Application of Interval Analysis Techniques to Linear Systems: Part I—Fundamental Results

EDWARD P. OPPENHEIMER, MEMBER, IEEE, AND ANTHONY N. MICHEL, FELLOW, IEEE

Abstract —In the present three-part paper we utilize interval analysis techniques to devise an algorithm which enables us to obtain estimates of bounds for the set of all solutions of initial-value problems of linear systems of autonomous first-order ordinary differential equations that linearly depend on a parameter belonging to an interval. Our results are of interest in many types of applications. We cite here as examples the tolerance problem in electric circuits (where components such as resistors, capacitors, etc., have tolerances associated with them), optimal control problems with large tolerances on a parameter (where sensitivity analysis methods fail), and the like.

In the *first part* of the present three-part paper we establish new results for continuous and rational interval functions which are of interest in their own right and which may find future applications. These results are used in the *second part* of the present three-part paper to study interval matrix exponential functions and to devise a method of constructing augmented partial sums which approximate interval matrix exponential functions as closely as desired. We apply the results of the first two parts of the present three-part paper in the third part to develop the algorithm mentioned at the outset (using machine bounding arithmetic to obtain true estimates of bounds). We demonstrate the applicability of our results by considering three specific examples: an *RLC* circuit, an instrument servo-mechanism, and the design of a minimum plant sensitivity optimal regulator.

I. Introduction

In THE present paper and in two companion papers [1], [2], we utilize interval analysis techniques to obtain estimates of bounds for the set of all solutions of initial-value problems of linear systems of autonomous first-order ordinary differential equations that linearly depend on a parameter belonging to an interval. The motivation for studying this problem includes many interesting applications. We cite here as examples the tolerance problem in electric circuits (where components such as resistors, capacitors, etc., have tolerances associated with them), optimal control problems with large tolerances on a parameter (where sensitivity analysis methods fail), and the like.

In the sequel, when referring to the "present three-part paper," we will actually have in mind three papers (Part I, which is the present paper; Part II, which refers to [1]; and Part III, which refers to [2]) while when speaking of the "present paper," we will have in mind the subject paper on hand.

Manuscript received January 21, 1987; revised December 2, 1987. The work of A. N. Michel was supported in part by the National Science Foundation under Grant ECS 88-02924. This paper was recommended by Associate Editor F. M. A. Salam.

E. P. Oppenheimer is with the Applied Physics Laboratory, Johns Hopkins University, Laurel, MD 20707.

A. N. Michel is with the Department of Electrical and Computer Engineering, University of Notre Dame, Notre Dame, IN 46556.

IEEE Log Number 8822398.

A. Summary of Results

In Part I of the present three-part paper we establish new results for continuous and rational interval functions which are of interest in their own right and which may find additional useful applications elsewhere. In doing so, we construct a complete metric space, $(\mathscr{F}, \mu)$, of continuous interval functions of an interval variable which includes the $C([a,b])$ real function space and the *united extensions* of its member functions f defined by $\bar{f}([a,b]) \triangleq \bigcup_{x \in [a,b]} f([x,x])$ ($[a,b]$ denotes the interval determined by $a \leqslant x \leqslant b$ and $[a,a]$ denotes the degenerate interval determined by $a \leqslant x \leqslant a$). We show that the rational interval functions belong to this space and exhibit the inclusion property $f([a,b]) \supset \bar{f}([a,b])$. Defining a partition of the interval $I=[a,b]$ by $I_i^n \triangleq [a(n-i+1)+b(i-1), (n-i)a+ib]/n$, $i=1,\cdots,n$, we establish the *convergence result* $f(I) \supset \lim_{n \to \infty} \bigcup_{i=1}^n f(I_i^n) = \bar{f}(I)$. (That is, the *convergence result* (Theorem 14) states that by using a sufficiently fine partition of I and by computing the union of the interval function over the partition sub-intervals, it is possible to approximate the exact range of the interval function for $x \in J, \bar{f}(J)$, *as closely as desired*.) Next, we extend the above results to continuous and rational interval matrix functions of an interval variable, including a *convergence result* for interval matrix functions of an interval variable (Theorem 14M) similar to Theorem 14 above.

In *Part II* of the present three-part paper (i.e., in [1]), we show that the sequence of partial sums obtained from the infinite series representation of the *interval exponential function* is a Cauchy sequence which converges to a member function of the complete metric space of continuous interval functions of an interval variable, $(\mathscr{F}, \mu)$, introduced in this paper. We devise a technique to compute an approximation $\tilde{g}$ of the interval exponential function $g([a,b]) \triangleq \exp([a,b])$ which, for $\epsilon > 0$, provides the error inclusive property $[1-\epsilon, 1+\epsilon] \cdot g([a,b]) \supset \tilde{g}([a,b]) \supset$

0098-4094/88/0090-1129$01.00

$g([a,b]) \supset \bar{g}([a,b])$ ($\bar{g}$ is an *augmented* kth-order partial sum for the exponential function g and ϵ depends on the size of k). Finally, we apply the *convergence result* of Part I (Theorem 14) to reduce conservativeness when obtaining estimates for $\bar{g}$ by the above results. Next, we extend the above development to continuous interval matrix functions of an interval variable (including the application of the convergence result of Part I (Theorem 14M)). In addition, in order to obtain optimal estimates of error bounds for the augmented truncated series representation of the interval matrix exponential function (augmented partial sum), we make use of Householder norms. Also, to reduce the inherent conservativeness of interval arithmetic operations, we utilize the nested form for interval polynomials and the centered form for interval arithmetic representations in computing partial sums for the interval matrix exponential. Finally, in order to obtain *true* estimates from our algorithmic results, we employ machine bounding arithmetic in computing partial sums for the interval matrix exponential.

In *Part III* of the present three-part paper (i.e., in [2]), we consider *initial-value problems* of the form $\dot{x} = (G_1 + \theta G_2)x$, $x(0) = x_0$, $\theta \in [-1.0, 1.0]$, where $x \in R^n$, $\dot{x} = dx/dt$, x_0 denotes an initial condition (a given interval vector), and G_1 and G_2 denote real $n \times n$ matrices. Using the partition $\theta_i = [(-M + 2(i-1))/M, (-M+2i)/M]$, $i = 1, \cdots, M$, we generate for the above initial-value problem M *subproblems* given by $\dot{x} = (G_1 + \theta G_2)x$, $x(0) = x_0$, $\theta \in \theta_i$, $i = 1, \cdots, M$. We use the results of Part I and Part II to establish an algorithm which enables us to obtain bounds at any desired point in time t for the interval solutions for the above M subproblems. The interval solution for the (entire) initial-value problem given above is then obtained by taking the union over the subproblem interval solutions, producing interval bounds or envelopes for the set of all solutions associated with the interval vector initial condition x_0 and the perturbation parameter θ, including the effects of algorithmic computer truncation or rounding errors. We demonstrate the applicability of our results by considering three specific examples: an *RLC* circuit, an instrument servo-mechanism, and the design of a minimum plant sensitivity optimal regulator.

B. Background Material and Related Results

The general mathematical tools employed in the present three part paper are rather modest: they include some algebra (see, e.g., [3], [4]), some basic results from metric spaces (see, e.g., [5, chap. 5], [6]) and some background in ordinary differential equations (see, e.g., [7]).

In [8] and [9], Moore applies interval techniques to initial-value problems of nonlinear ordinary differential equations to determine *numerical error* inclusive bounds for solution trajectories; he does not consider trajectory bounds which also include the effects of a perturbation parameter in the differential equations. We emphasize that whereas our results are certainly related to existing works on interval analysis (see, e.g., [8]–[15], as well as the extensive bibliographies given in these references), as stated earlier, to the best of our knowledge, the results of the present three-part paper have not been reported elsewhere in form, scope, or generality.

C. Outline of the Present Paper

The present paper (i.e., Part I of the present three part paper) consists of seven sections. In Section II, we establish some basic notation and we note that interval arithmetic is not endowed with a rich algebraic structure. In Section III, we introduce some additional essential notation and we introduce a metric space as our appropriate mathematical setting. In Section IV, we explore some pertinent properties of continuous interval functions while in Section V we examine properties of rational interval functions. In Section VI, we extend the results of Sections III–V to matrix interval functions. Finally, the present paper is concluded with pertinent comments in Section VII.

II. Algebraic Structure

Let $\mathscr{T}$ denote the set of all intervals $[a,b]$, a, $b \in R$, $a \leqslant b$. When $a = b$, then we call $I = [a,a]$ a "degenerate" interval. On $\mathscr{T}$ we define the interval arithmetic operations $+, -, \cdot, /$ by

$$\begin{aligned} &[a,b]+[c,d]=[a+c,b+d]\\ &[a,b]-[c,d]=[a-d,b-c]\\ &[a,b]\cdot[c,d]=[\min(ac,ad,bc,bd),\max(ac,ad,bc,bd)]\\ &[a,b]/[c,d]=[a,b][1/d,1/c] \text{ provided that } 0\notin[c,d]. \end{aligned} \tag{1}$$

Interval addition and multiplication are each associative and commutative operations and the intervals [0,0] and [1,1] are their respective identity elements. However, if an interval has distinct endpoints, its inverse with respect to $+$ or $\cdot$ does not exist.

The mathematical systems $\{\mathscr{T}, +\}$ and $\{\mathscr{T}, \cdot\}$ are commutative semigroups with identities but fail as groups. Hence, the mathematical system $\{\mathscr{T}, +, \cdot\}$ will fail as a ring. Indeed, $\cdot$ is not distributive with respect to $+$ and thus $\{\mathscr{T}, +, \cdot\}$ fails as a ring for a second reason. On the other hand, if $I, J, K \in \mathscr{T}$, then

$$I\cdot(J+K) \subset I\cdot J + I\cdot K \triangleq IJ + JK \tag{2}$$

which is called the "subdistributivity property" of interval arithmetic. Also, if $*$ denotes any one of the operations defined above, if I, J, K, $L \in \mathscr{T}$ and if $I \subset K$ and $J \subset L$, then

$$I * J \subset K * L \tag{3}$$

provided that in the case of $/$, $0 \notin L$. This property of interval arithmetic is called *monotonic inclusion*.

Since $\{\mathscr{T}, +\}$ is not an Abelian group, interval arithmetic does not yield the structure of a linear space. However, using the set inclusion relation, it is possible to partially order $\mathscr{T}$ and by defining binary interval operations of *meet* and *join* (which conceptually parallel the inf and sup, respectively) it is possible to satisfy the structure of a lattice.

III. Topological Structure: Metric Spaces

From above it is clear that the interval arithmetic operations do not give rise to a particularly rich algebraic structure. Things are a little better for topological structure. The workable abstract mathematical setting which we will employ is that of metric space. To the best of the authors' knowledge, the theoretical functional analysis development presented in this paper as a generalization and extension of the concepts introduced in [8], [9] (without the burden of advanced topological concepts), does not appear in the literature [10]–[15].

Let $X = R^2$ and let D_∞ denote the infinity metric defined on X. Thus if $x = (x_1, x_2)$, $y = (y_1, y_2)$ then

$$D_\infty(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|).$$

The space $\{X, D_\infty\}$ is complete.

Proposition 1: Let X_1 be the subset of X defined by $X_1 = \{x \in X: x_1 \leqslant x_2\}$. Then $\{X_1, D_\infty\}$ is a complete metric subspace of $\{X, D_\infty\}$.

Proof: X_1 is the half plane above and including the line $x_1 = x_2$. Let the local neighborhoods of a point $x_0 \in X$ be the open spheres defined by

$$S(x_0, r) = \{x \in X: D_\infty(x, x_0) < r\}, \qquad r \in R, \qquad r > 0.$$

The complement of X_1, X_1^c, is open in X in the topology induced by the metric space $\{X, D_\infty\}$, since for every point $x \in X_1^c$, there exists an $\epsilon > 0$ such that $S(x, \epsilon) \subset X_1^c$. Therefore, X_1, the complement of X_1^c, is closed in this topology and hence, $\{X_1, D_\infty\}$ is a complete subspace of $\{X, D_\infty\}$. ■

Proposition 2: For each $I = [a, b]$, $J = [c, d]$ in $\mathscr{T}$, define ρ by

$$\rho(I, J) = \max(|a - c|, |b - d|). \tag{4}$$

The space $\{\mathscr{T}, \rho\}$ is a complete metric space.

Proof: ρ is clearly a metric on $\mathscr{T}$. Let f be the mapping f: $\mathscr{T} \to X_1$ defined by $f([a, b]) = (a, b)$. Then for each $I, J \in \mathscr{T}$, $D_\infty(f(I), f(J)) = \rho(I, J)$ and f is an isometry. Since $\{\mathscr{T}, \rho\}$ is isometric to the complete space $\{X_1, D_\infty\}$, it is complete. ■

Proposition 3: Let $X_2 \subset X_1$ be defined by $X_2 = \{x \in X_1: a \leqslant x_1 \leqslant x_2 \leqslant b, a, b \in R^1\}$. Then $\{X_2, D_\infty\}$ is a complete and compact metric subspace of $\{X_1, D_\infty\}$ and $\{X, D_\infty\}$

Proof: Since $X_2 \subset X_1 \subset X = R^2$ and X_2 is closed and bounded in X and X_1 in the topology induced by $\{X, D_\infty\}$ it is compact by the Borel theorem. But since $\{X, D_\infty\}$ is complete and X_2 is a closed set of X, $\{X_2, D_\infty\}$ is also complete. ■

Proposition 4: Let $I = [a, b] \in \mathscr{T}$ and let

$$\mathscr{T}_I = \{J \in \mathscr{T}: J \subset I\}. \tag{5}$$

Then $\{\mathscr{T}_I, \rho\}$ is a complete and compact metric subspace of $\{\mathscr{T}, \rho\}$.

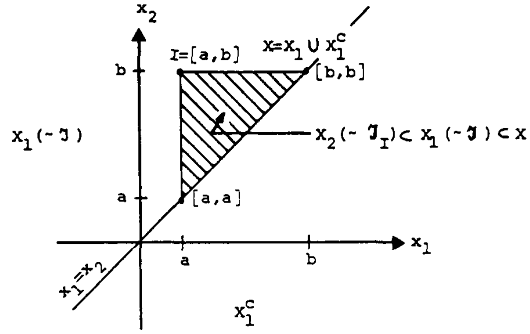


Fig. 1. Isometric representation of $\mathscr{T}$ and $\mathscr{T}_I$.

Proof: Since $\mathscr{T}_I \subset \mathscr{T}$, $\{\mathscr{T}_I, \rho\}$ is a subspace of $\{\mathscr{T}, \rho\}$. Using the isometry defined in Proposition 2, it follows that $\{\mathscr{T}_I, \rho\}$ is also complete and compact since it is isometric to the complete and compact metric space $\{X_2, D_\infty\}$ of Proposition 3. ■

Remark 1: a) Since the isometry f of Proposition 2 provides a convenient geometric interpretation for $\mathscr{T}$ and $\mathscr{T}_I$ in the plane $X = R^2$, this conveyance can be used to illustrate the various concepts involved and the notation for the elements of $\mathscr{T}$ will be used directly on the plane. The symbol $\tilde{}$ will occasionally be used to emphasize an isometric correspondence. Fig. 1 gives the isometric representation of $\mathscr{T}$ and $\mathscr{T}_I$ where $I = [a, b] \in \mathscr{T}$.

b) The real line R, when considered as the metric space $\{R, d\}$, where $d(a, b) = |a - b|$, $a, b \in R$, is isometrically embedded in $\{\mathscr{T}, \rho\}$ under the mapping g: $R \to \mathscr{T}$ defined by $g(a) = [a, a]$, an isometry because $\rho(g(a), g(b)) = d(a, b)$. In Fig. 1, the line $x_1 = x_2$ is the isometric image of the real line under fg and the degenerate intervals of $\mathscr{T}$ under f.

c) Fig. 2(a) and (b) respectively, illustrate interval addition and subtraction as geometric "vector additions" in $\{X_1, D_\infty\}$. The negative of the element $J \in \mathscr{T}$ provides an insight into the fact that, with respect to the operation of interval addition, the inverse of an element of $\mathscr{T}$ exists if and only if the element is "on the line" $x_1 = x_2$ (i.e., if the element is one of the degenerate intervals of $\mathscr{T}$).

d) Fig. 2(c) gives the geometric interpretation of the interval multiplication $I \cdot J \triangleq IJ$ as the unique element of $\mathscr{T}$ such that $tI \subset IJ$ for every $t \in J$ and $sI \not\subset IJ$ if $s \notin J$. For brevity in notation in the interval multiplication, tI, the degenerate interval $[t, t]$ has been indicated here simply by t. ■

IV. Continuous Interval Functions

We now address continuous interval functions. Specifically, let $\mathscr{F} = \{f | f: \mathscr{T}_I \to \mathscr{T}, f \text{ is continuous on } \mathscr{T}_I\}$. If $f, g \in \mathscr{F}$, let

$$\mu(f, g) = \sup_{J \in \mathscr{T}_I} \{\rho(f(J), g(J))\}. \tag{6}$$

Then μ is a metric on $\mathscr{F}$ (assume $f, g, h \in \mathscr{F}$) since

(i) by the symmetry of ρ, $\mu(f, g) = \mu(g, f)$;

(ii) since $\rho(J, K) = 0$ if and only if $J = K$, $\mu(f, g) = 0$ if and only if $f(J) = g(J)$ for every $J \in \mathscr{T}_I$ and then $f = g$;

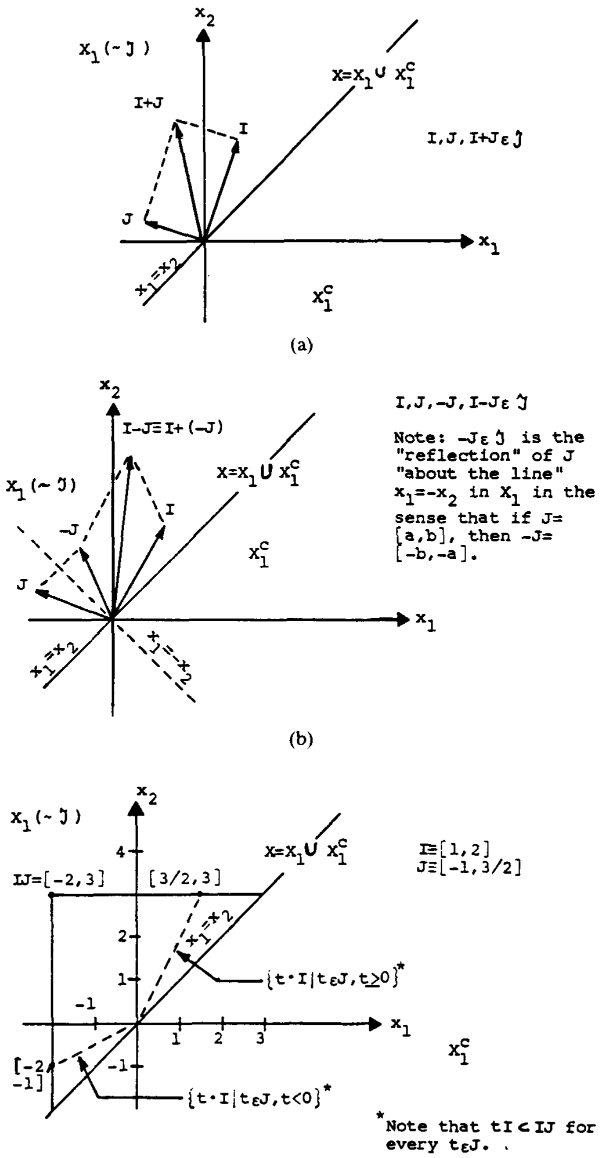


Fig. 2. (a) Isometric representation of interval addition. (b) Isometric representation of interval subtraction. (c) Isometric representation of interval multiplica.

and

(iii) by the triangle inequality for ρ,

$$\begin{aligned}\sup_{J \in \mathscr{T}_I} \{\rho(f(J), g(J))\} &\leq \sup_{J \in \mathscr{T}_I} \{\rho(f(J), h(J)) + \rho(h(J), g(J))\} \\ &\leq \sup_{J \in \mathscr{T}_I} \{\rho(f(J), h(J))\} \\ &\quad + \sup_{K \in \mathscr{T}_I} \{\rho(h(K), g(K))\}\end{aligned}$$

and, thus $\mu(f, g) \leq \mu(f, h) + \mu(h, g)$.

Thus $\{\mathscr{F}, \mu\}$ is a metric space of continuous interval functions.

Proposition 5: The metric space $\{\mathscr{F}, \mu\}$ is complete.

Proof: Let $\{f_n\}$ be an arbitrary Cauchy sequence in $\{\mathscr{F}, \mu\}$. This means that for any $\epsilon > 0$, there exists an $N_0(\epsilon)$ such that if $n, m \geq N_0(\epsilon)$, then

$$\rho(f_n(J), f_m(J)) \leq \mu(f_n, f_m) < \epsilon \qquad \text{for all } J \in \mathscr{T}_I.$$

In particular then, for fixed $J_0 \in \mathscr{T}_I$, $\{f_n(J_0)\}$ is a Cauchy sequence in $\{\mathscr{T}, \rho\}$. Since $\{\mathscr{T}, \rho\}$ is complete (Proposition 2), there exists an element $f(J_0) \in \mathscr{T}$ such that $\rho(f_n(J_0), f(J_0)) \to 0$ as $n \to \infty$. Applying the same argument for each $J \in \mathscr{T}_I$, obtain the function definition

$$f \triangleq f(J), J \in \mathscr{T}_I.$$

Since for $n, m \geq N_0(\epsilon)$, $\rho(f_n(J), f_m(J)) < \epsilon$ for each $J \in \mathscr{T}_I$, letting $m \to \infty$, it follows that

$$\rho(f_n(J), f(J)) \leq \epsilon \qquad \text{for each } J \in \mathscr{T}_I.$$

Since $N(\epsilon)$ is independent of J, it is therefore true that the sequence $\{f_n\}$ converges uniformly to f on $\mathscr{T}_I$.

By the triangle inequality, for any $J, J_0 \in \mathscr{T}_I$ and for any n,

$$\begin{aligned}\rho(f(J), f(J_0)) &\leq \rho(f(J), f_n(J)) \\ &\quad + \rho(f_n(J), f_n(J_0)) + \rho(f_n(J_0), f(J_0)).\end{aligned}$$

By the uniform convergence of the sequence $\{f_n\}$, the first and third terms can be made less than $\epsilon/3$ by choosing n sufficiently large. But since each f_n is continuous on $\mathscr{T}_I$ and $\mathscr{T}_I$ is compact, each f_n is also uniformly continuous on $\mathscr{T}_I$ and, therefore, there is a $\delta = \delta(\epsilon)$ such that if $\rho(J, J_0) < \delta(\epsilon)$, then $\rho(f_n(J_0), f_n(J)) < \epsilon/3$. Hence, $\rho(f(J), f(J_0)) < \epsilon$. This means that f is continuous at J_0 and since J_0 is arbitrary, it follows that $f \in \mathscr{F}$. Thus $\{\mathscr{F}, \mu\}$ is complete. ■

Remark 2: a) Since $\mathscr{T}_I$ and $\mathscr{T}$ include the degenerate intervals, embedded in $\mathscr{F}$ are the continuous real functions. Consequently, embedded in $\{\mathscr{F}, \mu\}$ is the metric space of continuous real functions $C[a, b]$ with sup metric $d, f, g \in C[a, b]$, $d(f, g) = \sup_{t \in [a, b]} |f(t) - g(t)|$.

b) Since each $f \in \mathscr{F}$ is continuous, since $\{\mathscr{T}_I, \rho\}$ and $\{\mathscr{T}, \rho\}$ are metric spaces and $\mathscr{T}_I$ is compact, then each $f \in \mathscr{F}$ is uniformly continuous on $\mathscr{T}_I$. ■

Remark 3: a) For any $f \in \{\mathscr{F}, \mu\}$, $J \in \mathscr{T}_I$, define the real values of $f^L(\cdot)$ and $f^R(\cdot)$ by

$$f(J) = [f^L(J), f^R(J)]. \tag{7}$$

b) For any $f \in \{\mathscr{F}, \mu\}$, $J \in \mathscr{T}_I$, define the *united extension of f* by

$$\bar{f}(J) \triangleq \bigcup_{x \in J} f([x, x]). \tag{8}$$

c) The above terminology in item b is used by Moore [8, p. 18]. He refers the reader to a definitive article on fixed point theorems for multivalued functions [16, p. 552]. Although this article has its foundations in the more advanced topological concepts of Peano spaces (c.f. compact Hansdorff spaces in [17]), it still does not precisely include our case considered herein. As will be seen, this matter takes on meaning at a considerably less abstract level of discussion.

d) In the following, it will be the practice to omit the terminology "united extension of f" and simply assume the correspondence f, $\bar{f}$ and $\{f_n\}$, $\{\bar{f}_n\}$. The following proposition gives significance to this definition in terms of the metric space adopted here. ■

Proposition 6: For arbitrary $f \in \{\mathscr{F}, \mu\}$ and $J \in \mathscr{T}_I$,

$$\bar{f}(J) \triangleq \bigcup_{x \in J} f([x,x]) = [p(J), q(J)] \in \mathscr{T} \quad (9)$$

where

$$p(J) = \inf_{x \in J} f^L([x,x]) \text{ and } q(J) = \sup_{x \in J} f^R([x,x]).$$

Proof: Clearly $\bigcup_{x \in J} f([x,x]) \subset [p(J), q(J)]$, since for every $x \in J$, it is true that $p(J) \leqslant f^L([x,x]) \leqslant f^R([x,x]) \leqslant q(J)$. Since f is continuous on $\mathscr{T}_I$ and $\mathscr{T}_I$ is compact, $J \in \mathscr{T}_I$ and $S \triangleq \{[x,x]: x \in J\} \subset \mathscr{T}_I$, the values

$$\inf_{t \in S} f^L(t) \triangleq p(J) \quad \text{and} \quad \sup_{t \in S} f^R(t) \triangleq q(J)$$

are each attained and $[p(J), q(J)] \in \mathscr{T}$. Thus it is true that

$$\bigcup_{x \in J} f([x,x]) \supset \{p(J), q(J)\}, \text{ the two points.}$$

It is then sufficient to show that every interior point of $[p(J), q(J)]$ is contained in $\bigcup_{x \in J} f([x,x])$. But this must be true since f is continuous on $\mathscr{T}_I$ and the subset S is connected (note $\bigcup_{t \in S} t = J$). Thus

$$\bigcup_{x \in J} f([x,x]) \supset [p(J), q(J)]$$

and since the set inclusion relation holds both ways, equality is obtained. ■

Proposition 7: For arbitrary $f \in \{\mathscr{F}, \mu\}$, $\bar{f} \in \{F, \mu\}$.

Proof: For continuity of $\bar{f}$ on $\mathscr{T}_I$, it is necessary and sufficient to show that for each $J \in \mathscr{T}_I$, each sequence $\{J_n\} \in \mathscr{T}_I$, $\{J_n\} \to J$, it always follows that $\{\bar{f}(J_n)\} \to \bar{f}(J)$.

Suppose that for some $J \in \mathscr{T}_I$, $\{J_n\} \in \mathscr{T}_I$, with $\{J_n\} \to J$, it occurs that $\{\bar{f}(J_n) - \} \nrightarrow \bar{f}(J)$. By Proposition 6, this means that $\{[p(J_n), q(J_n)\} \nrightarrow [p(J), q(J)]$. The sets $S \triangleq \{[x,x]: x \in J\}$ and $S_n \triangleq \{[x,x]: x \in J_n\}$, $n = 1, 2, \cdots$ are each individually connected subsets of the compact set $\mathscr{T}_I$. But f is continuous on $\mathscr{T}_I$ and also on its subsets and by the compactness of $\mathscr{T}_I$, the inf and sup of $f^L([x,x])$ and $f^R([x,x])$, respectively, are attained for each set S and S_n. Thus $\{[p(J_n), q(J_n)]\} \nrightarrow [p(J), q(J)]$ must mean that there are some points $[y,y]$ which do not belong to both S and the S_n's, for an infinite number of the n's. But since

$$\bigcup_{t \in S} t = J \quad \text{and} \quad \bigcup_{t \in S_n} t = J_n, \qquad n = 1, 2, \cdots$$

this implies that $\{J_n\} \nrightarrow J$, which is a contradiction. Hence $\{\bar{f}(J_n)\} \to \bar{f}(J)$ and this is true for every $J \in \mathscr{T}_I$, $\{J_n\} \in \mathscr{T}_I$, whenever $\{J_n\} \to J$. Thus $\bar{f}$ is continuous on $\mathscr{T}_I$ and belongs to $\{\mathscr{F}, \mu\}$. Since f is arbitrary, the above is true for every $\bar{f}$. ■

V. Rational Interval Functions

An interval function will be called a *rational interval function* if it is defined and can be expressed as a rational interval arithmetic expression in the interval variable and a finite set of constant coefficient intervals.

Proposition 8: The rational interval functions belong to $\{\mathscr{F}, \mu\}$.

Proof: Let $Y = \mathscr{T}_1 \times \mathscr{T}_2$ where $\mathscr{T}_1 = \mathscr{T}_2 = \mathscr{T}$ (i.e., the Cartesian product space $\mathscr{T} \times \mathscr{T}$). Denote the elements of Y by the 2-tuple (I, J), $I \in \mathscr{T}_1, J \in \mathscr{T}_2$. Then for $(I,J), (K,L) \in Y$, induce the metric

$$\rho_Y((I,J),(K,L)) = \max\{\rho(I,K), \rho(J,L)\}$$

where ρ is defined as before on $\mathscr{T}$. The completeness of $\{Y, \rho_Y\}$ as a metric space follows directly from the completeness of $\{\mathscr{T}, \rho\}$.

It is obvious that the interval arithmetic operations defined earlier are mappings h: $Y \to \mathscr{T}$ defined by $h((I,J)) = I * J$ and for arbitrary $(I,J) \in Y$, if $\{(I_n, J_n)\} \to (I,J)$, then $\{I_n * J_n\} \to I * J$ in $\{\mathscr{T}, \rho\}$. (*Remark*: If $*$ indicates division and 0 belongs to any of the intervals in the set $\{J, J_n, n = 1, 2, \cdots\} \subset \mathscr{T}$, the operation is not defined and will, therefore, be considered as an exception. Since it is assumed that the rational interval function is defined, this precludes the occurrence of such an exception.) Therefore, the interval arithmetic operations are continuous.

Since the rational interval functions may at most consist of a finite number of interval arithmetic operations, by repeated use of the continuity of the composition of the continuous interval arithmetic operations, the rational interval functions are continuous and hence belong to $\{\mathscr{F}, \mu\}$. ■

Proposition 9: For any rational interval function $f \in \{\mathscr{F}, \mu\}$ and arbitrary $J \in \mathscr{T}_I$,

$$f(J) \supset \bar{f}(J) \triangleq \bigcup_{x \in J} f([x,x]).$$

Proof: The result is inherently obvious from the monotonic inclusion property of the interval arithmetic operations and the definition of the rational interval function. Since a finite number of these operations is involved and since for every $x \in J$, $[x,x] \subset J$, $[x,x] \in \mathscr{T}_I$, then $f(J) \supset f([x,x])$.

To prove that equality may not be achieved, it is sufficient to demonstrate that the set inclusion may not go the other direction. Let $J = [-1/2, 1]$, $f(J) = J^2 = J \cdot J$. Then

$$f(J) = f([-1/2,1]) = [-1/2,1] \not\subset \bar{f}(J)$$
$$= \bar{f}([-1/2,1]) = [0,1]. \quad ■$$

Corollary 10: For any rational interval function $f \in \{\mathscr{F}, \mu\}$ and arbitrary $J \in \mathscr{T}_I$, if $K \in \mathscr{T}_I$, and $J \supset K$, then $f(J) \supset f(K) \supset \bar{f}(K)$.

Proof: By Proposition 9, $f(K) \supset \bar{f}(K)$. For the definition of the rational interval function and the monotonic inclusion property of interval arithmetic it follows that $f(J) \supset f(K)$. Equality may be precluded by demonstrating that for $J=[-1/2,1]$ $K=[-1/4,1]$ and $f(J)=J^2=J\cdot J$, $f(J)=f([-1/2,1])=[-1/2,1]\not\subset f(K)=f([-1/4,1])=[-1/4,1]$. ■

Proposition 10: Let $\{f_n\}\to f$ be an arbitrary Cauchy sequence in $\{\mathscr{F},\mu\}$. Then $\{\bar{f}_n\}$ is a Cauchy sequence in $\{\mathscr{F},\mu\}$ converging uniformly to $\bar{f}\in\{\mathscr{F},\mu\}$.

Proof: Since $\{f_n\}$ is a Cauchy sequence, given $\varepsilon>0$, there exists $N(\epsilon)$ such that for all $n,m\geqslant N(\epsilon)$, $\mu(f_n,f_m)<\epsilon$. But

$$\mu(f_n,f_m)=\sup_{J\in\mathscr{T}_I}\{\rho(f_n(J),f_m(J))\}<\epsilon.$$

Hence, for any $J\in\mathscr{T}_I$, whenever $n,m\geqslant N(\epsilon)$,

$$p(f_n(J),f_m(J))<\epsilon.$$

Now each $\bar{f}_n$ is a continuous function in $\{\mathscr{F},\mu\}$ (Proposition 7). For each $J\in\mathscr{T}_I$ and for each n,

$$\bar{f}_n(J)=\bigcup_{x\in J}f([x,x])=[p_n(J),q_n(J)]$$
$$=\left[\inf_{x\in J}f_n^L([x,x]),\ \sup_{x\in J}f_n^R([x,x])\right]$$

where

$$f_n([x,x])=[f_n^L([x,x]),f_n^R([x,x])]$$

Consider then for all $n,m\geqslant N(\epsilon)$ and for any $J\in\mathscr{T}_I$,

$$\rho(\bar{f}_n(J),\bar{f}_m(J))$$
$$=\max\left\{\left|\inf_{x\in J}f_n^L([x,x])-\inf_{x\in J}f_m^L([x,x])\right|,\right.$$
$$\left.\left|\sup f_n^R([x,x])-\sup f_m^R([x,x])\right|\right\}.$$

Since $\{[x,x]\colon x\in J\}\subset\mathscr{T}_I$, then for every element $[x,x]$ of this set (by virtue of $\mu(f_n,f_m)<\epsilon$), it is true that

$$\rho(f_n([x,x]),f_m([x,x]))$$
$$=\max\{|f_n^L([x,x])-f_m^L([x,x])|,$$
$$|f_n^R([x,x])-f_m^R([x,x])|\}<\epsilon.$$

Thus with respect to the set $\{[x,x]\colon x\in J\}$ and for any $J\in\mathscr{T}_I$, the pairs of real numbers $\{f_n^L([x,x]),\ f_m^L([x,x])\}$ and $\{f_n^R([x,x]),f_m^R([x,x])\}$ are individually never separated by a distance greater than ϵ. Thus it must be that $\rho(\bar{f}_n(J),\bar{f}_m(J))<\epsilon$ for any $J\in\mathscr{T}_I$ whenever $n,m\geqslant N(\epsilon)$. Therefore, $\{\bar{f}_n\}$ is a Cauchy sequence in $\{\mathscr{F},\mu\}$ and since the space is complete, this sequence converges to an element of the space. (Note that this convergence is true pointwise for every $J\in\mathscr{T}_I$ and that the sequence also converges uniformly.) Denote this element $\bar{g}$ and suppose that $\bar{g}\notin\bar{f}$. Then for at least one $\epsilon_1>0$, one $J\in\mathscr{T}_I$, and any $N_1>0$, for an infinite number of $n\geqslant N_1$ $\rho(\bar{f}_n(J),\bar{f}(J))>\epsilon_1$ and this occurs as $\{f_n\}\to f$ uniformly on $\mathscr{T}_I$. But for $\epsilon_1>0$, there exists an $N_2(\epsilon_1)$ such that if $n,m\geqslant N_2(\epsilon_1)$, then $\mu(f_n,f_m)<\epsilon_1$. Thus letting $m\to\infty$, $\mu(f_n,f)<\epsilon_1$. But this implies that for each $J\in\mathscr{T}_I$, $\rho(\bar{f}_n(J),\bar{f}(J))<\epsilon_1$ and this is a contradiction. Hence, it must be that $\bar{g}$ is arbitrarily close to $\bar{f}$ and, therefore, $\bar{g}=\bar{f}$. ■

Proposition 11: Let $\{f_n\}\to f$ be an arbitrary Cauchy sequence of rational interval functions in $\{\mathscr{F},\mu\}$. Then for each $J\in\mathscr{T}_I$, $f(J)\supset\bar{f}(J)$.

Proof: By Proposition 10, $\{\bar{f}_n\}$ is a Cauchy sequence in $\{\mathscr{F},\mu\}$ which converges uniformly to $\bar{f}\in\{\mathscr{F},\mu\}$. By Proposition 9, for each $J\in\mathscr{T}_I$ and for each n, $f_n(J)\supset\bar{f}_n(J)$. For any $J\in\mathscr{T}_I$, in the notation of Proposition 6, let

$$f_n(J)=[f_n^L(J),f_n^R(J)],\qquad f(J)=[f^L(J),f^R(J)]$$
$$\bar{f}_n(J)=[p_n(J),q_n(J)],\qquad \bar{f}(J)=[p(J),q(J)].$$

By Proposition 9 then, for every n,

$$f_n^L(J)\leqslant p_n(J)\leqslant q_n(J)\leqslant f_n^R(J).$$

Suppose $f(J)\not\supset\bar{f}(J)$. Then either $p(J)<f^L(J)$, $f^R(J)<q(J)$ or both occur. Suppose $p(J)<f^L(J)$. Then in fact there is an $\epsilon>0$ such that $p(J)+3\epsilon=f^L(J)$. But the sequences $\{p_n(J)\}$ and $\{f_n^L(J)\}$ are each convergent sequences in the reals since the sequences $\{f_n\}$ and $\{\bar{f}_n\}$ are Cauchy sequences in $\{\mathscr{F},\mu\}$ and for each $n,f_n^L(J)\leqslant p_n(J)$ and thus the contrary assumption above cannot occur and $f(J)\leqslant p(J)$. Similarly $q(J)\leqslant f^R(J)$ and both of these relations hold for each $J\in\mathscr{T}_I$. Thus $f(J)\supset\bar{f}(J)$ for every $J\in\mathscr{T}_I$. ■

Corollary 11: If $\{f_n\}\to f$ is an arbitrary Cauchy sequence of rational interval functions in $\{\mathscr{F},\mu\}$ then for each $J\in\mathscr{T}_I$, if $K\in\mathscr{T}_I$, and $J\supset K$,

$$f(J)\supset f(K)\supset\bar{f}(K).$$

Proof: The proof follows readily: $f(K)\supset\bar{f}(K)$ by Proposition 11 and employing Corollary 9, we argue in a manner similar to the proof of Proposition 11. ■

Proposition 12: Let $f\in\{\mathscr{F},\mu\}$ have the property that for any $J\in\mathscr{T}_I$, $f(J)\supset\bar{f}(J)$. Let $J=[d_1,d_{n+1}]$ and $J_i=[d_i,d_{i+1}]\in\mathscr{T}_I$, $d_i\leqslant d_{i+1}$, $i=1,\cdots,n$. Then

$$\bigcup_{i=1}^n f(J_i)\supset\bar{f}(J)\quad\text{and}\quad\bigcup_{i=1}^n f(J_i)\in\mathscr{T}.$$

Proof: By assumption,

$$\bigcup_{i=1}^n f(J_i)\supset\bigcup_{i=1}^n\bar{f}(J_i)=\bigcup_{i=1}^n\Big(\bigcup_{x\in J_i}f([x,x])\Big).$$

But $f\in\{\mathscr{F},\mu\}$ and therefore

$$\bigcup_{i=1}^n\Big(\bigcup_{x\in J_i}f([x,x])\Big)=\bigcup_{x\in J}f([x,x])=\bar{f}(J).$$

Also, since $f(J_i)\in\mathscr{T}$ and $f(J_i)\supset\bar{f}(J_i)\in\mathscr{T}$, $i=1,\cdots,n$ and

$$\bar{f}(J)=\bigcup_{i=1}^n f(J_i)\in\mathscr{T}$$

then obviously $\bigcup_{i=1}^n f(J_i)\in\mathscr{T}$. ■

Proposition 13: Let $\{f_k\}\to f$ be any Cauchy sequence of rational interval functions in $\{\mathscr{F},\mu\}$. For any $J\in\mathscr{T}_I$, let $J=[d_1,d_{n+1}]$ and $J_i=[d_i,d_{i+1}]\in\mathscr{T}_I$, $d_i\leqslant d_{i+1}$, $i=$

$1,\cdots,n$. Then

$$f_k(J) \supset \bigcup_{i=1}^{n} f_k(J_i) \supset \bar{f}_k(J), \qquad k=1,2,\cdots$$

and this relation converges to the relation

$$f(J) \supset \bigcup_{i=1}^{n} f(J_i) \supset \bar{f}(J).$$

Proof: By Corollary 9, for each $i=1,\cdots,n$, since $J \supset J_i$,

$$f_k(J) \supset f_k(J_i) \supset \bar{f}_k(J_i), \qquad k=1,2,\cdots$$

By Proposition 12 then,

$$f_k(J) \supset \bigcup_{i=1}^{n} f_k(J_i) \supset \bar{f}_k(J), \qquad k=1,2,\cdots$$

and this is true for any $J \in \mathscr{T}_I$ and any n in the assumed method of partitioning of J.

By Corollary 11, for each $i=1,\cdots,n$, since $J \supset J_i$, $f(J) \supset f(J_i) \supset \bar{f}(J_i)$. By Proposition 12 again,

$$f(J) \supset \bigcup_{i=1}^{n} f(J_i) \supset \bar{f}(J). \quad \blacksquare$$

We are now in a position to state and prove one of the principal results of this paper.

Theorem 14: Let $\{f_k\} \to f$ be any Cauchy sequence of rational interval functions in $\{\mathscr{F},\mu\}$. For any $J \in \mathscr{T}_I$, let $J=[c,d]$ and

$$J_i^n = \left[\frac{(n-i+1)c+(i-1)d}{n}, \frac{(n-i)c+id}{n}\right], \qquad i=1,\cdots,n;\ n=1,2,\cdots$$

Then

$$f(J) \supset \lim_{n\to\infty} \bigcup_{i=1}^{n} f(J_i^n) = \bar{f}(J).$$

Proof: Since for any $J \in \mathscr{T}_I$ and for any n, Proposition 13 provides that

$$f(J) \supset \bigcup_{i=1}^{n} f(J_i^n) \supset \bar{f}(J)$$

it is obvious that

$$f(J) \supset \lim_{n\to\infty} \bigcup_{i=1}^{n} f(J_i^n) \supset \bar{f}(J).$$

For arbitrary $\epsilon > 0$, denote the closed ϵ-sphere in $\mathscr{T}$ by $\bar{S}(\bar{f}(J),\epsilon) = \{K \in \mathscr{T}: \rho(f(J),K) \leqslant \epsilon\}$. Let

$$\bar{f}_\epsilon(J) = \{K \in \bar{S}(\bar{f}(J),\epsilon): t \subset K \text{ for every } t \in \bar{S}(\bar{f}(J),\epsilon)\}.$$

This is a unique element in $\mathscr{T}$, since $\bar{S}(\bar{f}(J),\epsilon)$ is a closed and bounded collection of closed intervals in $\mathscr{T}$ and as such, each of its members must be a subset of some maximal closed interval which belongs to the collection, namely, $\bar{f}_\epsilon(J)$.

Thus $\bar{f}_\epsilon(J)$ has been selected so that

$$\rho(\bar{f}_\epsilon(J),\bar{f}(J)) = \epsilon,\ \bar{f}_\epsilon(J) \supset \bar{f}(J) \text{ and } \bar{f}_\epsilon(J) \supset t$$

for every $t \in \bar{S}(\bar{f}(J),\epsilon)$. But it is now possible to claim that there is a number $N(\epsilon)$ such that if $n \geqslant N(\epsilon)$, then

$$\bar{f}_\epsilon(J) \supset \bigcup_{i=1}^{n} f(J_i^n) \supset \bar{f}(J)$$

and

$$\rho\left(\bigcup_{i=1}^{n} f(J_i^n), \bar{f}(J)\right) < \epsilon.$$

For by the uniform continuity of f on $\mathscr{T}_I$, it is possible to select $N(\epsilon)$ so that if $n \geqslant N(\epsilon)$, J has been partitioned into J_i^n's such that for each i and for every $x \in J_i^n$,

$$\rho(J_i^n,[x,x]) \leqslant \frac{d-c}{N(\epsilon)} < \delta(\epsilon)$$

and this implies that for each i and every $x \in J_i^n$,

$$\rho(f(J_i^n), f([x,x])) < \epsilon.$$

But the immediate consequence of this produces the second part of the claim,

$$\rho\left(\bigcup_{i=1}^{n} f(J_i^n), \bar{f}(J)\right) < \epsilon.$$

Since

$$\rho(\bar{f}_\epsilon(J),\bar{f}(J)) \triangleq \epsilon,\ \bar{f}_\epsilon(J) \supset \bar{f}(J)$$

$$\rho\left(\bigcup_{i=1}^{n} f(J_i^n), \bar{f}(J)\right) < \epsilon,$$

$$\bigcup_{i=1}^{n} f(J_i^n) \supset \bar{f}(J) \quad \text{and} \quad \bar{f}_\epsilon(J) \supset t$$

for every $t \in \bar{S}(\bar{f}(J),\epsilon)$, complete the claim obtaining

$$\bar{f}_\epsilon(J) \supset \cup_{i=1}^{n} f(J_i^n) \supset \bar{f}(J).$$

Thus it is always possible to select n sufficiently large so that no matter how small the number $\epsilon > 0$,

$$\bigcup_{i=1}^{n} f(J_i^n) \quad \text{is within } \epsilon \text{ of } \bar{f}(J)$$

and this is possible for each $J \in \mathscr{T}_I$. $\blacksquare$

Remark 4: In other words, Theorem 14 implies that by using a sufficiently large number in the specified partition of J and computing the union of the interval function over the partition subintervals, it is possible to approximate the exact range of the interval function for $x \in J$, $\bar{f}(J)$, as closely as desired.

VI. Matrix Interval Functions

We now extend the results presented thus far to matrix-valued interval functions. In the following, we identify the generalized results by the same corresponding number designations for propositions, corollaries, etc., except that the letter M will be appended to indicate the matrix case.

Define

$$\mathscr{T}^{n^2} = \{\text{set of all } n \times n \text{ matrices where each of its } n^2 \text{ elements belong to } \mathscr{T}\}. \quad (10)$$

Denote elements of $\mathscr{T}^{n^2}$ by the matrix notation

$$A = ((a_{ij})) = (([a_{ij}^L, a_{ij}^R])), \qquad i,j = 1,2,\cdots,n.$$

Let $A, B, C \in \mathscr{T}^{n^2}$ and let $\boldsymbol{u}$ be a fixed positive real vector

$$\boldsymbol{u} = (u_1, u_2, \cdots, u_n) \in R^n, u_i > 0, \qquad i = 1,\cdots,n.$$

Define

$$\sigma(A,B) = \max_i \left\{ \frac{1}{u_i} \sum_{j=1}^n u_j \rho(a_{ij}, b_{ij}) \right\} \quad (11)$$

where ρ is the metric of $\{\mathscr{T}, \rho\}$. Then σ is a metric and $\{\mathscr{T}^{n^2}, \sigma\}$ is a metric space, since

(i) by the symmetry of $\rho, \sigma(A,B) = \sigma(B,A)$

(ii) since $\rho(a_{ij}, b_{ij}) = 0$ if and only if $a_{ij} = b_{ij}$, $\sigma(A,B) = 0$ if and only if $A = B$, and

(iii) by the triangle property for ρ, we have

$$\sigma(A,B) \leqslant \sigma(A,B) + \sigma(C,B).$$

More will be said about the fixed real vector $\boldsymbol{u}$ in part 2 of the present three part paper (i.e., in [1]).

Proposition 2M: The metric space $\{\mathscr{T}^{n^2}, \sigma\}$ is complete.

Proof: Let $\{A_k\}$ be an arbitrary Cauchy sequence in $\{\mathscr{T}^{n^2}, \sigma\}$. Since $\boldsymbol{u}$ is a fixed positive vector, define the fixed constant

$$0 < Q = \min_{i,j} \left\{ \frac{u_j}{u_i} \right\} \leqslant 1.$$

For the Cauchy sequence $\{A_k\}$, given any $\epsilon > 0$ and letting $\epsilon_1 = Q\epsilon$, there exists an $N(\epsilon_1)$ such that for all $k, l \geqslant N(\epsilon_1)$,

$$\sigma(A_k, A_l) = \max_i \left\{ \frac{1}{u_i} \sum_{j=1}^n u_j \rho(a_{ij_k}, a_{ij_l}) \right\} < \epsilon_1.$$

But then for each i, j and for all $k, l \geqslant N(\epsilon_1)$,

$$\frac{u_j}{u_i} \rho(a_{ij_k}, a_{ij_l}) < \epsilon_1 \quad \text{and} \quad \rho(a_{ij_k}, a_{ij_l}) < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, for each $i, j, \{a_{ij_k}\}$ is a Cauchy sequence in $\{\mathscr{T}, \rho\}$ and since that metric space is complete,

$$\{a_{ij_k}\} \to a_{ij} \in \mathscr{T}.$$

Thus

$$\{A_k\} \to A \in \mathscr{T}^{n^2}$$

and therefore $\{\mathscr{T}^{n^2}, \sigma\}$ is complete since $\{A_k\}$ is an arbitrary Cauchy sequence in $\{\mathscr{T}^{n^2}, \sigma\}$. ■

Remark 5: The subsequent interval matrix function results can be proved in a manner similar to the preceding "scalar" cases. Accordingly, the proofs of the remaining results will not be given here.

Let

$$\mathscr{F}^{n^2} = \{F | F: \mathscr{T}_I \to \mathscr{T}^{n^2}, F \text{ is continuous on } \mathscr{T}_I\}.$$

If $F, G, H \in \mathscr{F}^{n^2}$, let

$$\zeta(F,G) = \sup_{J \in \mathscr{T}_I} \{\sigma(F(J), G(J))\}. \quad (12)$$

Then ζ is a metric, since

(i) by the symmetry of $\sigma, \zeta(F,G) = \zeta(G,F)$,

(ii) since $\sigma(F(J), G(J)) = 0$ if and only if $F(J) = G(J)$,

$\zeta(F,G) = 0$ if and only if $F = G$, and

(iii)

$$\begin{aligned}
&\sup_{J \in \mathscr{T}_I} \{\sigma(F(J), G(J))\} \\
&\quad \leqslant \sup_{J \in \mathscr{T}_I} \{\sigma(F(J), H(J)) + \sigma(H(J), G(J))\} \\
&\quad \leqslant \sup_{J \in \mathscr{T}_I} \{\sigma(F(J), H(J))\} \\
&\qquad + \sup_{K \in \mathscr{T}_I} \{\sigma(H(K), G(K))\}
\end{aligned}$$

that is, $\zeta(F,G) \leqslant \zeta(F,H) + \zeta(H,G)$. Therefore, $\{\mathscr{F}^{n^2}, \zeta\}$ is the metric space of continuous interval matrix functions.

Proposition 5M: The metric space $(\mathscr{F}^{n^2}, \zeta\}$ is complete. ■

Proposition 6M: For arbitrary $F \in \mathscr{F}^{n^2}$ and $J \in \mathscr{T}_I$,

$$\bar{F}(J) \triangleq \bigcup_{x \in J} F([x,x]) = (([p_{ij}(J), q_{ij}(J)])) \in \mathscr{T}^{n^2}$$

where for each i, j

$$F(J) = (([f_{ij}^L(J), f_{ij}^R(J)])),$$

$$p_{ij}(J) = \inf_{x \in J} f_{ij}^L([x,x])$$

and

$$q_{ij}(J) = \sup_{x \in J} f_{ij}^R([x,x]). \quad ■$$

Proposition 7M: For arbitrary $F \in \{\mathscr{F}^{n^2}, \zeta\}$, $\bar{F} \in \{\mathscr{F}^{n^2}, \zeta\}$. ■

Proposition 8M: The rational interval matrix functions belong to $\{\mathscr{F}^{n^2}, \zeta\}$. ■

Proposition 9M: For any rational interval matrix function $F \in \{\mathscr{F}^{n^2}, \zeta\}$ and arbitrary $J \in \mathscr{T}_I$

$$F(J) \supset \bar{F}(J) \triangleq \bigcup_{x \in J} F([x,x]). \quad ■$$

Corollary 9M: For any rational interval matrix function $F \in \{\mathscr{T}^{n^2}, \xi\}$ and arbitrary $J \in \mathscr{T}_I$, if $K \in \mathscr{T}_I$ and $J \supset K$, then

$$F(J) \supset F(K) \supset \bar{F}(K). \quad ■$$

Proposition 10M: Let $\{F_k\} \to F$ be an arbitrary Cauchy sequence in $\{\mathscr{F}^{n^2}, \zeta\}$. Then $\{\bar{F}_k\}$ is a Cauchy sequence in $\{\mathscr{F}^{n^2}, \zeta\}$ converging uniformly to $F \in \{\mathscr{F}^{n^2}, \zeta\}$. ■

Proposition 11M: Let $\{F_k\} \to F$ be an arbitrary Cauchy sequence of rational interval matrix functions in $\{\mathscr{F}^{n^2}, \zeta\}$. Then for each $J \in \mathscr{T}_I$, $F(J) \supset \bar{F}(J)$. ■

Corollary 11M: If $\{F_k\} \to F$ is an arbitrary Cauchy sequence of rational interval matrix functions in $\{\mathscr{F}^{n^2}, \zeta\}$, then for each $J \in \mathscr{T}_I$, if $K \in \mathscr{T}_I$ and $J \supset K$,

$$F(J) \supset F(K) \supset \bar{F}(K). \quad \blacksquare$$

Proposition 12M: Let $F \in \{\mathscr{F}^{n^2}, \zeta\}$ have the property that for any $J \in \mathscr{T}_I$, $F(J) \supset \bar{F}(J)$. Let $J = [d_1, d_{m+1}]$ and $J_i \triangleq [d_i, d_{i+1}] \in \mathscr{T}_I$, $d_i \leqslant d_{i+1}$, $i = 1, \cdots, m$. Then

$$\bigcup_{i=1}^{m} F(J_i) \supset \bar{F}(J) \quad \text{and} \quad \bigcup_{i=1}^{m} F(J_i) \in \mathscr{F}^{n^2}. \quad \blacksquare$$

Proposition 13M: Let $\{F_k\} \to F$ be any Cauchy sequence of rational interval matrix functions in $\{\mathscr{F}^{n^2}, \zeta\}$. For any $J \in \mathscr{T}_I$, let $J = [d_1, d_{m+1}]$ and $J_i = [d_i, d_{i+1}] \in \mathscr{T}_I$, $d_i \leqslant d_{i+1}$, $i = 1, \cdots, m$. Then

$$F_k(J) \supset \bigcup_{i=1}^{m} F_k(J_i) \supset \bar{F}_k(J), \qquad k = 1, 2, \cdots$$

and this relation converges to the relation

$$F(J) \supset \bigcup_{i=1}^{m} F(J_i) \supset \bar{F}(J). \quad \blacksquare$$

Theorem 14M: Let $\{F_k\} \to F$ be any Cauchy sequence of rational interval matrix functions in $\{\mathscr{F}^{n^2}, \zeta\}$. For any $J \in \mathscr{T}_I$, let $J \triangleq [c, d]$ and

$$J_i^m \triangleq \left[\frac{(m-i+1)c + (i-1)d}{m}, \frac{(m-i)c + id}{m} \right],$$
$$i = 1, \cdots, m; \text{ and } m = 1, 2, \cdots.$$

Then

$$F(J) \supset \lim_{m \to \infty} \bigcup_{i=1}^{m} F(J_i^m) = \bar{F}(J). \quad \blacksquare$$

Remark 6: Since the partial sums in the interval infinite matrix series representation of the interval matrix exponential function define a sequence of rational interval matrix functions, if this sequence is Cauchy, Theorem 14M provides a convergence philosophy for evaluating an interval fundamental matrix which element-wise contains the actual closed interval range of values for the perturbed fundamental matrix for all values of the perturbation parameter and does so as closely as desired.

While interval post-multiplication of this interval fundamental matrix by the interval vector initial condition in the linear homogeneous initial-value problem (described by ordinary differential equations) will certainly produce an interval vector solution bound or envelope for the set of all solutions associated with the parameter variation, it is not difficult to understand that this result will be unduly conservative.

This inconvenience may be recognized by observing that in forming the interval fundamental matrix, the union over the partition subintervals in Theorem 14M (or Proposition 13M) is elementwise independent and, therefore, the partition subinterval "signature" is lost within and between elements. Consequently, a less conservative interval vector bound solution for the set of all solutions associated with the perturbation parameter will be obtained if the union is performed before this "signature" is lost by taking the union over the interval vector solution bounds obtained for each partition subinterval. This in fact will be the technique that is used in the linear interval integration algorithm (in [1] and [2]) and since the interval vector initial condition is fixed, Theorem 14M provides the necessary convergence in this technique as well.

VII. Concluding Remarks

In the previous sections we first established some essential notation, we noted that the algebraic structure determined by the interval arithmetic operations is not particularly rich and we considered metric space as the appropriate setting for our work. Next, we studied continuous interval functions in general and rational interval functions in particular. In studying interval functions we considered the "scalar" interval functions and matrix interval functions. Although our primary goal was to establish Theorems 14 and 14M, we believe that all of the results of Sections III–VI are of interest in their own right.

In part 2 of the present three-part paper (i.e., in [1]), we employ Theorems 14 and 14M in studying "scalar" and matrix exponential interval functions, respectively. In part 3 of the present three-part paper (i.e., in [2]), we employ these results to study linear initial-value problems. As applications we consider worst-case studies of specific examples involving an *RLC* circuit, a servomechanism and an optimal regulator problem.

References

[1] E. P. Oppenheimer and A. N. Michel, "Application of interval analysis techniques to linear systems: Part II—The interval matrix exponential function," submitted for publication.

[2] ____, "Application of interval analysis techniques to linear systems: Part III—Initial-value problems," submitted for publication.

[3] N. Jacobsen, *Lectures in Abstract Algebra*. New York: Van Nostrand, 1951.

[4] C. B. Hanneken, *Introduction to Abstract Algebra*. Belmont, CA: Dickenson, 1968.

[5] A. N. Michel and C. J. Herget, *Mathematical Foundations in Engineering and Science: Algebra and Analysis*. Englewood Cliffs, NJ: Prentice-Hall, 1981.

[6] A. W. Naylor and G. R. Sell, *Linear Operator Theory in Engineering and Science*. New York: Holt, Rinehart and Winston, 1971.

[7] R. K. Miller and A. N. Michel, *Ordinary Differential Equations*. New York: Academic, 1982.

[8] R. E. More, *Interval Analysis*. Englewood Cliffs, NJ: Prentice-Hall, 1966.

[9] R. E. More, *Methods and Applications of Interval Analysis*, SIAM Studies in Applied Mathematics, Philadelphia, 1979.

[10] H. Ratschek and J. Rokne, *Computer Methods for Range of Functions*, Ellis Horwood, Ltd., Chicherter, West Sussex, England, N.Y., 1984.

[11] G. Alefeld and H. Herzberger, *Introduction to Interval Computations*. New York: Academic, 1983.

[12] U. W. Kulisch and W. L. Miranker, *Computer Arithmetic in Theory and Practice*. New York: Academic, 1981.

[13] Karl L. E. Nickel, Ed., *Interval Mathematics 1980*, in *Proc. Int. Symp. on Interval Mathematics*, Germany, May 27–31, 1980, New York: Academic, 1980.

[14] R. E. Moore, *Mathematical Elements of Scientific Computing*. New York: Holt, Rinehart and Winston, 1975.

[15] J. W. Daniel and R. E. Moore, *Computation and Theory in Ordinary Differential Equations*. San Francisco, CA, Freeman, 1970.

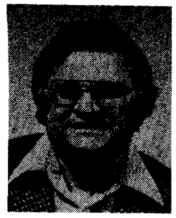
[16] W. Strother, "Fixed points, fixed sets and M-retracts," *Duke Math. J.*, vol. 22, no. 4, pp. 551–556, 1955.

[17] J. L. Kelly, *General Topology*. New York: Van Nostrand, 1955.

Edward P. Oppenheimer (M'57) was born in Moline, IL on June 6, 1932. He received the B.S., M.S., and Ph.D. degrees in electrical engineering from Iowa State University, Ames, in 1955, 1969, and 1974, respectively.

Dr. Oppenheimer joined the Johns Hopkins University Applied Physics Laboratory in 1974 where he performed analysis work on the Poseidon and Trident I weapons systems in the Strategic Systems Department. In 1979 he joined the Space Department where he was involved in early development analyses for the Trident II GPS satellite tracking system (SATRACK II). He is presently engaged in analysis work supporting various SDIO efforts.

Dr. Oppenheimer is a member of Sigma Xi.



Anthony N. Michel (S'55–M'59–SM'79–F'82) received the Ph.D. degree in electrical engineering from Marquette University and the D.Sc. degree in applied mathematics from the Technical University of Graz, Austria.

He has seven years of industrial experience. From 1968 to 1984 he was at Iowa State University, Ames. From 1984 to 1988 he was Frank M. Freimann Professor of Engineering and Chairman of the Department of Electrical Engineering, University of Notre Dame, IN. At present, he is Dean of the College of Engineering, University of Notre Dame, IN. He is author and co-author of three texts and several other publications.

Dr. Michel received the 1978 Best TRANSACTIONS Paper Award of the IEEE Control Systems Society (with R. D. Rasmussen), the 1984 Guillemin–Cauer Prize Paper Award of the IEEE Circuits and Systems Society (with R. K. Miller and B. H. Nam) and an IEEE Centennial Medal. He is a former Associate Editor and a former Editor of the IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS and a former Associate Editor for the IEEE TRANSACTIONS ON AUTOMATIC CONTROL. He was the Program Chairman of the 1985 IEEE Conference on Decision and Control. He is President-Elect of the IEEE Circuits and Systems Society.