, B)$ is inverse-positive on $H_2(\Omega)$. ∎

Theorem 3.5. *Let some $z \in H_2(\Omega)$ be given such that $\operatorname*{ess\,inf}_\Omega z > 0$, and $B[z] \geq 0$. Then, all real eigenvalues λ of L on $H_2^B(\Omega)$ satisfy*

$$\lambda \geq \operatorname*{ess\,inf}_\Omega \left\{ \frac{L[z]}{z} \right\}.$$

Proof. Let $\nu := \operatorname*{ess\,inf}_\Omega \{L[z]/z\}$, so that $L[z] - \nu z \geq 0$. For $\mu < \nu$, we have $L[z] - \mu z = L[z] - \nu z + (\nu - \mu)z \geq (\nu - \mu)z$ and thus, $\operatorname*{ess\,inf}_\Omega(L[z] - \mu z) > 0$. According to Theorem 3.2, $(L - \mu I, B)$ is therefore inverse-positive on $H_2(\Omega)$. Thus, Theorem 3.4 yields the positivity of all real eigenvalues of $L - \mu I$ on $H_2^B(\Omega)$. Consequently, $\lambda > \mu$ for all real eigenvalues λ of L on $H_2^B(\Omega)$. Since this is true for all $\mu < \nu$, the assertion $\lambda \geq \nu$ follows. ∎

Corollary 3.4. *All real eigenvalues λ of L on $H_2^B(\Omega)$ satisfy*

$$\lambda \geq \operatorname*{ess\,inf}_\Omega c$$

Proof. Apply Theorem 3.5 with $z \equiv 1$. ∎

4. "NON-MONOTONE" PROBLEMS

The "monotonicity" methods discussed in the previous section have proved, in a vast variety of examples, to be a very powerful tool for computing enclosures for various types of problems. The reason for their efficiency can be seen in the fact that, in monotone situations, the set inclusion (2.2) is equivalent to two single inequalities (2.11), which in turn are implied by differential inequalities (see (2.12), (3.4), (3.12), (3.14)).

However, as explained and illustrated at the end of Section 2, the applicability of monotonicity methods to problem (1.1) is severely restricted by the "almost" necessary condition (2.28) that the operator $L : H_2^B(\Omega) \to L_2(\Omega)$ given by

$$\begin{aligned} L[u] &:= -\Delta u + b \cdot \nabla u + cu \ , \\ b &:= F_z(\cdot, \omega, \nabla\omega) \ , \ \ c := F_y(\cdot, \omega, \nabla\omega) \ , \end{aligned} \tag{4.1}$$

with $\omega \in H_2^B(\Omega)$ denoting some approximate solution, shall be *inverse-positive* on $H_2^B(\Omega)$, which requires, e.g., all its real eigenvalues to be positive.

For ODE boundary value problems, this restrictive requirement can be avoided under preservation of the monotonicity ideas (see [14, 54], and the short description in Subsection 3.1). However, there seems to be no realistic possibility of carrying this approach over to elliptic boundary value problems, so that new ideas are required here.

4.1. The basic existence and inclusion theorem

We propose to use the fixed-point operator

$$Tu := L^{-1}[b \cdot \nabla u + cu - F(\cdot, u, \nabla u)], \tag{4.2}$$

with L from (4.1),which we assume to be *invertible* on $H_2^B(\Omega)$; the invertibility will be checked numerically (see Subsection 4.3.2). As discussed in Subsection 2.2, the "almost" necessary conditions derived there have no restrictive consequences for the choice of T in (4.2). T coincides with the simplified Newton operator (2.13), so that the *basis* of our approach is closely related to the Newton-Cantorovich Theorem. In order to be able to use the inverse operator $L^{-1} : L_2(\Omega) \to H_2^B(\Omega)$ in (4.2), we have to make sure, according to Theorem A.2 of the appendix, that $b \in L_\infty(\Omega)^n$ and $c \in L_p(\Omega)$ for some $p > n$, $p \geq 2$. For this purpose, it suffices to require that

$$\nabla \omega \in L_\infty(\Omega)^n \tag{4.3}$$

which implies, after (repeated) application of Sobolev's Imbedding Theorem, that $\omega \in C(\bar{\Omega})$, so that the continuity of F_y and F_z provides the *boundedness* of b and c. Condition (4.3) on the approximate solution ω is not very restrictive; it is satisfied for almost all "usual" approximations (obtained, e.g., from conformal finite element methods).

We now suppose that, for some Banach space $X \supset H_2^B(\Omega)$,

(X1) $\Phi : \left\{ \begin{array}{lll} X & \to & L_2(\Omega) \\ u & \mapsto & b \cdot \nabla u + cu - F(\cdot, u, \nabla u) \end{array} \right\}$ is continuous, bounded on bounded sets, and Fréchet-differentiable at ω with $\Phi'(\omega) \equiv 0$

(X2) the imbedding $H_2^B(\Omega) \hookrightarrow X$ is compact.

Together with the fact that $L^{-1} : L_2(\Omega) \to H_2^B(\Omega)$ is bounded (which follows from Theorem A.2 of the appendix and the Open Mapping Theorem), we obtain that $T : X \to X$ given by (4.2) is continuous, compact, and Fréchet-differentiable at ω with $T'(\omega) \equiv 0$. If $TU \subset U$ for some closed, bounded, convex set $U \subset X$, Schauder's Fixed-Point Theorem provides the existence of some fixed-point $u \in U$ of T, which is then a solution of problem (1.1) due to (4.1) and (4.2).

In Subsection 2.2, it was shown that a sufficient condition for the inclusion $TU \subset U$ is given by (2.20), with $V = U - \omega$. Since $T'(\omega) \equiv 0$, this condition now reads

$$L^{-1}\left[-d[\omega] + \varphi(V)\right] \subset V \tag{4.4}$$

where

$$d[\omega] := -\Delta\omega + F(\cdot, \omega, \nabla\omega), \tag{4.5}$$

$$\varphi(v) := -\left[F(\cdot, \omega + v, \nabla\omega + \nabla v) - F(\cdot, \omega, \nabla\omega) - b \cdot \nabla v - cv\right] \tag{4.6}$$

The major problem now is to construct some (closed, bounded, convex) set V satisfying (4.4) explicitly. No sufficient differential inequalities which provide a "small" enclosure set V exist unless L is inverse-positive on $H_2^B(\Omega)$ (see Section 2), and an explicit computation or a detailed enclosure of L^{-1} is impossible unless Ω, b, and c are very "simple".

We propose to bound the terms $L^{-1}, d[\omega], \varphi(V)$ (entering (4.4)) *normwise*. Suppose that constants $K > 0, \delta \geq 0$, and a monotonically nondecreasing function $G : [0,\infty) \to [0,\infty)$ can be computed such that

$$\|L^{-1}[r]\|_X \leq K\|r\|_2 \quad \text{for all } r \in L_2(\Omega), \tag{4.7}$$

$$\|d[\omega]\|_2 \leq \delta, \tag{4.8}$$

$$\|\varphi(v)\|_2 \leq G(\|v\|_X) \quad \text{for all } v \in X, \tag{4.9}$$

where $\|u\|_2 := \left[\frac{1}{\text{meas}(\Omega)} \int\limits_\Omega u^2\, dx\right]^{\frac{1}{2}}$ here and for the rest of this section. Since $\|\varphi(v)\|_2 = o(\|v\|_X)$ for $\|v\|_X \to 0$ due to the assumption $\Phi'(\omega) \equiv 0$ in (X1), we may assume that G has been constructed such that

$$G(t) = o(t) \quad \text{for } t \to 0. \tag{4.10}$$

We wish to remark that the concept of such a *majorizing function* G has first been introduced by Schröder (see, e.g. [49]).

We specify V in (4.4) to be some closed ball

$$V := \{v \in X : \|v\|_X \leq \alpha\},$$

with some $\alpha > 0$. Then, (4.7), (4.8), (4.9) provide, for each $v \in V$,

$$\begin{aligned}
\|L^{-1}[-d[\omega] + \varphi(v)]\|_X &\leq K\| - d[\omega] + \varphi(v)\|_2 \\
&\leq K[\delta + \|\varphi(v)\|_2] \\
&\leq K[\delta + G(\|v\|_X)] \\
&\leq K[\delta + G(\alpha)],
\end{aligned}$$

so that (4.4) is satisfied if $K[\delta + G(\alpha)] \leq \alpha$. We summarize the result in the following

Theorem 4.1 *Suppose that, for some $\alpha \geq 0$,*

$$\delta \leq \frac{\alpha}{K} - G(\alpha). \tag{4.11}$$

Then, there exists a solution $u \in H_2^B(\Omega)$ of problem (1.1) satisfying

$$\|u - \omega\|_X \leq \alpha. \tag{4.12}$$

Due to (4.10), the expression on the right-hand side of (4.11) is positive for small α. Therefore, (4.11) is satisfied for some small α (providing a tight enclosure (4.12)) if δ is sufficiently small, which means, according to (4.5) and (4.8), that the approximate solution ω has been computed with *sufficient accuracy.*

Thus, Theorem 4.1 yields a satisfactory result. There is no inverse-positivity or similar assumption on the linear operator L. We only need its *invertibility.* Expressed in eigenvalues, this assumption requires all eigenvalues of L on $H_2^B(\Omega)$ to be *non-zero,* while its inverse-positivity on $H_2^B(\Omega)$ requires all real eigenvalues to be *positive* (see Theorem 3.4).

Of course, the application of Theorem 4.1 requires explicit knowledge of terms K, δ, G satisfying (4.7), (4.8), (4.9). For this purpose, additional theoretical considerations and numerical computations are necessary, which we will describe later in this section and in Section 6.

First we will be concerned with suitable choices of the Banach space X satisfying assumptions (X1) and (X2), assuming from now on that

$$n \leq 3. \tag{4.13}$$

On one hand, X should be "broad" and its norm should be "weak", so that (X2) and (4.7) are easier to fulfill. On the other hand, a "strong" norm in X facilitates conditions (X1) and (4.9), and it provides a more accurate enclosure (4.12). Therefore, we choose the Banach space X with the strongest norm which still satisfies (X2) and allows explicit computation of a constant K satisfying (4.7). In this way, we obtain

$$X := C_1(\bar{\Omega}), \quad \|u\|_X := \max\{\|u\|_\infty,\ \gamma\|u'\|_\infty\} \quad \text{if } n = 1, \tag{4.14}$$

$$X := H_{1,4}(\Omega), \quad \|u\|_X := \max\{\|u\|_\infty,\ \gamma\|\nabla u\|_4\} \quad \text{if } n \in \{2, 3\}, \tag{4.15}$$

where $\|v\|_4 := \left[\dfrac{1}{\text{meas}(\Omega)} \displaystyle\int_\Omega |v|^4\, dx\right]^{\frac{1}{4}}$, and the constant $\gamma > 0$ will be specified later in (4.22). Observe that Sobolev's Imbedding Theorem provides, due to (4.13), the bounded imbedding $H_{1,4}(\Omega) \hookrightarrow C(\bar{\Omega})$, so that the norm given in (4.15) is in fact equivalent to the usual $H_{1,4}$-norm (involving $\|u\|_4$ instead of $\|u\|_\infty$). Moreover, the Imbedding Theorem verifies assumption (X2) for (4.14) and (4.15).

It should be noted that in the case where the nonlinearity F has the special form (3.1), so that ∇u does not actually occur in the operator Φ defined in (X1), one may also choose

$$X := C(\bar{\Omega})\ ,\ \|u\|_X := \|u\|_\infty \tag{4.16}$$

which facilitates some of the considerations below, but provides a weaker enclosure (4.12).

4.2. Verification of (X1), and computation of G

In order to verify assumption (X1), and to compute some function G satisfying (4.9) and (4.10), we first calculate some function $\tilde{G} : [0,\infty) \times [0,\infty) \to [0,\infty)$ with the following properties:

$(G1)$ $|F(x,\omega(x)+y,\nabla\omega(x)+z) - F(x,\omega(x),\nabla\omega(x)) - b(x)\cdot z - c(x)y| \leq \tilde{G}(|y|,|z|)$ for $x \in \Omega,\ y \in I\!R,\ z \in I\!R^n$,

$(G2)$ $\tilde{G}(\mu t, t) = o(t)$ for $t \to 0$ and each fixed $\mu > 0$,

$(G3)$ $\tilde{G}$ is monotonically nondecreasing in both variables,

$(G4)$ for each fixed $\alpha \geq 0$, $\left[\tilde{G}(\alpha, t^{\frac{1}{4}})\right]^2$ is a continuous and *concave* function of t.

Here, (G4) may be omitted in the ODE case $n = 1$. The remaining assumptions (G1) to (G3) can easily be satisfied with the aid of Taylor's Theorem due to the continuity of F, F_y, and F_z, and to (4.3). The same is true in the elliptic case $n \in \{2,3\}$. Here, however, one has to take the growth condition (1.2) into account in order to satisfy the additional assumption (G4). The easiest way to do so is to look for $\tilde{G}$ in the form

$$\tilde{G}(s,t) = \sum_{k=1}^{M} g_k(s)t^{\mu_k} ,$$

with exponents $\mu_k \in [0,2]$ and monotonically nondecreasing functions g_k satisfying $g_k(s) = o(s^{1-\mu_k})$ for $s \to 0$ (so that $\tilde{G}$ has the properties (G2) to (G4)), and to arrange the integer M, the exponents μ_k, and the functions g_k appropriately in order to satisfy property (G1), too. Obviously, the exponent restriction $\mu_k \in [0,2]$, which is necessary for condition (G4), requires the growth condition (1.2), via assumption (G1).

With such a function $\tilde{G}$, with φ from (4.6), and with the space X defined in (4.14) or (4.15), respectively, we have the following

Lemma 4.1 *For each $u \in X$,*

a) $\|\varphi(u)\|_2 \ \leq\ \tilde{G}(\|u\|_\infty, \|u'\|_\infty)$ *if* $n = 1$,

b) $\|\varphi(u)\|_2 \ \leq\ \tilde{G}(\|u\|_\infty, \|\nabla u\|_4)$ *if* $n \in \{2,3\}$.

Proof. **ad a):** Property (G1) provides $|\varphi(u)| \leq \tilde{G}(|u|,|u'|)$ pointwise, so that $\|\varphi(u)\|_\infty \leq \tilde{G}(\|u\|_\infty, \|u'\|_\infty)$ due to (G3). Moreover, $\|\varphi(u)\|_2 \leq \|\varphi(u)\|_\infty$ (regard the weight factor $\text{meas}(\Omega)^{-\frac{1}{2}}$ in $\|\cdot\|_2$).

ad b): Define $\Psi(t) := \left[\tilde{G}(\|u\|_\infty, t^{\frac{1}{4}})\right]^2$ for $t \geq 0$, and $v := |\nabla u|^4 \in L_1(\Omega)$. Properties (G1) and (G3) provide, for almost all $x \in \Omega$,

$$|\varphi(u)(x)|^2 \leq \left[\tilde{G}(|u(x)|, |\nabla u(x)|)\right]^2 \leq \left[\tilde{G}(\|u\|_\infty, v(x)^{\frac{1}{4}})\right]^2 = \Psi(v(x)),$$

so that it suffices to prove that $\Psi \circ v \in L_1(\Omega)$ and

$$\frac{1}{\text{meas}(\Omega)} \int_\Omega \Psi(v(x))dx \leq \Psi\left(\frac{1}{\text{meas}(\Omega)} \int_\Omega v(x)dx\right), \tag{4.17}$$

since the right-hand side equals $\left[\tilde{G}(\|u\|_\infty, \|\nabla u\|_4)\right]^2$. (4.17) is an immediate consequence of Jensen's inequality [5, Thm. 56.6, p. 310] since, due to (G4), Ψ is concave, continuous, and nonnegative. ∎

With $\|\cdot\|_X$ defined in (4.14), (4.15), we obtain from Lemma 4.1 and property (G3) that

$$\|\varphi(u)\|_2 \leq \tilde{G}(\|u\|_X, \frac{1}{\gamma}\|u\|_X) \text{ for } u \in X, \tag{4.18}$$

so that (4.9) is satisfied for

$$G(t) := \tilde{G}(t, \frac{1}{\gamma}t) \ \ (t \in [0, \infty)). \tag{4.19}$$

Moreover, this G is monotonically nondecreasing due to (G3), and (4.10) holds according to (G2).

Next we investigate assumption (X1). Since the operator $\Phi : X \to L_2(\Omega)$ defined there satisfies $\Phi(\omega + u) - \Phi(\omega) = \varphi(u)$ for all $u \in X$, we obtain from (4.18) that Φ is bounded on bounded sets, and that Φ is Fréchet-differentiable at ω with $\Phi'(\omega) \equiv 0$ (observe (G2)). The continuity of Φ follows, in the case $n = 1$, immediately from the uniform continuity of F on compact subsets of $\bar{\Omega} \times I\!R \times I\!R^n$, and from the boundedness of the coefficients b and c (regard the norm in (4.14)). In the elliptic case $n \in \{2, 3\}$, the proof of the continuity of Φ (resp., of φ) can be found in [37, Lemma 2]. It is based on inequalities similiar to that stated in Lemma 4.1 b), and on elementary measure theory.

4.3. Computation of K

In order to compute some constant K satisfying (4.7) we will derive the following inequalities:

$$\|u\|_\infty \leq K\,\|L[u]\|_2 \quad (\text{for all } u \in H_2^B(\Omega)) \tag{4.20}$$

$$\left.\begin{array}{ll} \|u'\|_\infty \leq K'\|L[u]\|_2 & (\text{for all } u \in H_2^B(\Omega)), \text{ if } n = 1 \\ \|\nabla u\|_4 \leq K'\|L[u]\|_2 & (\text{for all } u \in H_2^B(\Omega)), \text{ if } n \in \{2, 3\} \end{array}\right\} \tag{4.21}$$

with *known* constants K and K'. Then , (4.14) and (4.15) obviously provide $\|u\|_X \leq K\|L[u]\|_2$ (for all $u \in H_2^B(\Omega)$), if we choose

$$\gamma := \frac{K}{K'}, \tag{4.22}$$

so that (4.7) holds since $L : H_2^B(\Omega) \to L_2(\Omega)$ is one-to-one and onto.

We will provide the computation of such constants K and K' in the following steps: First we calculate constants C_0, C_1, C_2 such that

$$\|u\|_\infty \le C_0\|u\|_2 + C_1\|u'\|_2 \quad \text{(for all } u \in H_1(\Omega)\text{), if } n = 1, \tag{4.23}$$
$$\|u\|_\infty \le C_0\|u\|_2 + C_1\|\nabla u\|_2 + C_2\|u_{xx}\|_2 \quad \text{(for all } u \in H_2(\Omega)\text{), if } n \in \{2,3\}, \tag{4.24}$$

with $\|u_{xx}\|_2$ denoting the L_2-Frobenius-norm of the Hessian matrix u_{xx} (weighted by $\text{meas}(\Omega)^{-\frac{1}{2}}$). The determination of such constants obviously provides an *explicit* version of Sobolev's Imbedding Theorem for the special imbeddings $H_1(\Omega) \hookrightarrow C(\bar{\Omega})$ (if $n = 1$) and $H_2(\Omega) \hookrightarrow C(\bar{\Omega})$ (if $n \in \{2,3\}$).

In a second step, we compute constants K_0, K_1, K_2, and κ such that, for all $u \in H_2^B(\Omega)$,

$$\|u\|_2 \le K_0\|L[u]\|_2, \quad \|\nabla u\|_2 \le K_1\|L[u]\|_2, \tag{4.25 a, b}$$
$$\|\Delta u\|_2 \le \kappa\|L[u]\|_2, \quad \|u_{xx}\|_2 \le K_2\|L[u]\|_2, \tag{4.25 c, d}$$

where, of course, $\nabla u = u'$ and $\Delta u = u_{xx} = u''$ if $n = 1$. Once constants satisfying (4.23), (4.24), (4.25) are known, the following lemma shows how constants K and K' satisfying (4.20), (4.21) can be calculated very easily.

Lemma 4.2. *Let (4.23), (4.24), and (4.25) hold. Then, (4.20) and (4.21) are true for*

$$K := C_0K_0 + C_1K_1\,, \qquad K' := C_0K_1 + C_1K_2 \quad \textit{if } n = 1,$$
$$K := C_0K_0 + C_1K_1 + C_2K_2\,,\ K' := \sqrt{K(\kappa + 2K_2)} \quad \textit{if } n \in \{2,3\}.$$

Proof. (4.20) is an immediate consequence of (4.23), (4.25 a,b) (if $n = 1$), resp. of (4.24),(4.25 a,b,d) (if $n \in \{2,3\}$). In the case $n = 1$, (4.21) follows by inserting u' in place of u in (4.23), and using (4.25 b,d). To prove (4.21) in the case $n \in \{2,3\}$, let $u \in H_2^B(\Omega)$ satisfying $\nabla u \not\equiv 0$ be given. By partial integration we obtain, since $u \cdot (\partial u/\partial \nu) = 0$ on $\partial\Omega$,

$$\begin{aligned}
\int_\Omega |\nabla u|^4\, dx &= \int_\Omega \nabla u \cdot \left[|\nabla u|^2 \nabla u\right]\, dx \\
&= \int_{\partial\Omega} u|\nabla u|^2 \frac{\partial u}{\partial \nu} d\sigma - \int_\Omega u \,\mathrm{div}\left[(\nabla u \cdot \nabla u)\nabla u\right]\, dx \\
&= -\int_\Omega u\left[|\nabla u|^2 \Delta u + 2(\nabla u)^t u_{xx} \nabla u\right]\, dx \\
&\le \|u\|_\infty \int_\Omega |\nabla u|^2 \left[|\Delta u| + 2|u_{xx}|_{\text{Frobenius}}\right]\, dx \\
&\le \|u\|_\infty \sqrt{\int_\Omega |\nabla u|^4\, dx} \cdot \left[\sqrt{\int_\Omega (\Delta u)^2\, dx} + 2 \cdot \sqrt{\int_\Omega |u_{xx}|^2_{\text{Frobenius}}\, dx}\right].
\end{aligned}$$

Division by $\left[\operatorname{meas}(\Omega) \int\limits_{\Omega} |\nabla u|^4 dx\right]^{\frac{1}{2}}$ provides

$$\left[\frac{1}{\operatorname{meas}(\Omega)} \int\limits_{\Omega} |\nabla u|^4 dx\right]^{\frac{1}{2}} \leq \|u\|_\infty \left[\,\|\Delta u\|_2 + 2\|u_{xx}\|_2\,\right],$$

so that (4.20) and (4.25 c,d) yield the assertion. ■

In the following three sub-subsections, we briefly describe how constants satisfying (4.23), (4.24), (4.25) can be computed. We omit all proofs here, and refer to the papers [33, 34] (for the ODE case $n = 1$) and [35, 37] (for the elliptic case).

We wish to make the important remark that major *numerical* work has to be done only for the computation of K_0 satisfying (4.25 a). All other constants are obtained by theoretical means.

4.3.1 Calculation of C_0, C_1 (and C_2)

Lemma 4.3. ([33, Lemma 2]) *Let $n = 1$ and, without loss of generality, $\Omega = (0,1)$. Then, (4.23) holds for*

$$C_0 \geq 1 \text{ arbitrary, } C_1 = \frac{1}{\sqrt{3}C_0}.$$

The result in [33] is a little bit more general. It shows how the constants C_0 and C_1 can be improved in the case of *Dirichlet* boundary conditions at one or both endpoints, or in the case of *periodic* boundary conditions which are not considered in the present article due to their low importance for elliptic problems.

Lemma 4.4. ([35, Corollary 1]) *Let $n \in \{2,3\}$, and let $Q \subset I\!R^n$ denote some compact convex set with nonempty interior such that, for each $x_0 \in \bar{\Omega}$, there exists an affine-orthogonal linear transformation $\varphi : I\!R^n \to I\!R^n$ such that $x_0 \in \varphi(Q) \subset \bar{\Omega}$. Then, (4.24) holds true with*

$$C_\nu := \gamma_\nu \left[\frac{\operatorname{meas}(\Omega)}{\operatorname{meas}(Q)} \cdot \max_{x_0 \in Q} \left\{ \frac{1}{\operatorname{meas}(Q)} \int\limits_{Q} |x - x_0|^{2\nu} dx \right\}\right]^{\frac{1}{2}} \quad (\nu = 0, 1, 2),$$

where

$$\begin{array}{llll} \gamma_0 = 1 & \gamma_1 = 1.1548 & \gamma_2 = 0.22361 & \textit{if } n = 2, \\ \gamma_0 = 1.0708 & \gamma_1 = 1.6549 & \gamma_2 = 0.41413 & \textit{if } n = 3. \end{array}$$

Example: Let Ω be a rectangle with sidelengths $L_1, \ldots, L_n$, $n \in \{2,3\}$, and choose Q to be a rectangle with sidelength $l_i \in (0, L_i]$ $(i = 1, \ldots, n)$. Then, (4.24) holds true with

$$C_0 = \frac{\gamma_0}{\sqrt{l_1 \cdot l_2(\cdot l_3)}}, \quad C_1 = \frac{\gamma_1}{\sqrt{3}} \sqrt{\frac{l_1^2 + l_2^2(+l_3^2)}{l_1 \cdot l_2(\cdot l_3)}},$$

$$C_2 = \frac{\gamma_2}{3} \sqrt{\frac{[l_1^2 + l_2^2(+l_3^2)]^2 + \frac{4}{5}[l_1^4 + l_2^4(+l_3^4)]}{l_1 \cdot l_2(\cdot l_3)}}.$$

Here, the sidelengths $l_i \in (0, L_i]$ can be optimized in order to make the constant K obtained from Lemma 4.2 as small as possible.

4.3.2 Computation of K_0

In order to compute a constant K_0 satisfying (4.25 a) we consider the eigenvalue problem (in weak formulation)

$$u \in H_2^B(\Omega), \quad \langle L[u], L[\psi] \rangle = \lambda \langle u, \psi \rangle \quad \text{for all } \psi \in H_2^B(\Omega) \;, \tag{4.26}$$

with $\langle \cdot, \cdot \rangle$ denoting the canonical inner product in $L_2(\Omega)$. The variational characterization of the smallest eigenvalue λ_1 of problem (4.26) reads

$$\lambda_1 = \min_{u \in H_2^B(\Omega) \setminus \{0\}} \frac{\langle L[u], L[u] \rangle}{\langle u, u \rangle},$$

so that L is invertible on $H_2^B(\Omega)$ (as required) if and only if $\lambda_1 > 0$, and in that case (4.25a) obviously holds with arbitrary

$$K_0 \geq \frac{1}{\sqrt{\lambda_1}}. \tag{4.27}$$

Thus, we have to compute a positive lower bound for λ_1. Expressed slightly different, we need a positive lower bound for the *smallest singular value* $\sqrt{\lambda_1}$ of L on $H_2^B(\Omega)$.

Methods for computing the required lower bound for λ_1 (or better, for computing lower eigenvalue bounds in general) have been developed by Kato [19], Bazley and Fox [6], Lehmann [25], and Goerisch [15], where the latter method is certainly the most general and (together with Lehmann's method) the most accurate one.

For many eigenvalue problems one needs, in addition to such methods, a homotopy connecting the given eigenvalue problem to a "simple" one with known eigenvalues. Such homotopy algorithms have been developed independently in [16] and in [38].

For all details concerning eigenvalue bounds for selfadjoint eigenvalue problems (like (4.26)), we refer the reader to the article by Goerisch and Behnke [7] in the present volume.

In the particular case where the coefficient function b satisfies $b \equiv \nabla\varphi$ for some Lipschitz-continuous function φ, one may use the following alternative to (4.27). The operator L is now *symmetric* on $H_2^B(\Omega)$ with respect to the inner product $\langle\cdot,\cdot\rangle_\varphi$ weighted by $e^{-\varphi}$, so that an $\langle\cdot,\cdot\rangle_\varphi$ - orthonormal and complete system of eigenfunctions exists. Using eigenfunction series expansions one easily derives that, with $\|\cdot\|_\varphi := \sqrt{\langle\cdot,\cdot\rangle_\varphi}$,

$$\|L[u]\|_\varphi \geq \sigma\|u\|_\varphi \quad \text{for all } u \in H_2^B(\Omega),$$

where σ is some real number such that

$$\sigma \leq |\lambda| \quad \text{for each eigenvalue } \lambda \text{ of } L \text{ on } H_2^B(\Omega). \tag{4.28}$$

Consequently, with $\underline{\varphi}$ and $\overline{\varphi}$ denoting a lower and an upper bound for φ, (4.25 a) holds for

$$K_0 := \frac{1}{\sigma}\exp\left[\frac{1}{2}(\overline{\varphi}-\underline{\varphi})\right], \tag{4.29}$$

provided that a ***positive*** σ satisfying (4.28) can be computed. Since the eigenvalue problem

$$u \in H_2^B(\Omega)\ ,\ L[u] = \lambda u \tag{4.30}$$

is often easier to handle than problem (4.26) (in particular, with regard to the homotopy algorithms mentioned above), the computation of K_0 via (4.28), (4.29) has some advantages, compared with (4.27). However, (4.29) provides a less accurate K_0, due to the rough estimates $\underline{\varphi} \leq \varphi \leq \overline{\varphi}$. Of course, this disadvantage does not occur if φ is constant, i.e. , if $b \equiv 0$.

The eigenvalue bounds for problem (4.30) needed for (4.28) can again be computed by the general methods described above, combined with the Rayleigh-Ritz method providing *upper* eigenvalue bounds.

For the last time, we go back to the case where L is inverse-positive on $H_2^B(\Omega)$, and where some $z \in H_2(\Omega)$ satisfying

$$\operatorname*{ess\,inf}_{\Omega} z > 0, \qquad \operatorname*{ess\,inf}_{\Omega} L[z] > 0, \qquad B[z] \geq 0$$

(compare Theorem 3.2) is known. (We still assume that $b \equiv \nabla\varphi$.) Then, Theorem 3.5 shows that (4.28) holds true with

$$\sigma := \operatorname*{ess\,inf}_{\Omega}\left\{\frac{L[z]}{z}\right\}.$$

The simplicity of this expression, compared with the eigenvalue estimation methods mentioned above, reflects again the special possibilities for treating problems of monotone kind, which are completely absent in "non-monotone situations".

4.3.3 Calculation of K_1, κ, and K_2

While the computation of K_0 described in the previous sub-subsection requires a good deal of numerical work, constants K_1, κ, and K_2 satisfying the remaining inequalities in (4.25) can be calculated in a much more direct way, as shown in the following three lemmata. More general versions of these lemmata, and their proofs, can be found in [35, 37].

Lemma 4.5 [37, Lemma 5] *Let (4.25 a) hold with some constant K_0, and let $\underline{c}$ and $\overline{b}$ denote a lower bound for c, and an upper bound for $|b|$, respectively. Then, (4.25 b) holds with*

$$K_1 := \begin{cases} \frac{1}{2\sqrt{\underline{c}}-\overline{b}}, & \text{if } \underline{c} > 0 \text{ and } (2\sqrt{\underline{c}}-\overline{b}) \cdot \sqrt{\underline{c}} K_0 \geq 1 \\ \frac{1}{2}\overline{b}K_0 + \sqrt{(\frac{1}{4}\overline{b}^2 - \underline{c})K_0^2 + K_0}, & \text{otherwise.} \end{cases}$$

Lemma 4.6 [37, Lemma 6] *Let (4.25 a,b) hold with constants K_0 and K_1, respectively. Then, (4.25 c) is true for*

$$\kappa := 1 + \|b\|_\infty K_1 + \|c\|_\infty K_0.$$

For the following lemma, we suppose that $n \geq 2$ and make the additional assumption that the two boundary parts Γ_0 and Γ_1 both are *piecewise C_2-hypersurfaces*, i.e., that some measure-zero subset $Z \subset \partial\Omega$ exists such that $\Gamma_0 \setminus Z$ and $\Gamma_1 \setminus Z$ both are (relatively) open subsets of $\partial\Omega$, and are C_2-hypersurfaces of $\mathbb{R}^n$. (Of course, one of them may be empty.)

Consequently, the *mean curvature* H and the *maximal principal curvature* P are defined almost everywhere on $\partial\Omega$.

Lemma 4.7 [35, Section 4] and [37, Lemma 7] *Let (4.25 a,b,c) hold with constants K_0, K_1, and κ. Suppose that a Lipschitz-continuous function $g : \bar{\Omega} \to \mathbb{R}^n$ exists such that*

$$g \cdot \nu \geq (n-1)H \quad \text{a.e. on } \Gamma_0\,, \quad -g \cdot \nu \geq P \quad \text{a.e. on } \Gamma_1\,,$$

and that nonnegative constants G_0, G_1 are known such that

$$|g| \leq G_0\,, \quad -\operatorname{div} g + \lambda_{\max}\left[J[g] + J[g]^t\right] \leq G_1 \quad \text{a.e. on } \Omega\,,$$

with $J[g]$ denoting the Jacobian matrix of g, and $\lambda_{\max}$ indicating the maximal eigenvalue. Then, (4.25 d) holds with

$$K_2 := \left[\kappa^2 + 2\kappa G_0 K_1 + G_1 K_1^2\right]^{\frac{1}{2}}.$$

Corollary 4.1 *If Ω is convex and (4.25 c) holds for some constant κ, then (4.25 d) is true for*

$$K_2 := \kappa.$$

Proof. For convex domains, the curvatures H and P are nonpositive a.e. on $\partial\Omega$, so that the conditions of Lemma 4.7 hold true with $g \equiv 0, G_0 = G_1 = 0$. ∎

5. NAKAO'S AND OISHI'S ALTERNATIVES

In this section, we will give an impression of the methods proposed by Nakao and Oishi. We will *not* comment further on Nakao's earlier results using the fixed-point formulation (2.3), since this approach requires conditions which are even much more restrictive than those needed for the application of the monotonicity method, as we explained in Section 2. His new approach (2.16), however, is closely related to the "simplified Newton" formulation (2.14) proposed by the author (see Subsection 2.1 and Section 4), and is therefore applicable also in "non-monotone" situations. The same is true for Oishi's method proposed recently, which uses the fixed-point operator (2.14) *directly*. The approaches of both Nakao and Oishi have further in common that they avoid the (direct) computation of bounds for the inverse operator L^{-1}, which is required for the author's method (see Section 4, in particular Subsection 4.3). Instead, they deal with some finite-dimensional *projection* of L^{-1} (see (2.15)), and bound the (infinite-dimensional) "remainder" by other means. At first sight, the avoidance of bounding L^{-1} (directly) seems to be an important advantage. However, we will try to make clear that this is not really the case, and furthermore, that the disadvantages caused by the infinite-dimensional "remainder" are significant.

5.1. Nakao's method [30, 31]

Nakao uses the fixed-point formulation (2.1) for *Dirichlet* problems (1.1) ($\Gamma_0 = \partial\Omega, \Gamma_1 = \emptyset$), with T given by (2.16), and with $X := H_1^0(\Omega)$. In order to find some closed, bounded, convex subset $U \subset X$ satisfying (2.2) he introduces , for $V \subset X$, the "rounding" $R(V)$ and the "rounding error" $RE(V)$ with respect to some finite-dimensional (finite element) subspace $S \subset H_1^0(\Omega)$, by defining (essentially)

$$R(V) := PV \tag{5.1}$$

$$RE(V) := \left\{\phi \in S^\perp : \|\phi\|_{H_1^0} \leq \sup_{v\in V} \|(I-P)v\|_{H_1^0}\right\}, \tag{5.2}$$

with $P : H_1^0(\Omega) \to S$ denoting the H_1-orthogonal projection onto S. (In practice, $R(V)$ and $RE(V)$ are chosen a bit more refined, which however does not affect the main ideas and results.) Since obviously $V \subset R(V) \oplus RE(V)$, the crucial condition (2.2) is satisfied if

$$R(TU) \oplus RE(TU) \subset U. \tag{5.3}$$

With some approximate solution $\omega \in S$, and with $\phi_1, \ldots, \phi_m$ denoting some basis in S, Nakao looks for U satisfying (5.3) in the form

$$U = \omega + \sum_{j=1}^{m} [a_j]\phi_j + \left\{\phi \in S^\perp \ : \ \|\phi\|_{H_1^0} \leq \alpha\right\} \tag{5.4}$$

with interval coefficients $[a_j]$ $(j = 1, \ldots, m)$ and a "remainder" bound α to be determined numerically. With $[b_j]$ $(j = 1, \ldots, m)$ denoting interval coefficients satisfying

$$R(TU) \subset \omega + \sum_{j=1}^{m} [b_j]\phi_j, \tag{5.5}$$

and with some $\beta > 0$ such that

$$RE(TU) \subset \left\{\phi \in S^\perp : \|\phi\|_{H_1^0} \leq \beta\right\}, \tag{5.6}$$

(5.3) is obviously satisfied if

$$[b_j] \subset [a_j] \quad (j = 1, \ldots, m) \ , \beta \leq \alpha. \tag{5.7}$$

In order to find $[a_j]$ $(j = 1, \ldots, m)$ and α such that (5.7) holds under the side conditions (5.4), (5.5), (5.6), Nakao proposes an iterative procedure based on the method of ε-inflation introduced by Rump [45].

In each iteration step, interval coefficients $[b_j]$ $(j = 1, \ldots, m)$ and a constant β satisfying (5.5) and (5.6), with *given* $[a_j]$ $(j = 1, \ldots, m)$ and α defining U by (5.4), have to be determined. The computation of $[b_j]$ $(j = 1, \ldots, m)$ is essentially reduced to the following tasks:

(A) Computation of *verified* enclosures for the solutions of certain linear algebraic systems, with *interval* right-hand sides, and with system matrices

$$\mathcal{D} := \left(\langle -\Delta\phi_i, \phi_j\rangle\right)_{i,j=1,\ldots,m} \ , \ \mathcal{L} := \left(\langle L[\phi_i], \phi_j\rangle\right)_{i,j=1,\ldots,m} \ , \tag{5.8}$$

where L is the differential operator given by (2.6), and $\langle\cdot,\cdot\rangle$ indicates dual pairing of H_{-1}- and H_1^0-elements, so that $\langle -\Delta\phi_i, \phi_j\rangle = \langle \nabla\phi_i, \nabla\phi_j\rangle_{L_2(\Omega)^n}$, and $\langle L[\phi_i], \phi_j\rangle = \langle \nabla\phi_i, \nabla\phi_j\rangle_{L_2(\Omega)^n} + \langle b \cdot \nabla\phi_i + c\phi_i \ , \ \phi_j\rangle_{L_2(\Omega)}$;

(B) Computation of verified enclosures for

(B1) $\{\langle F(\cdot, u, \nabla u) - F(\cdot, \omega, \nabla\omega), \ \phi_i\rangle_{L_2(\Omega)} : u \in U\} \quad (i = 1, \ldots, m)$

(which of course requires growth conditions on F), and for the L_2-projection of the *defect* of ω into S:

(B2) $\langle \nabla\omega, \nabla\phi_i\rangle_{L_2(\Omega)^n} + \langle F(\cdot, \omega, \nabla\omega), \phi_i\rangle_{L_2(\Omega)} \quad (i = 1, \ldots, m).$

The computation of a constant β satisfying (5.6) is governed by the following estimate from finite element theory:

$$\|(I-P)u\|_{H_1^0} \leq Ch\|\Delta u\|_2 \quad (u \in H_2(\Omega) \cap H_1^0(\Omega)), \tag{5.9}$$

with h denoting the maximal diameter of the (C_0-) elements. (5.9) holds for convex polygonal domains $\Omega \subset I\!R^2$. Using (2.16), (5.4), (5.9) one finds, for $u \in U$,

$$\begin{aligned} \|(I-P)Tu\|_{H_1^0} &= \|(1-\varepsilon)(I-P)u \;-\; \varepsilon(I-P)(-\Delta)^{-1}F(\cdot,u,\nabla u)\|_{H_1^0} \\ &\leq (1-\varepsilon)\|(I-P)u\|_{H_1^0} + \varepsilon Ch\|F(\cdot,u,\nabla u)\|_2 \\ &\leq (1-\varepsilon)\alpha + \varepsilon\gamma\,, \\ \gamma &:= Ch\sup_{u\in U}\|F(\cdot,u,\nabla u)\|_2, \end{aligned} \tag{5.10}$$

so that, according to (5.2), condition (5.6) holds for $\beta := (1-\varepsilon)\alpha+\varepsilon\gamma$, and the inequality $\beta \leq \alpha$ in (5.7) is satisfied iff $\gamma \leq \alpha$. In addition to (A) and (B) above, one must therefore

(C) know the constant C in (5.9) *explicitly*,
(D) compute an upper bound for $\sup_{u\in U}\|F(\cdot,u,\nabla u)\|_2$,

in order to be able to test condition (5.7).

Obviously, (A) to (D) can be carried out without explicitly computing a bound for the inverse operator L^{-1}. However, in (A), the verified solution of linear systems with the matrix $\mathcal{L}$, which constitutes a finite-dimensional projection of L, is required (in each iteration step). Since the dimension m of $\mathcal{L}$ is large (more than 6000 in nontrivial examples), this task is *at least comparable with (if not harder than)* the computation of a few eigenvalue bounds for L, which is the only major computation needed for the required bound for L^{-1} in the author's method (see Subsection 4.3). Observe that, with the aid of the methods by Rayleigh-Ritz and Lehmann-Goerisch (see [7]), the computation of these eigenvalue bounds can be reduced to the verified solution of *matrix* eigenvalue problems of *small dimension*, after *a few approximate* eigenvectors of a large system have been computed.

Up to now, Nakao neglected the problems connected with the *verified* solution of the linear systems in (A), and solved them only approximately. He intends to use, in the future, a method developed by Rump [47] for the verified solution of large linear systems with band matrices. The application of this method requires, in particular, the computation of a verified positive lower bound for the smallest singular value of $\mathcal{L}$, which is a nontrivial task consuming a good deal of computing time! Here, one observes an interesting close connection to the author's method, which requires a positive lower bound for the smallest singular value of L (see Sub-subsection 4.3.2), as the final result of the computation of the eigenvalue bounds mentioned above.

Due to the above considerations, we believe that the avoidance of bounding L^{-1} in Nakao's method is not an advantage, at least not a significant one. Furthermore , Nakao's method has a clear disadvantage generated by the estimate (5.9), which is needed to treat the infinite-dimensional remainder ("rounding error"). In (5.9) (and thus in (5.10)), the "meshsize" h of the finite element mesh occurs only in ***first power.*** Thus, a very small h is needed in order to make γ in (5.10) sufficiently small, since the term $\sup\{\|F(\cdot, u, \nabla u)\|_2 \,:\, u \in U\}$ can be bounded only very roughly. This bound, as well as the bounds required in (B1) above, is computed with the aid of explicit Sobolev imbeddings and by use of the special polynomial form of the nonlinearity F which is present in Nakao's examples. Obviously, this can be carried out only roughly since the nonlinearity does not preserve the splitting of U into "rounding" and "rounding error" (see (5.4)).

It should be noted that the estimate (5.9) cannot be improved, with respect to the power of h, by use of higher order finite elements. The power of h only improves for functions u with higher regularity, which is usually not present for the exact solution u.

The roughness of (5.10) is the main reason for the fact that, on the one hand, very small meshsizes are needed, and on the other hand, the computed bounds are not very sharp even for small h. For example, Nakao admits in [30] that a meshsize $h \approx 10^{-3}$ (producing linear systems of dimension $\approx 10^6$) would be needed to make his method work for Emden's equation (see also the second Example in Section 7). Thus, Nakao searched for improvements of his method admitting larger meshsizes. In [30] he uses, for Emden's equation, the old idea (applied extensively, e.g., by the author) of considering the boundary value problem for the *error* $u - \omega$, which involves the *defect* of ω. In order to obtain an approximate solution $\omega \in H_2(\Omega) \cap \overset{0}{H}_1(\Omega)$, he applies Hermite-interpolation techniques. In this way, the method works for $h = 1/80$ (which produces systems of dimension larger than 6000), but the computed bounds are still not very precise. In [31], similar techniques are developed for approximate solutions ω which are only in $\overset{0}{H}_1(\Omega)$; the defect of ω is now bounded in the distributional H_{-1}-norm. Emden's equation can now be treated with $h = 1/60$ (system dimension 3481), but the computed solution bounds are rather coarse (see [31], p. 172).

Very recently, Nakao announced in personal communication an improvement of his method by use of higher order finite elements.

5.2. Oishi's method [32]

Oishi proposes an existence and inclusion method for operator equations of the form

$$L_0[u] + \mathcal{F}(u) = 0, \quad u \in D(L_0), \tag{5.11}$$

where, for some Banach spaces R and S , $L_0 : D(L_0) \subset R \to S$ is linear and closed, so that $X := D(L_0)$ becomes a Banach space with respect to the graph norm $\|u\|_X := \|u\|_R + \|L_0[u]\|_S$. $\mathcal{F} : X \to S$ is assumed to be Fréchet-differentiable, with derivative $\mathcal{F}'(u) : X \to S$ extendable to a bounded linear operator from R to S, for each $u \in X$.

(We changed Oishi's notation here, in order to facilitate the explanation of connections to other parts of the present article.)

A similar functional analytical setting for existence and inclusions results has been presented by the author on an international meeting [39], but the results are not yet published.

In the following, we will always have the boundary value problem (1.1) in mind, as a special case of (5.11). One can choose, for example,

$$\left.\begin{array}{lll} L_0[u] := -\Delta u, & \mathcal{F}(u) := F(\cdot, u, \nabla u), & \\ X := H_2^B(\Omega), & R := H_{1,p}(\Omega) \ (p \text{ suitable}), & S := L_2(\Omega) \end{array}\right\} \tag{5.12}$$

or, if F has the the special form (3.1),

$$\left.\begin{array}{ll} L_0[u] := -\Delta u + b \cdot \nabla u, & \mathcal{F}(u) := \tilde{F}(\cdot, u), \\ X := H_2^B(\Omega), & R = S := L_2(\Omega). \end{array}\right\} \tag{5.13}$$

Oishi uses the fixed-point formulation (2.1), with T defined by (2.14) (resp., (2.5), (2.6)), which is exactly the setting of the author's method. However, a few marginal and one major difference occur in the details of the methods. The marginal differences, which we will not comment further, are the following:

— Oishi uses, in (2.1), the space $X = D(L_0)$, in place of the spaces X proposed by the author (see (4.14), (4.15), (4.16)); this difference has almost no practical consequences.

— In place of a function G satisfying (4.9), (4.10), Oishi requires a local Lipschitz condition for $\mathcal{F}'$ (as in the original Newton-Cantorovich Theorem), which may provide slightly worse bounds.

— Oishi uses Banach's in place of Schauder's Fixed-Point Theorem, which again has no important practical consequences.

The main difference between Oishi's and the author's method consists in the avoidance of *directly* bounding L^{-1}. Instead, he has to bound $\mathcal{L}^{-1}$, with $\mathcal{L}$ denoting some finite-dimensional projection of L. Here, a strong connection to Nakao's method can be observed: the computation of a positive lower bound for the smallest singular value of $\mathcal{L}$, which will be needed if Nakao uses Rump's method for verified solution of large linear systems, is obviously equivalent to the computation of a bound for $\mathcal{L}^{-1}$ in the spectral norm.

In order to avoid the direct computation of a bound for L^{-1}, Oishi makes the following assumptions, which we will further analyze after Lemma 5.1.

Let $U \subset X$ and $V \subset S$ denote finite-dimensional subspaces of equal dimension, and let $P : X \to U$ and $Q : S \to V$ be projection operators onto U and V, respectively. Assume that

(A1) $\|\mathcal{L}^{-1}\|_{S\to R} \leq M$, where $\mathcal{L} := QL\big|_U$, with L given by (2.6), i.e., $L = L_0 + \mathcal{F}'(\omega)$;

(A2) $QL_0u = QL_0Pu$ for all $u \in X$;

(A3) $\|(I-P)u\|_R \leq \varepsilon\|L_0[u]\|_S$ for all $u \in X$, where the constant ε can be made arbitrarily small if the dimension of U and V is chosen sufficiently large;

(A4) $\|\mathcal{F}'(\omega)\|_{R\to S} \leq N$;

(A5) $\|Q\|_{S\to S} \leq 1$.

The following Lemma provides the needed bound for L^{-1}.

Lemma 5.1 [32, Thm.2.1] *Let (A1) to (A5) hold, and suppose that ε in (A3) is such that $\varepsilon N(1+MN) < 1$. Then,*

$$\|u\|_X \leq \frac{(1+\varepsilon)(1+MN)+M}{1-\varepsilon N(1+MN)}\|L[u]\|_S \quad \textit{for all } u \in X. \tag{5.14}$$

Proof. For $u \in X$ one finds, using (A1) to (A5),

$$\begin{aligned}
\|Pu\|_R = \|\mathcal{L}^{-1}\mathcal{L}Pu\|_R &\leq M\|\mathcal{L}Pu\|_S = M\|QLPu\|_S \\
&\leq M\Big[\|QL_0Pu + Q\mathcal{F}'(\omega)u\|_S + \|Q\mathcal{F}'(\omega)(Pu-u)\|_S\Big] \\
&= M\Big[\|QL[u]\|_S + \|Q\mathcal{F}'(\omega)(Pu-u)\|_S\Big] \\
&\leq M\Big[\|L[u]\|_S + N\varepsilon\|L_0[u]\|_S\Big]
\end{aligned}$$

so that we obtain, using (A3) again,

$$\|u\|_R \leq \|u-Pu\|_R + \|Pu\|_R \leq M\|L[u]\|_S + \varepsilon(1+MN)\|L_0[u]\|_S \tag{5.15}$$

(5.15) and (A4) imply

$$\|L_0[u]\|_S \leq \|L[u]\|_S + \|\mathcal{F}'(\omega)u\|_S \leq (1+MN)\|L[u]\|_S + \varepsilon N(1+MN)\|L_0[u]\|_S$$

which yields

$$\|L_0[u]\|_S \leq \frac{1+MN}{1-\varepsilon N(1+MN)}\|L[u]\|_S. \tag{5.16}$$

In the identity $\|u\|_X = \|u\|_R + \|L_0[u]\|_S$, we first estimate $\|u\|_R$ by (5.15), and then $\|L_0[u]\|_S$ by (5.16). The assertion follows. ∎

Concerning assumption (A1), the same remarks as in the discussion of Nakao's method hold. For "large" matrices $\mathcal{L}$ (occuring in PDE problems), the computation of M in (A1) is at least comparable with (if not harder than) the computation of a few eigenvalue bounds for the differential operator L.

Assumption (A3) is of course very closely related to Nakao's condition (5.9). Again, it represents the needed bound for the infinite-dimensional "remainder". However, Oishi's method has the advantage, compared with Nakao's approach, that the constant ε in (A3) enters the final solution bounds only via the bound (5.14) for L^{-1}, and not as a factor, as in Nakao's method (see (5.10)). Thus, the disadvantages of (5.9), (5.10) explained above are not present in Oishi's method; it suffices to choose the dimension of U and V such that, e.g., $\varepsilon N(1+MN) \leq 1/2$. The solution bounds become small via the *defect bound* (see (4.5), (4.8) and the remarks after Theorem 4.1).

However, Oishi has to pay for this advantage by the necessity of requiring assumption (A2), which severely restricts the applicability of the method, at least under the following additional assumption, as we show in the next lemma and the discussion following it.

(A6) R and S are Hilbert spaces; the projection $P : X \to U$ is bounded as a mapping from R to R; its bounded extension $\bar{P} : R \to U \subset R$, and $Q : S \to V \subset S$ are *orthogonal* projections.

Assumption (A6) does not seem to be too artificial, with regard to the settings (5.12) (with $p := 2$) and (5.13), and to applications. It remains an open question how restrictive assumption (A2) is if (A6) is not satisfied.

Lemma 5.2 *Let (A2) and (A6) hold, and let $L_0^* : D(L_0^*) \subset S \to R$ denote the adjoint of $L_0 : D(L_0) \subset R \to S$. Then,*

$$V \subset D(L_0 L_0^*) = \{u \in D(L_0^*) \,:\, L_0^*[u] \in D(L_0)\}, \text{ and } L_0^*(V) \subset U.$$

Proof. Let $\psi \in V$ be fixed. Using the orthogonality of the projection Q and assumption (A2) we obtain, for each $u \in X$,

$$\langle \psi, L_0[u] \rangle_S = \langle Q\psi, L_0[u] \rangle_S = \langle \psi, QL_0[u] \rangle_S = \langle \psi, QL_0 Pu \rangle_S. \tag{5.17}$$

The linear mapping $QL_0\big|_U : U \to V$ is bounded with respect to all norms in U and V since these spaces are finite-dimensional. In particular, it is bounded with respect to $\|\cdot\|_R$ and $\|\cdot\|_S$. Since $\bar{P} : R \to R$ is bounded we conclude that, for each $u \in X$,

$$\|QL_0 Pu\|_S = \|(QL_0\big|_U)\bar{P}u\|_S \leq C\|u\|_R,$$

so that (5.17) yields the boundedness of the linear functional $\ell : D(L_0) \subset R \to S, \ell u := \langle \psi, L_0[u] \rangle_S$. Consequently, Riesz' Representation Lemma provides some $\chi \in R$ such that

$$\langle \psi, L_0[u] \rangle_S = \langle \chi, u \rangle_R \quad \text{for all } u \in D(L_0),$$

so that $\psi \in D(L_0^*)$ (and $L_0^*[\psi] = \chi$). Furthermore, (5.17) and the orthogonality of Q imply, for $u \in X$,

$$\langle \psi, L_0[u] \rangle_S = \langle \psi, QL_0 Pu \rangle_S = \langle Q\psi, L_0 Pu \rangle_S = \langle \psi, L_0 Pu \rangle_S$$

and thus, $\langle\psi, L_0[u-Pu]\rangle_S = 0$. Consequently, since $\psi \in D(L_0^*)$ and $I-\bar{P}$ is orthogonal,

$$0 = \langle L_0^*[\psi], (I-\bar{P})u\rangle_S = \langle (I-\bar{P})L_0^*[\psi], u\rangle_R \text{ for all } u \in X.$$

Since X is dense in R we conclude that $L_0^*[\psi]$ is in the nullspace of $I-\bar{P}$ which equals the image space of $\bar{P}$ because $\bar{P}$ is an orthogonal projection. Consequently, $L_0^*[\psi] \in U \subset X = D(L_0)$. ■

The result of Lemma 5.2 requires that bases $\{\varphi_1, \ldots, \varphi_m\}$ of U and $\{\psi_1, \ldots, \psi_m\}$ of V have to be found such that

$$\psi_i \in D(L_0^*)\,,\ L_0^*[\psi_i] = \varphi_i\,(\in D(L_0)). \tag{5.18}$$

This is a severe restriction. One must be able to choose $\varphi_1, \ldots, \varphi_m \in U \subset D(L_0)$ such that the equations (5.18) can be solved *explicitly* for $\psi_1, \ldots, \psi_m$. This seems to be impossible for finite-element bases. It *is* possible if the eigenfunctions ψ_i of $L_0L_0^*$ are known explicitly (so that one can choose $\varphi_i := L_0^*[\psi_i]$), but this restricts the applicability, for instance in the setting (5.13), to simple domains Ω and to constant b. Moreover, the matrix $\mathcal{L}$, the inverse of which has to be bounded according to assumption (A1), is now *full* since the support of the eigenfunctions is the whole of $\bar{\Omega}$.

Up to now, Oishi applied his method to an ODE problem (see [32]), with $L_0[u] = -u''+bu'$ and constant b, so that in fact the eigenfunctions of $L_0L_0^*$ are known. Moreover, the matrix $\mathcal{L}$ is *not* "large" since the problem is an ODE. Therefore, the bound for $\mathcal{L}^{-1}$ required in (A1) can be computed by explicitly inverting $\mathcal{L}$ in rational arithmetic, and evaluating the Frobenius norm of $\mathcal{L}^{-1}$. Assumption (A3) is fulfilled by use of Fourier series expansions, with the mentioned eigenfunctions as basis.

6. Numerical procedures for the Computation of ω and δ

In the present section, we return to the context of Section 4 and give a brief description of the numerical procedures which were used, in our examples (with two-dimensional rectangular domains Ω), to compute an approximate solution ω of problem (1.1) and a defect bound δ satisfying (4.8), with $d[\omega]$ given by (4.5).

6.1. Computation of an approximate solution ω

ω is calculated by a *Newton-iteration*: Starting with some initial approximation $\omega^{(0)} \in H_2^B(\Omega)$, we compute iteratively approximate solutions $u^{(k)}$ of the linear boundary value problems

$$\begin{aligned} &-\Delta u + F_z(\cdot, \omega^{(k)}, \nabla\omega^{(k)}) \cdot \nabla u + F_y(\cdot, \omega^{(k)}, \nabla\omega^{(k)})u \\ &= \Delta\omega^{(k)} - F(\cdot, \omega^{(k)}, \nabla\omega^{(k)}), \qquad u \in H_2^B(\Omega), \end{aligned} \tag{6.1}$$

and define $\omega^{(k+1)} := \omega^{(k)} + u^{(k)}$.

To determine $u = u^{(k)}$ we use a *finite-element* procedure with rectangular elements for problem (6.1). On each element, u is put up as a bi-quintic polynomial. The local basis functions are chosen such that the 36 coefficients determining u coincide with the values of u, $\partial u/\partial x_1$, $\partial u/\partial x_2$, and $\partial^2 u/\partial x_1 \partial x_2$ in 9 knots of the element, namely the corners, the midpoints of the sides, and the midpoint of the element. Simple results from Hermite-interpolation theory show that the corresponding *global* representation

$$u(x_1, x_2) = \sum_{i=1}^{N} a_i \varphi_i(x_1, x_2) \tag{6.2}$$

provides a global C_1-function. (More precisely, the global basis functions φ_i are C_1-functions.) In particular, $u \in H_2(\Omega)$. To ensure that $u \in H_2^B(\Omega)$, i. e., that u satisfies the boundary condition in (1.1), one has to set to zero several of the coefficients a_i in (6.2). The remaining coefficients are determined by the usual Ritz-Galerkin method for problem (6.1). The occurring integrals are approximated by a composite trapezoidal or Simpson quadrature (product-) formula, applied on each element. The resulting linear algebraic system is solved (approximately) by a band-Gauss-algorithm with scaled partial pivoting.

It should be noted that a global C_1-function u could also be obtained by bi-*cubic* (local) basis functions. However, the bi-quintic functions described above proved to be more efficient in our context. Other alternatives are, e. g., triangular *Argyris* or *Bell* elements.

The Newton-iteration is terminated when, for some $k \in \mathbb{N}$, the coefficients $a_i^{(k)}$ of $u^{(k)}$ are (in modulus) below some tolerance. Then, the approximate solution $\omega := \omega^{(k+1)}$ is given in the form (6.2), provided that the starting approximation $\omega^{(0)}$ has that form. To find such a function $\omega^{(0)}$ we use a *homotopy* method, in our examples.

6.2. Computation of a defect bound δ

To compute a constant δ satisfying (4.8) we have to bound $\int_{\Omega_k} \Phi(x)\,dx$ $(k = 1, \ldots, K)$, with $\Omega_1, \ldots, \Omega_K$ denoting the rectangular finite elements described above, and $\Phi := \left[-\Delta\omega + F(\cdot, \omega, \nabla\omega)\right]^2$. In our examples, we used the following way of proceeding which is applicable for sufficiently smooth nonlinearities F.

Let $Q[\Phi]$ denote some quadrature formula (applicable on the rectangular elements Ω_k) such that a remainder term bound

$$\left| \int_{\Omega_k} \Phi(x)\,dx - Q[\Phi] \right| \leq R[\Phi] \tag{6.3}$$

is explicitly available. For instance, one may choose some composite Newton-Cotes product formula

$$Q[\Phi] = h_1 h_2 \sum_{\varrho=0}^{m_1} \sum_{\sigma=0}^{m_2} \mu_\varrho \mu_\sigma \Phi\left(x_1^{(\varrho)}, x_2^{(\sigma)}\right) \tag{6.4}$$

(with stepsizes h_1 in x_1- and h_2 in x_2-direction), e. g., the trapezoidal rule, where

$$R[\Phi] = \frac{1}{12}\,\mathrm{meas}(\Omega_k)\left[h_1^2 \left\|\frac{\partial^2 \Phi}{\partial x_1^2}\right\|_{\infty,\Omega_k} + h_2^2 \left\|\frac{\partial^2 \Phi}{\partial x_2^2}\right\|_{\infty,\Omega_k}\right], \tag{6.5}$$

or the Simpson formula, where

$$R[\Phi] = \frac{1}{180}\,\mathrm{meas}(\Omega_k)\left[h_1^4 \left\|\frac{\partial^4 \Phi}{\partial x_1^4}\right\|_{\infty,\Omega_k} + h_2^4 \left\|\frac{\partial^4 \Phi}{\partial x_2^4}\right\|_{\infty,\Omega_k}\right]. \tag{6.6}$$

In order to enclose $\int_{\Omega_k} \Phi(x)\,dx$ we are therefore left to compute

i) an enclosure for $Q[\Phi]$,

ii) rough bounds for some higher derivaties of Φ on Ω_k.

For both i) and ii) we use *interval arithmetical* subroutines (see [18, 23]), in order to take rounding errors into account.

ad i): For each quadrature point $(x_1^{(\varrho)}, x_2^{(\sigma)})$, we use the representation (6.2) (and the polynomial form of the basis functions φ_i on Ω_k) to compute enclosures for ω, $\nabla\omega$, and $\Delta\omega$ at $(x_1^{(\varrho)}, x_2^{(\sigma)})$. Supposing that an interval-evaluator for F is available in the program, we can now easily calculate an enclosure for $\Phi\big(x_1^{(\varrho)}, x_2^{(\sigma)}\big)$. Finally, an enclosure for $Q(\Phi)$ is computed according to (6.4).

ad ii): We describe how to compute a bound for $\|\partial^\mu \Phi/\partial x_1^\mu\|_{\infty,\Omega_k}$. Since the factor h_1^μ in (6.5) or (6.6), resp., can be made arbitrarily small by the choice of sufficiently many quadrature points, a very rough bound is sufficient.

First we compute upper and lower bounds for the x_1-derivatives (up to the μ-th order) of ω, $\nabla\omega$ and $\Delta\omega$ on Ω_k, using a two-dimensional version of a theorem by Ehlich and Zeller [11] which reduces the calculation of bounds for a polynomial (on a compact set) to its evaluation at finitely many points.

In the next step, we calculate (by hand or by automatic differentiation) all derivatives of $F(x, y, z)$ up to the μ-th order, and compute rough bounds for them on $\bar{\Omega}_k \times [\underline{y}, \overline{y}] \times [\underline{z}_1, \overline{z}_1] \times [\underline{z}_2, \overline{z}_2]$, with $\underline{y}, \overline{y}, \underline{z}_1, \overline{z}_1, \underline{z}_2, \overline{z}_2$ denoting the bounds for ω, $\partial\omega/\partial x_1$, and $\partial\omega/\partial x_2$ calculated before.

Now it is easy to compute rough bounds for $\|\partial^\nu d/\partial x_1^\nu\|_{\infty,\Omega_k}$ ($\nu = 0,\ldots,\mu$), where $d := -\Delta\omega + F(\cdot,\omega,\nabla\omega)$. Finally, the inequality

$$\left\|\frac{\partial^\mu \Phi}{\partial x_1^\mu}\right\|_{\infty,\Omega_k} \leq \sum_{\nu=0}^{\mu} \binom{\mu}{\nu} \left\|\frac{\partial^\nu d}{\partial x_1^\nu}\right\|_{\infty,\Omega_k} \cdot \left\|\frac{\partial^{\mu-\nu} d}{\partial x_1^{\mu-\nu}}\right\|_{\infty,\Omega_k}$$

is used to obtain the desired bound.

7. NUMERICAL EXAMPLES

Several examples with ordinary and elliptic differential equations have been treated to test our existence and inclusion method described in Section 4. In the present section, we report on some of the results obtained for *elliptic* boundary value problems. One ODE example was already presented at the end of Section 2.

Our first example is the well known parameter-dependent problem

$$-\Delta u = \lambda e^u \text{ on } \Omega := (0,1)^2, \quad u = 0 \text{ on } \partial\Omega \tag{7.1}$$

arising from combustion theory. The exponential nonlinearity is essentially generated by Arrhenius' law. We used the finite element method described in Subsection 6.1, with 8×8 square elements, to compute approximate solutions ω for several values $\lambda > 0$ (which is the relevant and "difficult" sign). However, a direct application of the Newton-finite element method to problem (7.1) is disadvantageous due to the corner-singularities which the exact solutions have and which cannot be represented by a finite element approximation. Therefore, we first *transformed* the problem to remove (or at least, weaken) these singularities; see [36] for the details of this transformation. The following bifurcation diagram results from several selected approximate solutions, and interpolation in between.

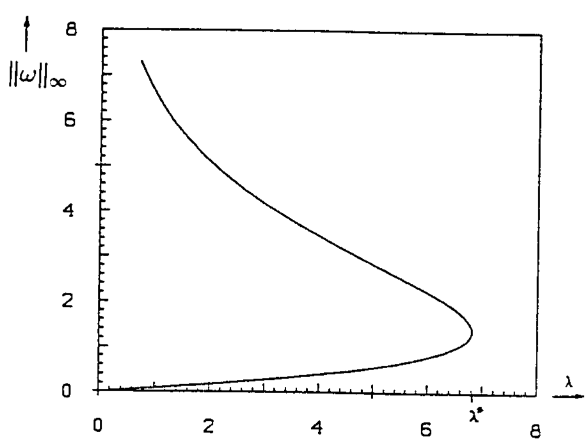


Figure 2: Bifurcation diagram for problem (7.1)

Our existence and inclusion method was successful on the "lower" branch (up to the turning point), and partly on the "upper" branch. In the further course of this branch, the method failed due to insufficient accuracy of the approximate solutions obtained with 8×8 elements; here, a smaller meshsize and/or some further reduction of the corner singularities would certainly extend the range of applicability. Furthermore, the method failed in some neighborhood of the turning-point, the reason for this being the non-invertibility of L in the turning-point, so that (4.7) resp. (4.20), (4.21) resp. (4.27), (4.28) could not be satisfied. In [40, 41], we extend our method by change-of-parameter techniques in order to overcome these difficulties with turning-points.

In our second example, we look for a nontrivial solution of *Emden's* equation

$$-\Delta u = u^2 \text{ on } \Omega := (0,\ell) \times (0,\ell^{-1}), \quad u = 0 \text{ on } \partial\Omega \tag{7.2}$$

for several positive values of ℓ. The solution u represents the stationary temperature distribution in a plate with an internal heat source proportional to u^2. The problem of finding an appropriate starting approximation $\omega^{(0)}$ for the Newton-iteration (described in Subsection 6.1) was solved as follows. Consider the auxiliary parameter-dependent problem

$$-\Delta u = u^2 + \lambda x_1(1-x_1)x_2(1-x_2) \text{ on } \Omega := (0,1)^2, \quad u = 0 \text{ on } \partial\Omega. \tag{7.3}$$

Starting at the trivial solution $u \equiv 0$ for $\lambda = 0$, changing λ in small steps, and using the approximate solution of the previous step as starting approximation for the Newton-iteration, we found the bifurcation diagram for problem (7.3) (with *approximate* solutions ω) plotted in Figure 3 below.

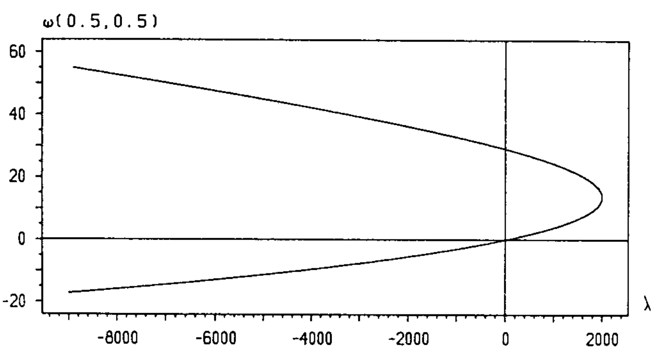
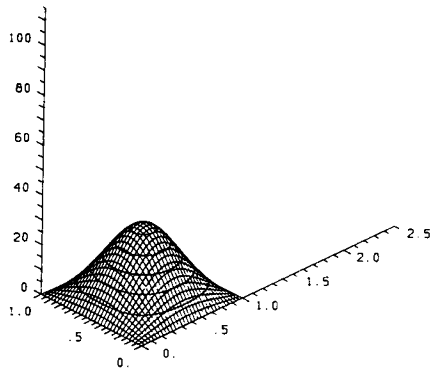


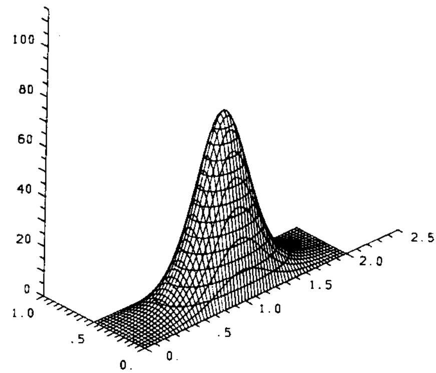
Figure 3: Bifurcation diagram for problem (7.3)

In particular, we obtained a nontrivial approximate solution for $\lambda = 0$, i. e., for problem (7.2) (with $\ell = 1$). To compute approximate solutions for other values of ℓ, we used a (stepwise) homotopy in ℓ, starting with $\ell = 1$.

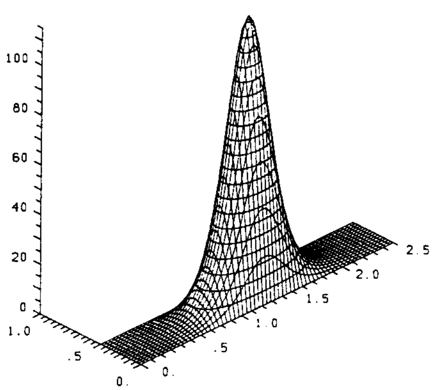
We only looked for solutions which are symmetric with respect to reflection at the two symmetry axes of the rectangle Ω, which we realized by replacing Ω by $\tilde{\Omega} := (0, \frac{1}{2}\ell) \times (0, \frac{1}{2}\ell^{-1})$, and requiring *Neumann* boundary conditions on $\partial\tilde{\Omega} \cap \Omega$. This way of proceeding has two consequences. First, the numerical effort is considerably reduced, and second, several eigenvalues of the linearization L of the original problem are absent in the reduced problem, which may provide a smaller constant K_0 (see Subsection 4.3.2). In fact, the latter effect is dramatic for problem (7.2) with $\ell = 2$ or $\ell = 2.5$, where the second eigenvalue of L for the original problem is very close to zero, but this eigenvalue is absent in the reduced problem. We wish to remark that the *first* eigenvalue of L is (in any case) *negative,* so that monotonicity methods cannot be applied due to the considerations at the end of Section 2.



$\ell = 1$



$\ell = 2$



$\ell = 2.5$

ℓ	$\|\omega\|_\infty \approx$	δ	K	α
1	29.26	0.2207E–2	0.3832	0.8467E–3
2	71.52	0.8183E–1	0.3820	0.3167E–1
2.5	111.73	0.2231	0.3147	0.7188E–1

Figure 4 and **Table 1:** Emden's equation

We used the finite-element procedure described in Subsection 6.1 with 64 rectangular elements (on $\tilde{\Omega}$). For the three values $\ell = 1$, $\ell = 2$, and $\ell = 2.5$, our existence and inclusion algorithm was applied to problem (7.2). Figure 4 shows plots of the three approximate solutions ω, and Table 1 contains the approximate sizes of $\|\omega\|_\infty = \omega(\frac{1}{2}\ell, \frac{1}{2}\ell^{-1})$, the defect bounds δ (see (4.8)), the constants K (see (4.7), (4.20)), and the error bounds α (see Theorem 4.1) for $\|u-\omega\|_\infty$. A constant K' needed not be computed here since properties G1 to G4 (see Subsection 4.2) are satisfied for $G(s,t) := s^2$ which is independent of t.

Our third example is the parameter-dependent boundary value problem

$$-\Delta u = u\left(\lambda - \frac{1}{2}|\nabla u|^2\right) \text{ on } \Omega := (0,1)^2, \quad u = 0 \text{ on } \partial\Omega \tag{7.4}$$

which has an infinite number of potential bifurcation points at the eigenvalues $\lambda_{k,\ell} = (k^2+\ell^2)\pi^2$ of the problem obtained by linearization at the trivial solution $u \equiv 0$.

For several values of λ, we computed approximate solutions ω on the first two nontrivial branches bifurcating from the trivial branch at $\lambda_{1,1} = 2\pi^2$ and $\lambda_{2,1} = \lambda_{1,2} = 5\pi^2$, respectively. In fact, there are *two* branches bifurcating from $5\pi^2$ passing into each other by exchange of x_1 and x_2. In the diagram plotted in Figure 5 below, these two branches coincide due to the chosen projection $\omega_{\max} := \max\{\omega(x) : x \in \bar{\Omega}\}$.

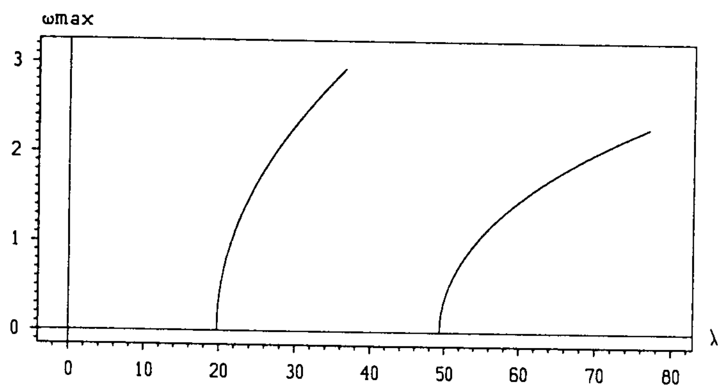


Figure 5: Bifurcation diagram for problem (7.4)

As a starting approximation for the Newton iteration we used, at $\lambda = 2.1\pi^2$ and $\lambda = 5.1\pi^2$, respectively, appropriate multiples of eigenfunctions corresponding to the eigenvalues $2\pi^2$ and $5\pi^2$ of the linearized problem. To obtain approximate solutions for other values of λ, we followed the branches by a (stepwise) homotopy in λ.

As in the second example, we exploited the symmetry of the expected solutions by treating the problem on $\tilde{\Omega} := (0, \frac{1}{2}) \times (0, \frac{1}{2})$ (first branch) or $\tilde{\Omega} := (0, \frac{1}{4}) \times (0, \frac{1}{2})$ (second branch), respectively, and requiring Neumann boundary conditions on $\partial\tilde{\Omega} \cap \Omega$.

The finite-element procedure was used with 9×9 elements on the first, and with 7×14 elements on the second branch.

Our existence and inclusion method was successful for λ up to $3.7\pi^2$ on the first, and for λ up to $7.8\pi^2$ on the second branch. On the latter, the method failed for $\lambda = 5.0001\pi^2$ since, due to the immediate neighborhood of the singular point $5\pi^2$, the constants K and K' are very large. Table 2 below contains the approximate sizes of $\|\omega\|_\infty$, the defect bounds δ (see (4.8)), the constants K and K' (see (4.20), (4.21)), and the error bounds α and $(K'/K)\alpha$ (see Theorem 4.1 and (4.12), (4.15), (4.22)) for $\|u-\omega\|_\infty$ and $\|\nabla u - \nabla\omega\|_4$, respectively.

We wish to remark that, on the second branch, the linearized operator L is not inverse-positive, so that monotonicity methods cannot be applied there.

λ/π^2	$\|\omega\|_\infty$	δ	K	K'	α	$(K'/K)\cdot\alpha$
			First Branch			
2.0001	0.02309	0.3059E-7	2030.	7844.	0.6615E-4	0.2557E-3
2.001	0.07303	0.9676E-7	201.5	779.4	0.1963E-4	0.7591E-4
2.01	0.2309	0.3235E-6	20.51	79.9	0.6644E-5	0.2591E-4
2.1	0.7277	0.3475E-5	2.441	10.15	0.8490E-5	0.3530E-4
2.5	1.611	0.5503E-4	0.9232	4.472	0.5095E-4	0.2468E-3
3.0	2.264	0.2774E-3	0.9377	5.009	0.2646E-3	0.1414E-2
3.5	2.767	0.8215E-3	1.273	7.278	0.1204E-2	0.6887E-2
3.7	2.945	0.1159E-2	1.500	8.786	0.3047E-2	0.1786E-1
3.8	3.030	0.1354E-2	1.639	9.713	—	—
			Second Branch			
5.0001	0.01461	0.1500E-6	2438.	13950.	—	—
5.001	0.04619	0.4743E-6	230.9	1321.	0.1173E-3	0.6709E-3
5.01	0.1460	0.1525E-5	23.14	133.0	0.3553E-4	0.2043E-3
5.1	0.4608	0.9968E-5	2.488	14.90	0.2486E-4	0.1489E-3
5.5	1.021	0.1130E-3	0.6649	4.444	0.7542E-4	0.5041E-3
6.0	1.427	0.4424E-3	0.4628	3.394	0.2069E-3	0.1517E-2
6.5	1.728	0.1145E-2	0.4184	3.292	0.4923E-3	0.3872E-2
7.0	1.974	0.2406E-2	0.4184	3.485	0.1084E-2	0.9023E-2
7.5	2.184	0.4375E-2	0.4406	3.852	0.2382E-2	0.2082E-1
7.8	2.297	0.5915E-2	0.4615	4.141	0.4959E-2	0.4449E-1
7.9	2.333	0.6484E-2	1.4696	4.248	—	—

Table 2: Existence and inclusion results for problem (7.4)

8. APPENDIX: SOLVABILITY AND REGULARITY RESULTS

In this final section, we briefly refer to some results concerned with the theory of elliptic boundary value problems and with Sobolev spaces, which are needed in the course of this article. In particular, we will mention solvability results for linear problems in the space $H_2^B(\Omega)$.

8.1. Sobolev's Imbedding Theorem

In various places of this article, Sobolev's Imbedding Theorem (providing bounded imbeddings) and its extension by Rellich and Kondrachov (which yields compact imbeddings) is needed. Here, we will give a formulation for bounded domains Ω with Lipschitz-continuous boundary $\partial\Omega$, which is sufficient for our present purposes. A general formulation can be found in [2, Thm. 5.4 and Thm. 6.2].

Theorem A.1. *(Sobolev-Rellich-Kondrachov): Let $\Omega \subset \mathbb{R}^n$ denote a bounded domain with Lipschitz-continuous boundary $\partial\Omega$, and let $j, k \in \mathbb{N}$, $j \geq k$, and $p, q \in (1, \infty)$, $\alpha \in [0, 1)$.*

a) If

$$\frac{1}{q} - \frac{k}{n} \geq \frac{1}{p} - \frac{j}{n}, \tag{A.1}$$

then $H_{j,p}(\Omega) \subset H_{k,q}(\Omega)$, and the identity map $H_{j,p}(\Omega) \to H_{k,q}(\Omega)$ is bounded. (These two statements together are usually abbreviated by saying that the imbedding $H_{j,p}(\Omega) \hookrightarrow H_{k,q}(\Omega)$ is bounded.)

If $j > k$, and if the inequality in (A.1) is strict, then the imbedding $H_{j,p}(\Omega) \hookrightarrow H_{k,q}(\Omega)$ is compact.

b) If (A.1) holds with $q = \infty$, with k replaced by $k + \alpha$, and with strict inequality if $\alpha = 0$, then the imbedding $H_{j,p}(\Omega) \hookrightarrow C_{k+\alpha}(\bar{\Omega})$ is bounded. The same imbedding is compact if the inequality is strict.

8.2. Solvability in $H_2^B(\Omega)$

In this subsection, we will prove a simple solvability result for the boundary value problem

$$u \in H_2^B(\Omega), \quad L[u] \equiv -\Delta u + b \cdot \nabla u + cu = r \quad \text{on } \Omega, \tag{A.2}$$

under the assumption that the tripel $(\Omega, \Gamma_0, \Gamma_1)$ is regular, which means that, for some $\sigma \in \mathbb{R}$, the boundary value problem

$$u \in H_2^B(\Omega), \quad -\Delta u + \sigma u = r \quad \text{on } \Omega \tag{A.3}$$

is (uniquely) solvable for each $r \in L_2(\Omega)$ (see Section 1).

Theorem A.2. *Let $(\Omega, \Gamma_0, \Gamma_1)$ be regular, and let $b \in L_\infty(\Omega)^n$, $c \in L_p(\Omega)$ for some $p > n$, $p \geq 2$. Suppose that L is invertible on $H_2^B(\Omega)$, i. e., that the homogeneous problem $(A.2)$ $(r \equiv 0)$ has only the trivial solution. Then, problem $(A.2)$ has a unique solution for each $r \in L_2(\Omega)$.*

Proof. Due to the regularity assumption, the operator $-\Delta + \sigma : H_2^B(\Omega) \to L_2(\Omega)$ is one-to-one and onto. Moreover, it is obviously bounded. Thus, the Open Mapping Theorem provides the boundedness of $(-\Delta + \sigma)^{-1} : L_2(\Omega) \to H_2(\Omega)$.

Furthermore, the mapping $\varphi : H_1(\Omega) \to L_2(\Omega)$, $\varphi(u) := b \cdot \nabla u + cu$, is bounded since Hölder's inequality and Sobolev's Imbedding Theorem provide, with $q := 2p/(p-2)$ $(:= \infty$ if $p = 2)$, that

$$\|\varphi(u)\|_2 \leq \|b\|_\infty \|\nabla u\|_2 + \|c\|_p \|u\|_q \leq C\|u\|_{H_1} \quad \text{for } u \in H_1(\Omega).$$

Finally, the imbedding $H_2(\Omega) \hookrightarrow H_1(\Omega)$ is compact due to Sobolev's Imbedding Theorem. Altogether, the operator $K : H_1(\Omega) \to H_1(\Omega)$, $Ku := (-\Delta + \sigma)^{-1}[\sigma u - \varphi(u)]$, is (linear and) compact, so that Fredholm's alternative holds for the problem

$$u \in H_1(\Omega), \quad u = Ku + (-\Delta + \sigma)^{-1} r,$$

which is obviously equivalent to problem $(A.2)$. Since the homogeneous problem has only the trivial solution, the assertion follows. ∎

8.3. Regular triples $(\Omega, \Gamma_0, \Gamma_1)$

The solvability result of the previous subsection relies on the fact that $(\Omega, \Gamma_0, \Gamma_1)$ is regular. As already mentioned in Section 1, the complete class of regular triples seems to be unknown. After Lemma A.1, we will list some general examples of regular triples which show that this concept has sufficient generality.

Lemma A.1. *Suppose that Γ_0 and Γ_1 are piecewise C_2-hypersurfaces, with principal curvatures bounded from above, and that, for some $\sigma > 0$, problem $(A.3)$ is solvable for a set of functions r which is dense in $L_2(\Omega)$. Then, $(\Omega, \Gamma_0, \Gamma_1)$ is regular.*

Proof. Integration by parts shows that $\|-\Delta u + \sigma u\|_2^2 \geq \sigma^2 \|u\|_2^2$ for $u \in H_2^B(\Omega)$, so that (4.25 a) holds for $K_0 := \sigma^{-1}$, with $L := -\Delta + \sigma$. Lemmata 4.5, 4.6, 4.7 show that also (4.25 b, c, d) hold for suitable constants. Consequently, $(-\Delta + \sigma)^{-1} : D \subset L_2(\Omega) \to H_2^B(\Omega)$ is bounded, with D denoting the dense subset of $L_2(\Omega)$ provided by our assumption. $(-\Delta + \sigma)^{-1}$ can therefore be continuously extended to a bounded linear operator $T : L_2(\Omega) \to H_2^B(\Omega)$. Since $(-\Delta + \sigma)T : L_2(\Omega) \to L_2(\Omega)$ is bounded and $(-\Delta + \sigma)T|_D = \mathrm{id}_D$, it follows that $(-\Delta + \sigma)T = \mathrm{id}_{L_2(\Omega)}$. Thus, for $r \in L_2(\Omega)$, $Tr \in H_2^B(\Omega)$ solves problem $(A.3)$. ∎

Examples of regular triples

1) $(\Omega, \Gamma_0, \Gamma_1)$ is regular if $\partial\Omega$ is a global C_2-hypersurface and, moreover, $\partial\Omega = \Gamma_0$ or $\partial\Omega = \Gamma_1$. See [12], Lemma 18.2 (and Lemma 19.1 in connection with problem (6) after Theorem 19.4) for the case $\partial\Omega = \Gamma_0$, and Theorem 19.3 for both cases $\partial\Omega = \Gamma_0$ and $\partial\Omega = \Gamma_1$.

2) The results just mentioned may be carried over to "regular" mixed boundary value problems where each connected component C of $\partial\Omega$ satisfies $C \subset \Gamma_0$ or $C \subset \Gamma_1$, and to domains with a $C_{1,1}$-boundary (which may locally be parametrized by a C_1-function with Lipschitz-continuous first derivatives).

3) Let $\Omega, \Gamma_0, \Gamma_1$ have the property that the eigenvalue problem $\varphi \in H_2^B(\Omega)$, $-\Delta\varphi = \lambda\varphi$ on Ω, has a complete system $(\varphi_j)_{j\in\mathbb{N}}$ of orthonormal eigenfunctions $\varphi_j \in H_2^B(\Omega)$, and that Γ_0, Γ_1 are piecewise C_2-hypersurfaces with principal curvatures bounded from above. Then, $(\Omega, \Gamma_0, \Gamma_1)$ is regular due to Lemma A.1 since the set of all functions $r = \sum_{j=1}^{N} \alpha_j\varphi_j$ (with $N \in \mathbb{N}$, $\alpha_j \in \mathbb{R}$) is dense in $L_2(\Omega)$, and the boundary value problem (A.3) is solved, for such r, by $u := \sum_{j=1}^{N}(\lambda_j+\sigma)^{-1}\alpha_j\varphi_j$, with $(\lambda_j)_{j\in\mathbb{N}}$ denoting the sequence of corresponding eigenvalues and $\sigma \neq -\lambda_j$ for all j. (Compare [24, Chap. 3, Sec. 9].)

In particular, this assumption holds for many domains with *known* eigenfunctions, such as rectangles (in arbitrary dimension) with each side belonging completely either to Γ_0 or to $\overline{\Gamma}_1$, circular disks, balls and shells, circular sectors (in two dimensions) and circular cones (in higher dimensions) with each "side" (including the spherical part) belonging completely either to Γ_0 or to $\overline{\Gamma}_1$ and with interior angle $\vartheta \in (0, \pi]$. If $n = 2$ and Γ_0 and $\overline{\Gamma}_1$ "meet" at the angular point, ϑ must further be restricted to $(0, \frac{\pi}{2}]$. Moreover, each cylinder $\Omega := \widetilde{\Omega} \times (0,T) \subset \mathbb{R}^n$, with $\widetilde{\Omega} \subset \mathbb{R}^{n-1}$ denoting a domain of one of the types considered above (for instance, a domain with C_2-smooth boundary), has the desired properties.

4) Suppose that $\partial\Omega = \Gamma_0$ and that $\bar{\Omega}$ may be mapped by a $C_{1,1}$-diffeomorphism ϕ (i.e., a C_1-diffeomorphism with Lipschitz-continuous first derivatives) onto $\bar{\Omega}_0$, with Ω_0 denoting a domain such that $(\Omega_0, \partial\Omega_0, \emptyset)$ is regular. Moreover, let $\partial\Omega$ be a piecewise C_2-hypersurface, with maximal principal curvature bounded from above. Then, $(\Omega, \partial\Omega, \emptyset)$ is regular. This can be seen as follows: The boundary value problem (A.3) is equivalent to the following problem for $v := u \circ \phi^{-1}$, $s := r \circ \phi^{-1}$

$$
\begin{aligned}
&v \in H_2^B(\Omega_0) \quad , \\
&L_0[v] := -\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j} - \sum_{i=1}^{n} b_i \frac{\partial v}{\partial x_i} + \sigma v = s \quad \text{on } \Omega_0
\end{aligned}
\tag{A.4}
$$

where $A = (a_{ij}) := (J[\phi] \cdot J[\phi]^t) \circ \phi^{-1}$, $b = (b_i) := (\Delta\phi) \circ \phi^{-1}$. Without going into details we state that estimates of the type (4.25) may also be derived with $L^{(\tau)} :=$

$-\Delta + \sigma + \tau(L_0 + \Delta - \sigma)$ $(0 \leq \tau \leq 1)$ and Ω_0 in place of L and Ω, with uniform constants for $\tau \in [0,1]$. Using this a priori estimate and the regularity of $(\Omega_0, \partial\Omega_0, \emptyset)$, and applying the usual continuation process along $\tau \in [0,1]$ (compare [24, p. 111 ff.]) we obtain that problem (A.4) has a unique solution and thus, $(\Omega, \partial\Omega, \emptyset)$ is regular.

For example, $(\Omega, \partial\Omega, \emptyset)$ is therefore regular for parallelepipeds (in arbitrary dimension), triangles (in two dimensions) and cones, which may be mapped $C_{1,1}$-diffeomorphically onto rectangles, circular sectors and spherical cones, respectively.

5) Let $\Omega \subset I\!R^2$ denote a *convex polygonal* domain, and let $\Gamma_0, \Gamma_1 \subset \partial\Omega$ be such that each "side" of $\partial\Omega$ belongs completely (with possible exception of its endpoints) either to Γ_0 or to Γ_1. Suppose further that, in angular points where Γ_0 and Γ_1 "meet", the interior angle is not larger than $\pi/2$. Then, $(\Omega, \Gamma_0, \Gamma_1)$ is regular, which can be seen as follows.

According to Lemma A.1, it suffices to prove that problem (A.3), with $\sigma := 1$, has a solution for each $r \in C_\infty(\bar{\Omega})$. Let

$$u \in H_1^B(\Omega) := \{w \in H_1(\Omega) : w\big|_{\Gamma_0} = 0\}$$

denote the (weak) solution of the problem

$$\int\limits_\Omega [\nabla u \cdot \nabla\varphi + u\varphi]\,dx = \int\limits_\Omega r\varphi\,dx \quad \text{for all } \varphi \in H_1^B(\Omega). \tag{A.5}$$

Let $\xi_1, \ldots, \xi_N \in \partial\Omega$ denote the corners of $\bar{\Omega}$, and choose circular sectors $\Omega_1, \ldots, \Omega_N \subset \Omega$, with vertices $\xi_1, \ldots, \xi_N$, with sides being parts of the sides of $\bar{\Omega}$ which are adjacent to the respective corner of $\bar{\Omega}$, and with radii such (small) that $\bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset$ for $i \neq j$. Moreover, choose some subdomain $\Omega_0 \subset \Omega$ such that $\Omega_0 \supset \Omega \setminus \bigcup_{i=1}^N \Omega_i$, $\xi_i \notin \bar{\Omega}_0$ for $i = 1, \ldots, N$, and that $\partial\Omega_0 \setminus \partial\Omega$ has positive distance from the circular parts of $\partial\Omega_i$ $(i = 1, \ldots, N)$.

Due to well known regularity results (see, e. g., [13, 24]), $u\big|_{\bar{\Omega}_0} \in C_\infty(\bar{\Omega}_0)$, and u satisfies the differential equation $-\Delta u + u = r$ on Ω_0 and the boundary condition $B[u] = 0$ on $\partial\Omega \cap \partial\Omega_0$ in the classical sense. Since Ω_0 may be chosen arbitrarily "close" to Ω, the differential equation holds on Ω, and we are left to show that $u \in H_2^B(\Omega)$.

For $i \in \{1, \ldots, N\}$, let $\psi_i \in C_\infty(\bar{\Omega}_i)$ be chosen such that $\partial\psi_i/\partial\nu = 0$ on $\partial\Omega_i \cap \Gamma_1$, $\psi_i = 0$ in some neighborhood of the vertex point ξ_i, and $\psi_i = 1$ on the circular boundary part of $\bar{\Omega}_i$. Let $\Gamma_{0,i}$ denote the union of $\Gamma_0 \cap \partial\Omega_i$ and the circular part of $\partial\Omega_i$, and $\Gamma_{1,i} := \partial\Omega_i \setminus \Gamma_{0,i}$. The boundary operator B_i for $\partial\Omega_i$ is defined according to this subdivision of $\partial\Omega_i$. The boundary value problem

$$v_i \in H_2^{B_i}(\Omega_i), \quad (-\Delta + 1)v_i = r - (-\Delta + 1)(\psi_i u) \quad \text{on } \Omega_i, \tag{A.6}$$

has a solution due to example **3)** above. Consequently, since $\partial(v_i + \psi_i u)/\partial\nu = 0$ on $\Gamma_{1,i}$,

$$\int_{\Omega_i} [\nabla(v_i + \psi_i u) \cdot \nabla\varphi + (v_i + \psi_i u)\varphi]\, dx = \int_{\Omega_i} r\varphi\, dx \qquad (A.7)$$

for all $\varphi \in H_1(\Omega_i)$ vanishing on $\Gamma_{0,i}$. Since each such φ may be extended (by zero) to Ω, so that a function $\varphi \in H_1^B(\Omega)$ arises, (A.5) shows that (A.7) also holds with u in place of $v_i + \psi_i u$. Since $u - (v_i + \psi_i u)$ vanishes on $\Gamma_{0,i}$ (regard that $\psi_i = 1$ on the circular part of $\partial\Omega_i$), it follows that

$$u = v_i + \psi_i u \text{ on } \Omega_i. \qquad (A.8)$$

(A.6) implies, via the definition of $H_2^{B_i}(\Omega_i)$, the existence of a sequence $\left(v_i^{(k)}\right)_{k\in\mathbb{N}}$ in $C_2(\bar{\Omega}_i)$ such that $B_i[v_i^{(k)}] = 0$ on $\partial\Omega_i$, and $\left(v_i^{(k)}\right)$ converges to v_i in $H_2(\Omega_i)$. Since $\psi_i u \in C_\infty(\bar{\Omega}_i)$ and $B[\psi_i u] = 0$ on $\partial\Omega_i \cap \partial\Omega$ (observe that ψ_i vanishes in some neighborhood of the vertex ξ_i), the sequence

$$u_i^{(k)} := v_i^{(k)} + \psi_i u$$

in $C_2(\bar{\Omega}_i)$ satisfies $B[u_i^{(k)}] = 0$ on $\partial\Omega_i \cap \partial\Omega$, and converges to $u\big|_{\Omega_i}$ in $H_2(\Omega_i)$ due to (A.8).

Now let $(\Phi_0, \ldots, \Phi_N)$ denote a C_∞-partition of unity subordinate to the covering $(\Omega_0, \ldots, \Omega_N)$ of Ω, such that $\partial\Phi_i/\partial\nu = 0$ on Γ_1 for $i = 0, \ldots, N$. Then,

$$u^{(k)} := \Phi_0 u + \sum_{i=1}^{N} \Phi_i u_i^{(k)}$$

defines a sequence in $C_2(\bar{\Omega})$ satisfying $B[u^{(k)}] = 0$ on $\partial\Omega$, and converging to u in $H_2(\Omega)$. Thus, $u \in H_2^B(\Omega)$.

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