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Input-Output Planning With Inexact Data

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INPUT-OUTPUT PLANNING WITH INEXACT DATA

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1. Introduction. In this paper, we deal with the input-output model from the point of view of the interval mathematics. No preliminary knowledge neither from economics, nor from the region of interval mathematics are required, because all the necessary facts are given below. After a short review of the used symbolics in the section 2, we give a brief description of the classical input-output model in the section 3. The next section 4 contains the formulation of the problem and some basic assumptions, too. The proper mathematical part of this paper begins in the section 5, where the fundamental definition of feasible solutions and two different ways of their description are given. In the section 6, we discuss optimal feasible solutions. Then in the section 7 some further properties of feasible solutions are given.

2. Symbolics. Some symbols used throughout the paper are explained in this section. Let $A = (a_{ij})$, $B = (b_{ij})$ be two mxn--matrices. We write $A \leq B$ ($A < B$) if $a_{ij} \leq b_{ij}$ ($a_{ij} < b_{ij}$) for $i = 1,\dots,m$, $j = 1,\dots,n$. The relation $B \geq A$ ($B > A$) is equivalent to $A \leq B$ ($A < B$). A matrix A is called nonnegative (positive) if $A \geq 0$ ($A > 0$), 0 being the zero matrix. Let $A \leq B$; then, the set $\{C \mid A \leq C \leq B\}$ is called interval and is denoted by $[A,B]$. Thus,

due to the definition, any interval $[A,B]$ is nonempty, but may be reduced to a single matrix if $A = B$. An interval $[A,B]$ is said to be nonnegative if A is nonnegative. A^T means the transpose of A. The same symbols are also used for vectors, which are treated as one-column matrices. E denotes the unit matrix, whereas e denotes the vector $(1,1,\dots,1)^T$.

3. The input-output model. Here, we give a brief account of the input-output model. For more detailed information, see [1] , [2].

The model describes a national economy which is supposed to be disaggregated to n sectors, $n \geq 2$, labeled by $1,2, \dots,n$. For each i, $i = 1, \dots,n$, the i-th sector produces a single kind of goods, which is also labeled by i. A part (possibly zero) of the gross output of each sector is consumed by some other sectors as inputs for their own productions; the amount of the i-th goods consumed by the j-th sector is supposed to be proportional to the gross output x_j of the j-th sector with the coefficient of proportionality a^o_{ij}. Thus the total amount of the i-th goods consumed for production purposes within the national economy is equal to $\sum_{j=1}^{n} a^o_{ij} x_j$. Hence, we have

$$x_i = \sum_{j=1}^{n} a^o_{ij} x_j + y_i , \qquad i = 1,2, \dots,n,$$

where, for each i, x_i denotes the gross output and y_i the net output of the i-th sector (measured, as usual, in monetary units). The a^o_{ij}'s are called input coefficients and are assumed to be constant. Taking $A_o = (a^o_{ij})$, $x = (x_i)$, $y = (y_i)$, we can put the above system into the form

$$(E - A_o)x = y \tag{1}$$

which is the basic equation in the Leontief's input-output analysis. Clearly, A_o and x are nonnegative and so is y if the workability of all sectors is supposed. The profitability of the j-th sector means that the value of its production is greater than that of components used, i.e.

$$x_j > \sum_{i=1}^{n} a_{ij}^{o} x_i,$$

which leads to

$$1 > \sum_{i=1}^{n} a_{ij}^{o}.$$

Thus, using the vector $e = (1,1,\dots,1)^T$, introduced in the section 2, the assumption of profitability of all sectors can be written as

$$e^T(E - A_o) > o^T.$$

As we shall see in the next section, the two conditions

$$(I_o) \quad A_o \geqslant 0$$

$$(II_o) \quad e^T(E - A_o) > o^T$$

imply the nonnegative invertibility of the matrix $E - A_o$.

For a given net output $y = (y_i)$, the number $\sum_{i=1}^{n} y_i = e^T y$ is called the national income.

The model (1) is used mainly for planning to solve the two main problems: (i) to find a gross output x which yields a given net output y, (ii) to find a net output y corresponding to a given gross output x. In the sequel, we shall consider the problem (i) only.

4. Basic assumptions. From the point of view of the model (1), the national economy is completely characterized by the matrix A_o. But, in practice, it appears to be very difficult to find out the exact values of input coefficients, because the data from which they are determined often fail to be both exact and complete. In this paper, we make an attempt to take this fact into account assuming for each $i,j = 1,2, \dots, n$ we know only a real interval

$[\underline{a}_{ij}, \overline{a}_{ij}]$ containing a^o_{ij}, but not the value a^o_{ij} itself. Taking $\underline{A} = (\underline{a}_{ij})$, $\overline{A} = (\overline{a}_{ij})$, we thus have an interval $[\underline{A},\overline{A}]$containing the unknown matrix A_o. We assume this interval to be sufficiently "narrow" so that the conditions (I_o), (II_o) from the preceding section remain valid for any $A \in [\underline{A},\overline{A}]$. It can be easily seen that an interval $[\underline{A},\overline{A}]$ has this property if and only if it satisfies the assumptions

(I) $[\underline{A},\overline{A}]$ is a nonnegative interval containing A_o

(II) $e^T(E - \overline{A}) > 0^T$.

Now, for a given gross output x, we are not able to say more than that the corresponding net output y belongs to the set $\{(E - A)x \mid A \in [\underline{A},\overline{A}]\}$, so that the problem (i) mentioned at the end of section 3 is no more solvable. Therefore, we assume that instead of a fixed desired net output y, an interval $[\underline{y},\overline{y}]$ satisfying the assumption

(III) $[\underline{y},\overline{y}]$ is a nonnegative interval

is given and we want to find out such a gross output x to be able to be sure that the corresponding net output $y = (E - A_o)x$ will lie in $[\underline{y},\overline{y}]$. This leads to the definition of a feasible solution, which is given in the next section.

The assumptions (I), (II), (III) are called basic assumptions. We draw some conclusions of them in the following

<u>Corollary.</u> Let (I), (II) be satisfied. Then, (i), (ii), (iii) hold:

(i) $E - A$ is nonnegatively invertible for each $A \in [\underline{A},\overline{A}]$

(ii) $(E - A)^{-1} \in [(E - \underline{A})^{-1}, (E - \overline{A})^{-1}]$ for each $A \in [\underline{A},\overline{A}]$

(iii) $(E - \underline{A})(E - \overline{A})^{-1} \geqslant E$.

<u>Proof.</u> Using the norm $\|A\| = \max_j \sum_i |a_{ij}|$, we have $\|A\| \leqslant \leqslant \max_j \sum_i \overline{a}_{ij} = \max_j (e^T\overline{A})_j < 1$ for each $A \in [\underline{A},\overline{A}]$. Hence $(E - A)^{-1} =$

$= \sum_{j=0}^{\infty} A^j$ due to the well-known theorem, which in view of the nonnegativity of A implies (i). Because the inequalities $0 \leq \underline{A} \leq \leq A \leq \overline{A}$ imply $\underline{A}^j \leq A^j \leq \overline{A}^j$ for $j = 1,2,\ldots$, we have $(E - \underline{A})^{-1} \leq \leq (E - A)^{-1} \leq (E - \overline{A})^{-1}$, which is (ii). At last, $(E - \underline{A})(E - \overline{A})^{-1} = E + (\overline{A} - \underline{A})(E - \overline{A})^{-1} \geq E$ and the proof is complete. ■

Using the Kronecker symbol δ_{ij} (i.e. $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ otherwise), we define the matrices

$$\underline{A}_1 = ((1 - \delta_{ij})\underline{a}_{ij})$$
$$\underline{A}_2 = (\delta_{ij}\underline{a}_{ij})$$
$$\overline{A}_1 = ((1 - \delta_{ij})\overline{a}_{ij})$$
$$\overline{A}_2 = (\delta_{ij}\overline{a}_{ij}).$$

Then, $\underline{A}_1$ and $\underline{A}_2$ are nonnegative, $\underline{A}_2$ is diagonal and $\underline{A}_1 + \underline{A}_2 = \underline{A}$; for $\overline{A}_1$, $\overline{A}_2$ the same is valid. Further, $E - \underline{A}_2$ and $E - \overline{A}_2$ are diagonal matrices with positive diagonal elements, since $\underline{a}_{ii} \leq \overline{a}_{ii} \leq (e^T\overline{A})_i < 1$ for $i = 1,2, \ldots,n$. Now we define the matrices

$$F = (E - \underline{A})(E - \overline{A})^{-1}$$
$$H = E - \underline{A}_2 + F\overline{A}_1$$

which both are nonnegative according to the Corollary and the previous note. Finally, we define the vectors

$$y' = \overline{y} - F\underline{y}$$
$$x_0 = (E - \overline{A})^{-1}\underline{y}$$
$$x_1 = (E - \underline{A})^{-1}\overline{y}$$

the last two of them being obviously nonnegative. The introduced matrices and vectors will frequently appear in the following theorems.

5. Feasible solutions. In the sequel, we always suppose the basic assumptions (I), (II), (III) to be satisfied, without

mentioning them in the theorems. We start with the fundamental definition:

Definition. A nonnegative interval $[\underline{x},\overline{x}]$ is said to be a feasible solution if $(E - A)x \in [\underline{y},\overline{y}]$ for any $A \in [\underline{A},\overline{A}]$, $x \in \in [\underline{x},\overline{x}]$.

It would be more correct to speak of "feasible gross output solution subject to $[\underline{y},\overline{y}]$ ", but, for the sake of brevity, we say "feasible solution" only. Thus, if $[\underline{x},\overline{x}]$ is a feasible solution and $x \in [\underline{x},\overline{x}]$, then the definition combined with the basic assumption (I) ensures that the corresponding net output $y = (E - A_o)x$ lies in $[\underline{y},\overline{y}]$ although the matrix A_o is not known. In Theorem 1 and Theorem 2 we give two different ways of description of feasible solutions.

Theorem 1. An interval $[\underline{x},\overline{x}]$ is a feasible solution if and only if $\underline{x}$ and $\overline{x}$ satisfy the system

$$\begin{aligned} (E - \underline{A}_2)\overline{x} - \underline{A}_1\underline{x} &\leq \overline{y} \\ (E - \overline{A}_2)\underline{x} - \overline{A}_1\overline{x} &\geq \underline{y} \qquad (2) \\ \underline{x} &\leq \overline{x}. \end{aligned}$$

Proof. Let $[\underline{x},\overline{x}]$ be a feasible solution. Set $\overline{w} = (E - \underline{A}_2)\overline{x} - \underline{A}_1\underline{x}$, $\underline{w} = (E - \overline{A}_2)\underline{x} - \overline{A}_1\overline{x}$ and suppose $\overline{w}_i > \overline{y}_i$ for some i, $1 \leq i \leq n$. Then, taking $A = \underline{A}$, $x = (\underline{x}_1,\ldots,\underline{x}_{i-1},\overline{x}_i,\underline{x}_{i+1},\ldots,\underline{x}_n)^T \in \in [\underline{x},\overline{x}]$, we have $((E - A)x)_i = \overline{w}_i > \overline{y}_i$, which contradicts the feasibility of $[\underline{x},\overline{x}]$. Hence $\overline{w} \leq \overline{y}$; the inequality $\underline{w} \geq \underline{y}$ can be proved in a similar way. Conversely, let $\underline{x}$ and $\overline{x}$ satisfy (2). First, the second and third inequality in (2) imply $(E - \overline{A})\underline{x} \geq \geq (E - \overline{A}_2)\underline{x} - \overline{A}_1\overline{x} \geq \underline{y}$ and premultiplying this by the nonnegative matrix $(E - \overline{A})^{-1}$ gives $\underline{x} \geq (E - \overline{A})^{-1}\underline{y} = x_o \geq 0$, so that $[\underline{x},\overline{x}]$ is a nonnegative interval. Second, taking $\underline{w}$, $\overline{w}$ as above, for any $A \in [\underline{A},\overline{A}]$, $x \in [\underline{x},\overline{x}]$ we have $(E - A)x \leq (E - \underline{A})x \leq \overline{w} \leq \overline{y}$ and

$(E - A)x \geqslant (E - \overline{A})x \geqslant \underline{w} \geqslant \underline{y}$, hence $[\underline{x},\overline{x}]$ is a feasible solution.■

For the purpose of another description of feasible solutions, it is useful to introduce the vector $d = \overline{x} - \underline{x}$, which is called the length of the interval $[\underline{x},\overline{x}]$. With the help of the identity $\overline{x} = \underline{x} + d$, the system (2) can be rearranged to the system

$$\begin{aligned} (E - \underline{A})\underline{x} &\leqslant \overline{y} - (E - \underline{A}_2)d \\ (E - \overline{A})\underline{x} &\geqslant \underline{y} + \overline{A}_1 d \\ 0 &\leqslant d. \end{aligned} \tag{3}$$

<u>Theorem 2.</u> An interval $[\underline{x},\overline{x}]$ is a feasible solution if and only if

$$\begin{aligned} \underline{x} &= x_o + (E - \overline{A})^{-1}(\overline{A}_1 d + k) \\ \overline{x} &= \underline{x} + d \end{aligned} \tag{4}$$

where d and k are nonnegative vectors satisfying the system

$$Hd + Fk \leqslant y'. \tag{5}$$

<u>Proof.</u> Let $[\underline{x},\overline{x}]$ be a feasible solution. Then the vectors $d = \overline{x} - \underline{x}$, $k = (E - \overline{A})\underline{x} - \underline{y} - \overline{A}_1 d$ are nonnegative in view of (3) and we obtain immediately $\overline{x} = \underline{x} + d$, $\underline{x} = (E - \overline{A})^{-1}(\underline{y} + \overline{A}_1 d + k)$, which is (4). To see (5), we apply this expression of $\underline{x}$ to the first inequality in (3), which yields (5). Conversely, if (4) and (5) hold, then d is the length of $[\underline{x},\overline{x}]$ and we have $(E - \overline{A})\underline{x} = \underline{y} + \overline{A}_1 d + k \geqslant \underline{y} + \overline{A}_1 d$, $(E - \underline{A})\underline{x} = Hd + F(\underline{y} + k) - (E - \underline{A}_2)d \leqslant \overline{y} - (E - \underline{A}_2)d$, thus $\underline{x}$ and d satisfy (3), hence $[\underline{x},\overline{x}]$ is a feasible solution.■

Using this result, we can describe lengths of feasible solutions.

Theorem 3. A nonnegative vector d is the length of a feasible solution if and only if it satisfies the system

$$Hd \leqslant y' . \qquad (6)$$

Proof. If d is the length of a feasible solution $[\underline{x},\overline{x}]$, then Theorem 2 ensures the inequality $Hd + Fk \leq y'$ for some $k \geqslant 0$, which, by virtue of $F \geqslant 0$, implies $Hd \leqslant y'$. On the other hand, if $Hd \leqslant y'$ and $d \geqslant 0$, then Theorem 2 as applied to $k = 0$ ensures the existence of a feasible solution of the length d. ■

For a nonnegative vector d, there can be infinitely many feasible solutions of the length d in the general case, but it is possible to estimate their lower and upper bounds:

Theorem 4. Let $[\underline{x},\overline{x}]$ be a feasible solution of the length d. Then, we have

$$\underline{x} \in [x_o + (E - \overline{A})^{-1}\overline{A}_1 d \ , \ x_1 - (E - \underline{A})^{-1}(E - \underline{A}_2)d] \qquad (7)$$
$$\overline{x} \in [x_o + (E - \overline{A})^{-1}(E - \overline{A}_2)d \ , \ x_1 - (E - \underline{A})^{-1}\underline{A}_1 d] \ . \qquad (8)$$

Proof. Premultiplying the first inequality in (3) by nonnegative matrix $(E - \underline{A})^{-1}$ and the second one by nonnegative matrix $(E - \overline{A})^{-1}$, we obtain

$$x_o + (E - \overline{A})^{-1}\overline{A}_1 d \leqslant \underline{x} \leq x_1 - (E - \underline{A})^{-1}(E - \underline{A}_2)d$$

which proves (7). Then, by adding d to these inequalities, we obtain (8). ■

6. Optimization. It can be easily seen from (6) that if $y' > 0$, then infinitely many feasible solutions exist, so that now the question arises, which of them is to be chosen for practical use. Perhaps the most natural criterion of optimality is the maximization of the national income, which represents the summary profit of a national economy. Let us recall that the national income formed by a net output $y = (E - A_o)x$ is the number $e^T y =$

$= e^T(E - A_0)x$. First, we shall evaluate the lower and upper bound of national income at a given feasible solution.

Theorem 5. Let $[\underline{x},\overline{x}]$ be a feasible solution and let $M = \left\{ e^T(E - A)x \mid A \in [\underline{A},\overline{A}],\ x \in [\underline{x},\overline{x}] \right\}$. Then

$$M = \left[e^T(E - \overline{A})\underline{x}\ ,\ e^T(E - \underline{A})\overline{x} \right].$$

Proof. Take $r = e^T(E - \overline{A})\underline{x}$, $s = e^T(E - \underline{A})\overline{x}$. If $A \in [\underline{A},\overline{A}]$, then $0^T < e^T(E - \overline{A}) \leq e^T(E - A) \leq e^T(E - \underline{A})$, therefore for any $x \in [\underline{x},\overline{x}]$ we have $e^T(E - A)x \in [r,s]$, hence $M \subset [r,s]$. To prove the converse inclusion, we define the real function $f(t) = e^T(E - \overline{A} + t(\overline{A} - \underline{A}))(\underline{x} + t(\overline{x} - \underline{x}))$ for $t \in [0,1]$. Clearly, f is continuous in $[0,1]$ and $f(0) = r$, $f(1) = s$. Let $v \in [r,s]$. Then, there exists a $t \in [0,1]$ such that $f(t) = v$. Now, taking $A = \overline{A} + t(\underline{A} - \overline{A})$, $x = \underline{x} + t(\overline{x} - \underline{x})$, we have $A \in [\underline{A},\overline{A}]$, $x \in [\underline{x},\overline{x}]$ and $v = e^T(E - A)x \in M$. Hence $[r,s] \subset M$, which completes the proof. ■

Under maximization of the national income in the interval case we will understand the maximization of its lower bound. This leads, in the light of Theorem 5, to the problem

$$\max \left\{ e^T(E - \overline{A})\underline{x} \mid [\underline{x},\overline{x}] \text{ is a feasible solution} \right\}. \tag{9}$$

But, from the point of view of the interval approach, the problem in this formulation has a great disadvantage. Namely, if $[\underline{x},\overline{x}]$ is a feasible solution which yields the optimal value in (9), then $\underline{x} = \overline{x}$. In fact, if $\underline{x} \neq \overline{x}$, then the basic assumption II implies $e^T(E - \overline{A})\underline{x} < e^T(E - \overline{A})\overline{x}$, hence the feasible solution $[\overline{x},\overline{x}]$ yields a greater value of objective function, which is a contradiction. For the practical use, however, the relation $\underline{x} \neq \overline{x}$ in the goal solution $[\underline{x},\overline{x}]$ may be desirable. To remove this difficulty, we shall suppose the length of the desired optimal feasible solution to be given beforehand. Then, we have a new problem: for a given nonnegative vector d, find

$$\max\left\{ e^T(E - \bar{A})\underline{x} \;\middle|\; [\underline{x}, \underline{x} + d] \text{ is a feasible solution}\right\}. \quad (10)$$

Clearly, (9) is equivalent to (10) for $d = 0$. According to (3), the problem (10) is equivalent to the linear programming problem

$$\max\left\{ e^T(E - \bar{A})\underline{x} \;\middle|\; (E - \underline{A})\underline{x} \leq \bar{y} - (E - \underline{A}_2)d,\; (E - \bar{A})\underline{x} \geq \underline{y} + \bar{A}_1 d \right\} \quad (11)$$

which, according to Theorem 3 and Theorem 4, has an optimal solution for any nonnegative d satisfying $Hd \leq y'$. The optimal solution $\underline{x}$ of the problem (11) is called the d-optimal solution.

<u>Theorem 6.</u> A vector $\underline{x}^o$ is a d-optimal solution if and only if

$$\underline{x}^o = x_o + (E - \bar{A})^{-1}(\bar{A}_1 d + k_o), \quad (12)$$

where k_o is an optimal solution of the problem

$$\max\left\{ e^T k \;\middle|\; k \geq 0,\; Fk \leq y' - Hd \right\}. \quad (13)$$

Moreover, if m_1 is the optimal value of (11) and m_2 the optimal value of (13), then

$$m_1 = e^T(\underline{y} + \bar{A}_1 d) + m_2. \quad (14)$$

<u>Proof.</u> According to the Theorem 2, $\underline{x}$ satisfies the constraints of the problem (11) if and only if

$$\underline{x} = x_o + (E - \bar{A})^{-1}(\bar{A}_1 d + k) \quad (^x)$$

where k satisfies the constraints of the problem (13) and this correspondence between $\underline{x}$ and k is obviously one-to-one. Moreover, $(^x)$ implies

$$e^T(E - \bar{A})\underline{x} = e^T(\underline{y} + \bar{A}_1 d) + e^T k. \quad (^{xx})$$

Hence if $\underline{x}^o$ is an optimal solution of (11), then the vector k_o, corresponding to $\underline{x}^o$ in the sense of $(^x)$, is an optimal solution of (13), and conversely. Taking the optimal values in $(^{xx})$, we obtain (14). ■

This theorem gives more than a mere reformulation of the problem (11), because the number of rows in the constraints matrix of (13) is twice smaller than that of (11).

For a length d, satisfying an additional condition, the d-optimal solution can be expressed explicitly.

<u>Theorem 7.</u> If a nonnegative vector d satisfies the condition

$$F^{-1}Hd \leqslant F^{-1}y', \tag{15}$$

then the vector

$$\underline{x}^{o} = x_1 - (E - \underline{A})^{-1}(E - \underline{A}_2)d \tag{16}$$

is the unique d-optimal solution.

<u>Proof.</u> First of all, the basic assumption (II) implies $e^T F^{-1} = e^T(E - \bar{A})(E - \underline{A})^{-1} > 0^T$. Further, the vector $k_o = F^{-1}(y' - Hd)$ obviously satisfies the constraints of the problem (13). Now, if k is any vector satisfying the constraints of (13), then taking $h = y' - Hd - Fk$, we have $h \geqslant 0$ and $k = k_o - F^{-1}h$, hence $e^T k = e^T k_o - e^T F^{-1} h \leqslant e^T k_o$ and $e^T k = e^T k_o$ if and only if $h = 0$, which is equivalent to $k = k_o$. Hence k_o is the unique optimal solution of (13), so that the vector $\underline{x}^o = x_o + (E - \bar{A})^{-1}(\bar{A}_1 d + k_o) = x_1 - (E - \underline{A})^{-1}(E - \underline{A}_2)d$ is the unique d-optimal solution. ■

Note that the condition (15) is stronger than (6), which follows from (15) by premultiplying it by the nonnegative matrix F.

In the last section 7, we shall deal with some further properties of feasible solutions without any connection with the matter of section 6.

7. Properties. Another characterization of feasible solutions is given in the following

Theorem 8. An interval $[\underline{x},\overline{x}]$ is a feasible solution if and only if $[\underline{x},\overline{x}]\subset X$, where

$$X = \left\{x \mid (E - \underline{A})x \leq \overline{y},\ (E - \overline{A})x \geq \underline{y}\right\}. \tag{17}$$

Proof. First of all, if $x \in X$, then $(E - \overline{A})x \geq \underline{y}$, which implies $x \geq x_0 \geq 0$. Whence, if $[\underline{x},\overline{x}] \subset X$, then $[\underline{x},\overline{x}]$ is a nonnegative interval and for any $A \in [\underline{A},\overline{A}]$, $x \in [\underline{x},\overline{x}]$ we have $(E - A)x \in [(E - \overline{A})x\ ,\ (E - \underline{A})x] \subset [\underline{y},\overline{y}]$, so that $[\underline{x},\overline{x}]$is a feasible solution. Conversely, if $[\underline{x},\overline{x}]$ is a feasible solution and $x \in [\underline{x},\overline{x}]$, then in the light of Theorem 1 we have $(E - \underline{A})x \leq$ $\leq (E - \underline{A}_2)\overline{x} - \underline{A}_1\underline{x} \leq \overline{y}$ and $(E - \overline{A})x \geq (E - \overline{A}_2)\underline{x} - \overline{A}_1\overline{x} \geq \underline{y}$, hence $x \in X$ and since x has been chosen arbitrarily in $[\underline{x},\overline{x}]$, it implies $[\underline{x},\overline{x}] \subset X$. ■

Note that X is the union of all feasible solutions. The next theorems of this section will be devoted to various properties of X. First, we give some equivalent necessary and sufficient conditions for existence of a feasible solution.

Theorem 9. The following conditions are mutually equivalent:

(i) There exists a feasible solution

(ii) $y' \geq 0$

(iii) $(E - \underline{A})(x_1 - x_0) \geq 0$

(iv) $(\overline{A} - \underline{A})x_0 \leq \overline{y} - \underline{y}$

(v) $x_0 \in X$

(vi) $X \neq \emptyset$.

Proof. We prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). If there exists a feasible solution $[\underline{x},\overline{x}]$,then, by virtue of the nonnegativity of its length d and of the matrix H, Theorem 3 implies $y' \geq Hd \geq 0$.

(ii) $\Rightarrow$(iii) $\Rightarrow$(iv). Both these implications follow from the identities $y' = (E - \underline{A})(x_1 - x_0) = \overline{y} - \underline{y} - (\overline{A} - \underline{A})x_0$, which can be proved easily.

(iv) $\Rightarrow$ (v). By definition of x_0, we have $(E - \overline{A})x_0 = \underline{y}$. Then $(E - \underline{A})x_0 = \underline{y} + (\overline{A} - \underline{A})x_0 \leq \overline{y}$, so that $x_0 \in X$ according to (17).

(v) $\Rightarrow$(vi) is obvious.

(vi) $\Rightarrow$(i). If $x \in X$, then $[x,x]$ is a feasible solution according to Theorem 8.■

Perhaps (iv) is the most illustrative of these six conditions, showing that the matrix $\overline{A} - \underline{A}$ must be "sufficiently smaller" than the vector $\overline{y} - \underline{y}$ to ensure the existence of a feasible solution.

Theorem 10. Let $X \neq \emptyset$. Then, we have the following:

(i) $X \subset [x_0, x_1]$

(ii) $x_0 \in X$

(iii) $x_1 \in X$ if and only if $F^{-1}y' \geqslant 0$

(iv) If $e^T\overline{A}_1 > 0^T$, then X is an interval if and only if $X = \{x_0\}$.

Proof. (i) If $x \in X$, then $[x,x]$ is a feasible solution of the length $d = 0$, hence Theorem 4 ensures $x \in [x_0, x_1]$. Thus $X \subset [x_0, x_1]$.

(ii) The implication $X \neq \emptyset \Rightarrow x_0 \in X$ follows from Theorem 9.

(iii) Because $(E - \underline{A})x_1 = \overline{y}$ due to the definition of x_1, (17) gives that $x_1 \in X$ if and only if $(E - \overline{A})x_1 \geqslant \underline{y}$. But $(E - \overline{A})x_1 - \underline{y} = F^{-1}\overline{y} - \underline{y} = F^{-1}y'$, hence $x_1 \in X$ if and only if $F^{-1}y' \geqslant 0$.

(iv) If $X = \{x_0\}$, then $X = [x_0, x_0]$. Conversely, let $X = [\underline{x}, \overline{x}]$. Then (i), (ii) imply $\underline{x} = x_0$. Take $d = \overline{x} - x_0$. Then

x_o and d satisfy the system (3), especially the second inequality in (3) implies $\bar{A}_1 d \leqslant 0$, hence $\bar{A}_1 d = 0$ and thus also $e^T\bar{A}_1 d = 0$, which gives $d = 0$, since $e^T\bar{A}_1 > 0^T$ and $d \geqslant 0$. Hence $\bar{x} = x_o$ and $X = \{x_o\}$. ■

The condition $e^T\bar{A}_1 > 0^T$, appearing in (iv), requires for each j to exist an i such that $i \neq j$ and $\bar{a}_{ij} > 0$. It means, that this condition is fulfilled if each sector needs the products of at least one another sector as input for its production.

An alternative holds for the segment connecting x_o with x_1.

Theorem 11. Let $X \neq \emptyset$ and let $S = \{x_o + t(x_1 - x_o) \mid t \in \in [0,1]\}$. Then, either $S \subset X$, or $S \cap X = \{x_o\}$.

Proof. For $t \in [0,1]$ denote $x_t = x_o + t(x_1 - x_o)$. Now, $x_t \in X$ if and only if $(E - \underline{A})x_t \leq \bar{y}$ and $(E - \bar{A})x_t \geqslant \underline{y}$. These inequalities, if rearranged, are equivalent to $(1 - t)y' \geqslant 0$ and $tF^{-1}y' \geqslant 0$. But $(1 - t)y' \geqslant 0$ holds for any $t \in [0,1]$ since $y' \geqslant 0$ (Theorem 9, (ii)). Now, if $x_1 \in X$, then $F^{-1}y' \geqslant 0$ (Theorem 10, (iii)), which implies $tF^{-1}y' \geqslant 0$ for any $t \in [0,1]$, hence $S \subset X$. If $x_1 \notin X$, then $(F^{-1}y')_i < 0$ for some i, $1 \leq i \leq n$, hence $t(F^{-1}y')_i \geqslant 0$ if and only if $t = 0$. Thus $S \cap X = \{x_o\}$ in this case. ■

At last, we give some facts concerning the interior of X, denoted by X^o.

Theorem 12. We have (i), (ii), (iii):

(i) $X^o = \{x \mid (E - \underline{A})x < \bar{y},\ (E - \bar{A})x > \underline{y}\}$

(ii) $X^o \neq \emptyset$ if and only if $y' > 0$

(iii) If $\underline{A}_1 e > 0$, $[\underline{x}, \bar{x}] \subset X$ and $\underline{x} < \bar{x}$, then $\underline{x} \in X^o$ and $\bar{x} \in X^o$.

Proof. (i) Take $Y = \{x \mid (E - \underline{A})x < \bar{y},\ (E - \bar{A})x > \underline{y}\}$. Clearly, Y is open and $Y \subset X$, hence $Y \subset X^o$. To prove (i), it is sufficient to show that $X^o \subset Y$. To this end, let $x = (x_i) \in X^o$. Then there exists a $\delta_1 > 0$ such that $[x - \delta_1 e,\ x + \delta_1 e] \subset X$. Suppose $x \notin Y$, let e.g. $((E - \underline{A})x)_i = \overline{y}_i$ for some i. Then taking $x' = (x_1, \ldots, x_i + \delta_1, \ldots, x_n)^T$, we have $x' \in [x - \delta_1 e, x + \delta_1 e] \subset X$ and $((E - \underline{A})x')_i = \overline{y}_i + (1 - \underline{a}_{ii})\delta_1 > \overline{y}_i$, hence $x' \notin X$, which is a contradiction. Thus $x \in Y$, hence $X^o \subset Y$, which proves $X^o = Y$.

(ii) Let $X^o \neq \emptyset$. Then there exists an interval $[\underline{x}, \overline{x}] \subset X$ such that $d = \overline{x} - \underline{x} > 0$. Using Theorem 3, we obtain $y' \geq Hd \geq$ $\geq (E - \underline{A}_2)d > 0$, since $E - \underline{A}_2$ is diagonal matrix with positive diagonal elements. On the other hand, if $y' > 0$, then there exists a $\delta_2 > 0$ such that $\delta_2(He) < y'$. Then, according to Theorem 3, there exists an interval $[\underline{x}, \overline{x}] \subset X$ such that $\overline{x} - \underline{x} = \delta_2 e > 0$, hence $\emptyset \neq [\underline{x}, \overline{x}]^o \subset X^o$.

(iii) Let $[\underline{x}, \overline{x}] \subset X$, $\underline{x} < \overline{x}$. Set $d = \overline{x} - \underline{x} = (d_i)$, $\delta_3 = \min_i d_i$. Then $d > 0$, $\delta_3 > 0$ and we have $\overline{A}_1 d \geq \underline{A}_1 d \geq \delta_3 \underline{A}_1 e > 0$, $(E - \underline{A}_2)d \geq (E - \overline{A}_2)d > 0$. First, $\underline{x}$ and d satisfy the system (3), hence $(E - \underline{A})\underline{x} \leq \overline{y} - (E - \underline{A}_2)d < \overline{y}$ and $(E - \overline{A})\underline{x} \geq \underline{y} + \overline{A}_1 d > \underline{y}$ so that (i) ensures $\underline{x} \in X^o$. Second, rearranging the system (3) by putting $\underline{x} = \overline{x} - d$, we obtain $(E - \underline{A})\overline{x} \leq \overline{y} - \underline{A}_1 d < \overline{y}$ and $(E - \overline{A})\overline{x} \geq \underline{y} + (E - \overline{A}_2)d > \underline{y}$, which implies $\overline{x} \in X^o$. ∎

Some further results concerning the interval approach to the input-output model are contained in [3], [4] .

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