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# INTEGRATION OF INTERVAL FUNCTIONS*

OLE CAPRANI†, KAJ MADSEN‡ AND L. B. RALL§

**Abstract.** An interval function $Y$ assigns an interval $Y(x)=(\underline{y}(x), \bar{y}(x)]$ in the extended real number system to each $x$ in its interval $X=[a, b]$ of definition. The integral of $Y$ over $[a, b]$ is taken to be the interval $\int_a^b Y(x)\,dx=[\underline{\int}_a^b \underline{y}(x)\,dx, \bar{\int}_a^b \bar{y}(x)\,dx]$, where $\underline{\int}_a^b \underline{y}(x)\,dx$ is the lower Darboux integral of the lower endpoint function $\underline{y}$, and $\bar{\int}_a^b \bar{y}(x)\,dx$ is the upper Darboux integral of the upper endpoint function $\bar{y}$. Since these Darboux integrals always exist in the extended real number system, it follows that all interval functions are integrable, no matter how nasty the endpoint functions $\underline{y}$, $\bar{y}$ are. The interval integral defined in this way includes the interval integral of R. E. Moore as the special case that $\underline{y}$, $\bar{y}$ are continuous, and hence Riemann integrable.

In addition to a construction of the interval integral in a form suitable for numerical approximation, some of its basic properties and other implications and applications of its definition are presented. The theory of interval integration given here supplies a previously lacking mathematical foundation for the numerical solution of integral equations by interval methods.

## 1. Intervals in the extended real number system.

In ordinary interval analysis [5], [6], the term *interval* refers to *closed* intervals of real numbers,

$$X=[a, b]=\{x \mid a \leqq x \leqq b\}, \tag{1.1}$$

with finite *endpoints* $a$, $b$. The *width*

$$w(X)=w([a, b])=b-a, \tag{1.2}$$

of an interval with real endpoints is consequently finite. To develop the theory of integration of interval functions given below, it is convenient to use the *extended* real number system, which includes the values $\pm\infty$ [3]. Thus, in addition to *finite* intervals of the form (1.1) with $a$, $b$ finite, there will be *infinite* intervals in the system of one of the following types:

(i) *semi-infinite* intervals

$$S_a=[a, +\infty], \qquad S^b=[-\infty, b], \qquad a, b \text{ finite}; \tag{1.3}$$

(ii) the *real line*

$$R=[-\infty, +\infty]; \tag{1.4}$$

and

(iii) the *indegenerate* intervals

$$S^{-\infty}=[-\infty, -\infty], \qquad S_{+\infty}=[+\infty, +\infty]. \tag{1.5}$$

(In what follows, "$+\infty$" will often be written simply as "$\infty$".)

All the infinite intervals will be defined to be of *infinite width*, that is,

$$w(S_a)=w(S^b)=w(R)=w(S^{-\infty})=w(S_{+\infty})=+\infty, \tag{1.6}$$

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† Department of Computer Science, University of Copenhagen, Copenhagen, Denmark.

‡ Institute for Numerical Analysis, Technical University of Denmark, DK-2800 Lyngby, Denmark.

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in the extended real number system. This definition is consistent with the type of limiting process used to define "improper" integrals, that is,

$$S_a = \lim_{b\to\infty} [a, b] = [a, \infty], \qquad S_\infty = \lim_{a\to\infty} [a, \infty] = [\infty, \infty], \tag{1.7}$$

and hence,

$$w(S_\infty) = \lim_{a\to\infty} w(S_a) = \lim_{a\to\infty} (\infty) = \infty, \tag{1.8}$$

so that it is reasonable in this sense to assign infinite widths to the indegenerate intervals $S^{-\infty}$ and $S_\infty$.

In what follows, a closed interval in the extended real number system will be called simply an *interval.*

**2. Interval arithmetic.** Interval arithmetic [5], [6] as defined for finite intervals may also be performed in the system of intervals in the extended real number system defined in § 1 if suitable rules are adopted for computing with the values $\pm\infty$. In essence, these "rules" are a shorthand notation for the results of the types of limiting processes to be encountered in the theory of integration presented below. McShane [3, p. 21] gives the following rules:

$$\begin{array}{rl}
\text{(i)} & -\infty < a < \infty \text{ for every real number } a; \\
\text{(ii)} & \infty \cdot a = a \cdot \infty = \infty \text{ if } 0 < a \leqq \infty; \\
\text{(iii)} & \infty \cdot a = a \cdot \infty = -\infty \text{ if } -\infty \leqq a < 0; \\
\text{(iv)} & (-\infty) \cdot a = a \cdot (-\infty) = -\infty \text{ if } 0 < a \leqq \infty; \\
\text{(v)} & (-\infty) \cdot a = a \cdot (-\infty) = \infty \text{ if } -\infty \leqq a < 0; \\
\text{(vi)} & a/\infty = a/(-\infty) = 0 \text{ if } a \text{ is real}; \\
\text{(vii)} & \infty + a = a + \infty = \infty \text{ if } a > -\infty; \\
\text{(viii)} & -\infty + a = a + (-\infty) = -\infty \text{ if } a < \infty; \\
\text{(ix)} & \infty \cdot 0 = 0 \cdot \infty = (-\infty) \cdot 0 = 0 \cdot (-\infty) = 0.
\end{array} \tag{2.1}$$

Thus, rule (2.1ix) takes care of the "indeterminant" form "$0 \cdot \infty$" which can arise if one of the factors in a multiplication is an infinite interval. The product of two intervals will be *defined* to be

$$[a, b] \cdot [c, d] = [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}] \tag{2.2}$$

in the extended real number system. In ordinary interval arithmetic [5, p. 9], (2.2) is a consequence of the definition $[a, b] \cdot [c, d] = \{z \mid z = x \cdot y, x \in [a, b], y \in [c, d]\}$ of multiplication of intervals. In the extended real number system, however, one has $\{z \mid z = x \cdot y, x \in [-1, 1], y \in [\infty, \infty]\} = \{-\infty, 0, \infty\}$ by (2.1ii, iii, ix), and the result is not an interval. Use of the rule (2.2) gives $[-1, 1] \cdot [\infty, \infty] = [-\infty, \infty]$, which circumvents this problem.

As in ordinary interval arithmetic, division by intervals containing 0 will not be defined. The *reciprocal* of an interval,

$$[c, d]^{-1} = \left[\frac{1}{d}, \frac{1}{c}\right], \qquad 0 \notin [c, d], \tag{2.3}$$

is defined for all zero-free intervals, with rule (2.1vi) used if $[c, d]$ is an infinite interval.

One has

(2.4)
$$S_a^{-1}=[a,\infty]^{-1}=\left[0,\frac{1}{a}\right],\quad a>0;\qquad S_\infty^{-1}=[\infty,\infty]^{-1}=[0,0];$$
$$(S^b)^{-1}=[-\infty,b]^{-1}=\left[\frac{1}{b},0\right],\quad b<0;\qquad (S^{-\infty})^{-1}=[-\infty,-\infty]^{-1}=[0,0].$$

The indeterminant form "$\infty/\infty$" will thus not occur in the interval arithmetic under discussion, since division is defined by

(2.5)
$$\frac{[a,b]}{[c,d]}=[a,b]\cdot[c,d]^{-1},\qquad 0\notin[c,d],$$

and $[c,d]^{-1}$, if it exists, will have only finite endpoints by (2.3) and (2.4).

The indeterminant form "$\infty-\infty$" can appear in addition or subtraction according to the usual rules [5, pp. 8–9],

(2.6)
$$[a,b]+[c,d]=[a+c,b+d],$$
$$[a,b]-[c,d]=[a-d,b-c],$$

but only if at least one of the terms is an indegenerate interval. Thus, an additional rule to augment the list (2.1) is needed, which is

(2.7) (x)
$$[a,\infty]+[-\infty,-\infty]=[a,\infty]-[\infty,\infty]=[-\infty,\infty],$$
$$[-\infty,b]+[\infty,\infty]=[-\infty,b]-[-\infty,-\infty]=[-\infty,\infty],$$

where $a$, $b$ may be finite or infinite. Thus, rule (2.7x) assigns the value $+\infty$ to $\infty-\infty$ as an *upper* endpoint of an interval, and $-\infty$ as a *lower* endpoint.

Thus, the total collection of rules for interval arithmetic in the system of intervals defined over the extended real numbers consists of (2.1i–ix), (2.7x), (2.2), (2.3), (2.5) and (2.6). The interval arithmetic constructed in this way contains ordinary interval arithmetic on finite intervals [5], [6] in the sense that it gives the same results for finite intervals. The operations on infinite intervals are defined in such a way as to be convenient in the sequel for the construction of a theory of integration of interval functions. Other extensions of interval arithmetic to infinite intervals are possible, but will not be considered here.

**3. Interval functions.** $Y$ is said to be an *interval function* of $x$ on $[a,b]$ if it assigns a nonempty interval

(3.1)
$$Y(x)=[\underline{y}(x),\bar{y}(x)]=\{y\mid \underline{y}(x)\leqq y\leqq \bar{y}(x)\},$$

to each $x\in[a,b]$. The (extended) real-valued functions $\underline{y}$, $\bar{y}$ are called the *endpoints* or *boundary functions* of $Y$, and the notation

(3.2)
$$Y=[\underline{y},\bar{y}]$$

will be used, as well as the alternative notation

(3.3)
$$Y(x)=[\underline{y},\bar{y}](x),$$

for the interval (3.1).

The interval function $Y$ can also be identified with its graph, which is the set of points

$$Y = [a, b] \times Y(x) = \{(x, y) | x \in [a, b], y \in Y(x)\} \tag{3.4}$$

in the $x$, $y$-plane. Geometrically, the graph (3.4) extends from the "lines" $x = a$ on the left to $x = b$ on the right, and from the "curves" defined by $y = \underline{y}(x)$ below to $y = \bar{y}(x)$ above (recall that extended real values are permitted).

In the context of interval functions, a real-valued (or extended real-valued) function $f$ is considered to be the *degenerate* interval function

$$f = [f, f]. \tag{3.5}$$

In the extended real number system, numbers $c \leqq d$ exist such that the graph (3.4) of $Y$ is contained in the *rectangle* $R = [a, b] \times [c, d] = \{(x, y) | x \in [a, b], y \in [c, d]\}$ in the $x$, $y$-plane; that is,

$$Y \subset [a, b] \times [c, d] = R. \tag{3.6}$$

The set of all rectangles $R$ for which (3.6) holds will be denoted by $R(Y)$ or by $R_{[a,b]}(Y)$ if it is desired to specify the *interval of definition* $[a, b]$ of $Y$.

If $[a, b]$ is a finite interval, then $Y$ is said to be *finitely defined.* If (3.6) holds with $c$, $d$ finite, then $Y$ is called a *bounded* interval function. A bounded and finitely defined interval function is said to be finite; the graph of a finite interval function is obviously contained in a finite rectangle $R$ with area $w([a, b]) \cdot w([c, d]) = (b - a) \cdot (d - c)$.

DEFINITION 3.1. For

$$c = \inf_{x \in [a,b]} \{\underline{y}(x)\}, \qquad d = \sup_{x \in [a,b]} \{\bar{y}(x)\}, \tag{3.7}$$

the interval

$$\nabla Y_{[a,b]} = [c, d] \tag{3.8}$$

is called the *vertical extent* of the interval function $Y$ on $[a, b]$. If the interval of definition of $Y$ is understood, then $\nabla Y_{[a,b]}$ may be abbreviated as $\nabla Y$. The rectangle

$$R(\nabla Y) = R_{[a,b]}(\nabla Y) = [a, b] \times \nabla Y_{[a,b]} \tag{3.9}$$

is the "smallest" containing the graph of $Y$. One has

$$R(\nabla Y) = \bigcap_{R \in R(Y)} R; \tag{3.10}$$

that is, $R(\nabla Y)$ is the intersection of all rectangles (3.6) which contain the graph of $Y$.

Vertical extent of an interval function, as defined above, has the important property of being inclusion monotone with respect to the interval of definition of the interval function and inclusion of interval functions; $Y \subset Z$ means that the graph of $Z$ contains the graph of $Y$ considered to be point sets in the $x$, $y$-plane. More precisely, (3.7) and the definition (3.8) of vertical extent lead directly to the following result:

LEMMA 3.1. *If $I$, $J$ are intervals on the $x$-axis with $I \subset J$, then*

$$\nabla Y_I \subset \nabla Y_J; \tag{3.11}$$

*if $Y$, $Z$ are interval functions on $X = [a, b]$ such that $Y \subset Z$, then*

$$\nabla Y_{[a,b]} \subset \nabla Z_{[a,b]}. \tag{3.12}$$

**4. Vertical measure and Darboux sums.**

DEFINITION 4.1. The interval

$$W_{[a,b]}(Y)=w([a,b])\cdot\nabla Y_{[a,b]} \tag{4.1}$$

is called the *vertical measure* of the interval function $Y$ on $[a, b]$. Note that this quantity is interval-valued, and specifies the interval of definition of the interval function $Y$ on which its vertical extent $\nabla Y$ is obtained.

(The goal in this paper is to construct a theory of Riemann-type integrals of interval functions. The *horizontal measure*

$$H_{[a,b]}(Y)=[a,b]\cdot w(\nabla Y_{[a,b]}) \tag{4.2}$$

of $Y$ on $[a, b]$ may be useful in a Lebesgue-type integration theory, but this will not be pursued further here.)

*Remark* 4.1. Vertical measure is inclusion monotone with respect to inclusion of interval functions: If $Y\subset Z$, then $W_{[a,b]}(Y)\subset W_{[a,b]}(Z)$.

The assertion of Remark 4.1 follows immediately from Lemma 3.1.

As usual, a set of points $\{x_0, x_1, \cdots, x_n\}$ such that

$$a=x_0\leqq x_1\leqq\cdots\leqq x_{n-1}\leqq x_n=b \tag{4.3}$$

defines a *partition*,

$$\Delta_n=(X_1, X_2, \cdots, X_n), \tag{4.4}$$

of the interval $X=[a, b]$, where

$$X_i=[x_{i-1}, x_i], \qquad i=1,2,\cdots,n. \tag{4.5}$$

Obviously,

$$X=\bigcup_{i=1}^{n} X_i, \qquad w(X)=\sum_{i=1}^{n} w(X_i). \tag{4.6}$$

DEFINITION 4.2. The interval

$$\Sigma_{\Delta_n}Y=\sum_{i=1}^{n} w(X_i)\cdot\nabla Y_i=\sum_{i=1}^{n} W_{[x_{i-1},x_i]}(Y) \tag{4.7}$$

is called the *Darboux sum* of the interval function $Y$ corresponding to the partition $\Delta_n$ of $X=[a, b]$, where $\nabla Y_i=\nabla Y_{X_i}=\nabla Y_{[x_{i-1},x_i]}$ has been written for brevity. For

$$\nabla Y_i=[c_i, d_i], \tag{4.8}$$

one has

$$c_i=\inf_{x\in X_i}\{\underline{y}(x)\}, \qquad d_i=\sup_{x\in X_i}\{\bar{y}(x)\} \tag{4.9}$$

and

$$\Sigma_{\Delta_n}Y=\left[\sum_{i=1}^{n} c_i\cdot w(X_i), \sum_{i=1}^{n} d_i\cdot w(X_i)\right]; \tag{4.10}$$

the endpoints of $\Sigma_{\Delta_n}Y$ are thus, respectively, the *lower Riemann sum* of the function $\underline{y}$ and the *upper Riemann sum* of the function $\bar{y}$ corresponding to the partition $\Delta_n$ of $X$ [7].

The upper and lower limits of the interval (4.10) may also be interpreted as (elementary) integrals of *step-functions* [3], p. 54,

$$\sum_{i=1}^{n} c_i \cdot w(X_i) = {\int_a^b}' \underline{s}(x)\,dx = {\int_X}' \underline{s}(x)\,dx \tag{4.11}$$

and

$$\sum_{i=1}^{n} d_i \cdot w(X_i) = {\int_a^b}' \bar{s}(x)\,dx = {\int_X}' \bar{s}(x)\,dx. \tag{4.12}$$

In (4.11), the step function $\underline{s}(x)$ will have the values

$$\underline{s}(x) = c_i = \inf_{x \in X_i} \{\underline{y}(x)\}, \qquad x_{i-1} < x < x_i, \tag{4.13}$$

in all nondegenerate intervals $X_i$ of the partition $\Delta_n$. At each of the *partition points* $x_i$ listed in (4.3), there will be a finite number of intervals $X_{i-j_i}$, $X_{i-j_i+1}, \cdots, X_i, X_{i+1}, \cdots, X_{i+k_i}$ which contain $x_i$. Define

$$\underline{s}(x_i) = \min \{c_j | x_i \in X_j\}, \qquad i = 0, 1, \cdots, n. \tag{4.14}$$

Similarly,

$$\bar{s}(x) = d_i = \sup_{x \in X_i} \{\bar{y}(x)\}, \qquad x_{i-1} < x < x_i \tag{4.15}$$

in nondegenerate intervals $X_i$ of the partition $\Delta_n$, and

$$\bar{s}(x_i) = \max \{d_j | x_i \in X_j\}, \qquad i = 0, 1, \cdots, n, \tag{4.16}$$

at the partition points $x_0, x_1, \cdots, x_n$. It follows that

$$\underline{s}(x) \leqq \underline{y}(x) \leqq \bar{y}(x) \leqq \bar{s}(x), \qquad a \leqq x \leqq b. \tag{4.17}$$

The properties of integrals of step-functions are well-documented [3, pp. 54–57]; for example, if $s_1$ and $s_2$ are step-functions on an interval $X$, and $k$ is a finite constant, then

(4.18)

(a) $${\int_X}' k \cdot s_1(x)\,dx = k \cdot {\int_X}' s_1(x)\,dx;$$

(b) $${\int_X}' (s_1(x) + s_2(x))\,dx = {\int_X}' s_1(x)\,dx + {\int_X}' s_2(x)\,dx;$$

(c) if $s_1(x) \leqq s_2(x)$ for all $x \in X$, then

$${\int_X}' s_1(x)\,dx \leqq {\int_X}' s_2(x)\,dx.$$

Furthermore, if $s(x)$ is a step-function on $X$, then for each partition $\Delta_m$ of $X$

$$\sum_{j=1}^{m} {\int_{X_j}}' s(x)\,dx = {\int_X}' s(x)\,dx. \tag{4.19}$$

The integral of a step function is also invariant under translation [3, p. 57].

The above results may be used to prove corresponding assertions about the Darboux sums (4.7), taking into account the differences between real and interval arithmetic.

THEOREM 4.1. *If $Y$, $Z$ are interval functions on $X=[a, b]$ and $k$ is a constant, then*

$$\text{(4.20)}\quad \begin{aligned} &\text{(a)} \quad \Sigma_{\Delta_n} k \cdot Y = k \cdot \Sigma_{\Delta_n} Y;\\ &\text{(b)} \quad \Sigma_{\Delta_n}(Y+Z) \subset \Sigma_{\Delta_n} Y + \Sigma_{\Delta_n} Z;\\ &\text{(c)} \quad \textit{if } Y \subset Z \textit{ on } X, \textit{ then } \Sigma_{\Delta_n} Y \subset \Sigma_{\Delta_n} Z \ (\textit{inclusion monotonicity}). \end{aligned}$$

*Proof.* For finite $k$, (4.20a) follows directly from (4.18a); rule (2.7x) allows one to drop the restriction of $k$ to finite values. For $Y=[\underline{y}, \bar{y}]$, $Z=[\underline{z}, \bar{z}]$, the inequalities

$$\text{(4.21)}\quad \begin{aligned} &\text{(a)} \quad \inf_{X_i}\{\underline{y}+\underline{z}\} \geqq \inf_{X_i}\{\underline{y}\} + \inf_{X_i}\{\underline{z}\},\\ &\text{(b)} \quad \sup_{X_i}\{\bar{y}+\bar{z}\} \leqq \sup_{X_i}\{\bar{y}\} + \sup_{X_i}\{\bar{z}\} \end{aligned}$$

on the intervals $X_i$, $i=1, 2, \cdots, n$, [3, p. 25] give

$$\text{(4.22)}\qquad \nabla(Y+Z)_{X_i} \subset \nabla Y_{X_i} + \nabla Z_{X_i},$$

from which

$$\text{(4.23)}\qquad W_{X_i}(Y+Z) \subset W_{X_i}(Y) + W_{X_i}(Z),$$

$i=1, 2, \cdots, n$, and (4.20b) follows. Finally, the inclusion monotonocity of the vertical measure $W$ (see Remark 4.1) with respect to inclusion of interval functions gives (4.20c). Q.E.D.

An analogue of (4.19) is also available immediately. For $m=2$, suppose that $a \leqq p \leqq b$, and that

$$\text{(4.24)}\qquad \Delta_{n_1}^{(1)} = (X_{11}, X_{12}, \cdots, X_{1n_1})$$

is a partition of $X_1=[a, p]$; similarly,

$$\text{(4.25)}\qquad \Delta_{n_2}^{(2)} = (X_{21}, X_{22}, \cdots, X_{2n_2})$$

is a partition of $X_2=[p, b]$. For $n=n_1+n_2$, it follows that

$$\text{(4.26)}\qquad \Delta_n = (X_{11}, X_{12}, \cdots, X_{1n_1}, X_{21}, \cdots, X_{2n_2}),$$

will be a partition of $X=[a, b]$, and

$$\text{(4.27)}\qquad \Sigma_{\Delta_n^{(1)}} Y + \Sigma_{\Delta_n^{(2)}} Y = \Sigma_{\Delta_n} Y.$$

This may be extended by induction to any positive integer $m>2$.

A type of mean value (or mean interval-value) theorem holds for the Darboux sums (4.7).

THEOREM 4.2. *If $X=[a, b]$ is a finite, nondegenerate interval, then an interval $\bar{Y}(\Delta_n) \subset \nabla Y_X$ exists for each partition $\Delta_n$ of $X$ such that*

$$\text{(4.28)}\qquad \Sigma_{\Delta_n} Y = w(X) \cdot \bar{Y}(\Delta_n).$$

*Proof.* By (4.7) and (4.20a),

$$\text{(4.29)}\qquad \frac{1}{w(X)} \cdot \Sigma_{\Delta_n} Y = \sum_{i=1}^{n} \left(\frac{w(X_i)}{w(X)}\right) \cdot \nabla Y_i = [r, s],$$

say, where for $\alpha_i = w(X_i)/w(X)$, $i=1, 2, \cdots, n$,

$$\text{(4.30)}\qquad r = \sum_{i=1}^{n} \alpha_i c_i, \qquad s = \sum_{i=1}^{n} \alpha_i d_i, \qquad \sum_{i=1}^{n} \alpha_i = 1.$$

Thus,

$$(4.31)\qquad \begin{aligned} c = \min_{(i)}\{c_i\} \le r \le \max_{(i)}\{c_i\}, \\ \min_{(i)}\{d_i\} \leqq s \leqq \max_{(i)}\{d_i\} = d, \end{aligned}$$

and hence $[r, s] \subset \nabla Y_X$. Thus, by (4.29), (4.28) holds with $\bar{Y}(\Delta_n) = [r, s]$. Q.E.D.

**5. Step-functions and Riemann sums.** For each positive integer $n$, let $S_n$ denote the set of all step-functions $s_n$ on $X$ having $n+1$ partition points $x_0, x_1, \cdots$, disposed according to (4.3). Furthermore, let

$$(5.1)\qquad \begin{aligned} \underline{S}_n(Y) = \{\underline{s}_n \mid \underline{s}_n \in S_n, \underline{s}_n(x) \leqq \underline{y}(x), a \leqq x \leqq b\}, \\ \bar{S}_n(Y) = \{\bar{s}_n \mid \bar{s}_n \in S_n, \bar{s}_n(x) \geqq \bar{y}(x), a \leqq x \leqq b\}. \end{aligned}$$

The sets $\underline{S}_n(Y)$, $\bar{S}_n(Y)$ are nonempty, as $\underline{s}_n \equiv -\infty$ belongs to $\underline{S}_n(Y)$, and $\bar{\bar{s}}_n \equiv +\infty$ to
The sets $\underline{S}_n(Y)$, $\bar{s}_n(Y)$ are nonempty, as $\bar{S}_n \equiv -\infty$ belongs to $\underline{S}_n(Y)$ and $\bar{\bar{s}}_n \equiv +\infty$ to

$$(5.2)\qquad \int_a^{'b} \underline{s}_n(x)\,dx \leqq \int_a^{'b} \bar{s}_n(x)\,dx,$$

and consequently

$$(5.3)\qquad \bar{c}_n = \sup_{\underline{s}_n \in \underline{S}_n} \left\{ \int_a^{'b} \underline{s}_n(x)\,dx \right\} \leqq \inf_{\bar{s}_n \in \bar{S}_n} \left\{ \int_a^{'b} \bar{s}_n(x)\,dx \right\} = \underline{d}_n.$$

DEFINITION 5.1. For each positive integer $n$, let $\mathscr{D}_n$ denote the set of partitions (4.4). The interval

$$(5.4)\qquad \Sigma_n Y = \bigcap_{\Delta_n \in \mathscr{D}_n} \Sigma_{\Delta_n} Y = [\bar{c}_n, \underline{d}_n],$$

is called the *Riemann sum of order n of the interval function Y over* $[a, b]$.

LEMMA 5.1. *The interval $\Sigma_n Y$ is nonempty; furthermore, if $m > n$, then*

$$(5.5)\qquad \Sigma_m Y \subset \Sigma_n Y.$$

*Proof.* The assertion of the nonemptiness of the interval (5.4) is simply a restatement of (5.3). Denoting the set of Darboux sums (4.7) by $\mathscr{S}_n$, if $m > n$, then

$$(5.6)\qquad \mathscr{S}_n \subset \mathscr{S}_m,$$

as if $\Sigma_{\Delta_n} \subset \mathscr{S}_n$; then one may take the partition $\Delta_m$ defined by

$$(5.7)\qquad a = x_0 \leqq x_1 \leqq \cdots \leqq x_n = x_{n+1} = \cdots = x_m = b,$$

for which $\Sigma_{\Delta_n} Y = \Sigma_{\Delta_n} Y$, and thus $\Sigma_{\Delta_n} Y \in \mathscr{S}_m$ for $m > n$. The inclusion (5.5) then follows from (5.6) by the definition (5.4). Q.E.D.

The properties of Darboux sums listed in Theorem 4.1 survive the intersection (5.4) and thus become properties of Riemann sums, giving immediately the following result.

THEOREM 5.1. *If $Y$, $Z$ are interval functions on $X = [a, b]$ and $k$ is a constant, then*

$$(5.8)\qquad \begin{aligned} &\text{(a)}\quad \Sigma_n k \cdot Y = k \cdot \Sigma_n Y; \\ &\text{(b)}\quad \Sigma_n (Y \times Z) \subset \Sigma_n Y + \Sigma_n Z; \\ &\text{(c)}\quad \textit{if } Y \subset Z \textit{ on } X, \textit{ then } \Sigma_n Y \subseteq \Sigma_n Z \textit{ (inclusion monotonicity).} \end{aligned}$$

The additivity of Riemann sums with respect to the intervals over which they are defined will now be investigated. In order to be definite, the notation

$$\Sigma_n Y_{[a,b]} = \Sigma_n Y_X, \tag{5.9}$$

will be used to indicate the *interval of summation* $X = [a, b]$. Suppose that $a \leqq p \leqq b$ and $X_1 = [a, p]$, $X_2 = [p, b]$. The following results apply to the expression of the sum of an interval function $Y$ over $X$ in terms of its sums over $X_1$ and $X_2$.

LEMMA 5.2. *If $X = [a, b]$ is finite and nondegenerate, then*

$$W_{[a,p]}(Y) + W_{[p,b]}(Y) \subset W_{[a,b]}(Y), \tag{5.10}$$

*that is,*

$$w([a, p]) \cdot \nabla Y_{[a,p]} + w([p, b]) \cdot \nabla Y_{[p,b]} \subset w([a, b]) \cdot \nabla Y_{[a,b]}. \tag{5.11}$$

*Proof.* Let $\nabla Y_{[a,p]} = [c_1, d_1]$, $\nabla Y_{[p,b]} = [c_2, d_2]$, $\nabla Y_{[a,b]} = [c, d]$. Then, by the definition of vertical extent,

$$c = \min\{c_1, c_2\}, \qquad d = \max\{d_1, d_2\}. \tag{5.12}$$

For $\alpha = w([a, p])/w([a, b])$, one has $1 \geqq \alpha \geqq 0$ and $w([p, b])/w([a, b]) = 1 - \alpha \geqq 0$. Thus,

$$\begin{aligned} w([a, p]) \cdot \nabla Y_{[a,p]} &+ w([p, b]) \cdot \nabla Y_{[p,b]} \\ &= w([a, b]) \cdot [\alpha c_1 + (1-\alpha)c_2, \alpha d_1 + (1-\alpha)d_2] \end{aligned} \tag{5.13}$$

and, as

$$c \leqq \alpha c_1 + (1-\alpha)c_2 \leqq \alpha d_1 + (1-\alpha)d_2 \leqq d, \tag{5.14}$$

by (5.12), (5.11) follows. Q.E.D.

THEOREM 5.2. *If $X = [a, b]$ is finite and nondegenerate, then for each positive integer $n \geqq 2$*

$$\Sigma_n Y_{[a,b]} \subset \bigcap_{n_1+n_2=n} \{\Sigma_{n_1} Y_{[a,p]} + \Sigma_{n_2} Y_{[p,b]}\} \subset \Sigma_{n-1} Y_{[a,b]}. \tag{5.15}$$

*Proof.* The set $\mathscr{S}_n$ of Darboux sums (4.7) may be decomposed into two disjoint subsets for each positive integer $n \geqq 2$: the set $\mathscr{S}_n^p$ of sums corresponding to partitions $\Delta_n^p$ which have $p$ as a partition point, the set of which will be denoted by $\mathscr{D}_n^p$, and the complement of $\mathscr{S}_n^p$ relative to $\mathscr{S}_n$, $\mathscr{S}_n^{p'} = \mathscr{S}_n \backslash \mathscr{S}_n^p$, that is, the set of all Darboux sums corresponding to partitions $\Delta_n$ for which $p$ is not a partition point. As $\mathscr{S}_n^p \subset \mathscr{S}_n$, one has

$$\Sigma_n Y_{[a,b]} \subset \bigcap_{\Delta_n^p \in \mathscr{D}_n^p} \Sigma_{\Delta n}^p Y_{[a,b]}. \tag{5.16}$$

By (4.27), for $\Sigma_{\Delta_n^p} Y \in \mathscr{S}_n^p$ one can write

$$\Sigma_{\Delta_n} Y_{[a,b]} = \Sigma_{\Delta_{n_1}} Y_{[a,p]} + \Sigma_{\Delta_{n_2}} Y_{[p,b]}, \tag{5.17}$$

where $n_1 + n_2 = n$. Consequently, as

$$\bigcap_{\Delta_n^p \in \mathscr{D}_n^p} \Sigma_{\Delta_n} Y_{[a,b]} = \bigcap_{n_1+n_2=n} \{\Sigma_{n_1} Y_{[a,p]} + \Sigma_{n_2} Y_{[p,b]}\}, \tag{5.18}$$

the first inclusion of (5.15) follows.

Now, consider a partition $\Delta_{n-1}$ of $X = [a, b]$ for $n \geqq 2$, and let $\Delta_n^p$ denote the partition of $X$ obtained by adding the point $p$ to the set $\{x_0, x_1, \cdots, x_{n-1}\}$. Either $p = x_i$

for some $i$, $0 \leqq i \leqq n-1$, in which case

(5.19) $$\Sigma_{\Delta_{n-1}} Y_{[a,b]} = \Sigma_{\Delta_n^p} Y_{[a,b]} = \Sigma_{\Delta_{n_1}} Y_{[a,p]} + \Sigma_{\Delta_{n_2}} Y_{[p,b]},$$

$n_1 + n_2 = n$, or a nondegenerate interval $X_i = [x_{i-1}, x_i] \subset \Delta_{n-1}$ exists such that $x_{i-1} < p < x_i$. As

(5.20) $$w([x_{i-1}, p]) \cdot \nabla Y_{[x_{i-1},p]} + w([p, x_i]) \cdot \nabla Y_{[p,x_i]} \subset w([x_{i-1}, x_i]) \cdot \nabla Y_{[x_{i-1},x_i]},$$

by Lemma 5.2 one has

(5.21) $$\Sigma_{\Delta_n^p} Y_{[a,b]} \subset \Sigma_{\Delta_{n-1}} Y_{[a,b]}$$

in this case. Thus, as $\{\Sigma_{\Delta_n^p} Y_{[a,b]}\} = \mathcal{S}_n^p$,

(5.22) $$\bigcap_{\Delta_n^p \in \mathcal{D}_n^p} \Sigma_{\Delta_n^p} Y_{[a,b]} \subset \bigcap_{\Delta_{n-1} \in \mathcal{D}_{n-1}} \Sigma_{\Delta_{n-1}} Y_{[a,b]} = \Sigma_{n-1} Y_{[a,b]}$$

by (5.20) and (5.21), and the second inclusion in (5.15) now follows from (5.18). Q.E.D.

A mean interval-value theorem also holds for Riemann sums.

THEOREM 5.3. *If $X = [a, b]$ is finite and nondegenerate then, for each positive integer $n$, an interval $\bar{Y}_n \subset \nabla Y_X$ exists such that*

(5.23) $$\Sigma_n Y = w(X) \cdot \bar{Y}_n.$$

*Proof.* As before, let $\mathcal{D}_n$ denote the set of all partitions $\Delta_n$ for each positive integer $n$. Then, by (4.28),

(5.24) $$\Sigma_n Y = \bigcap_{\Delta_n \in \mathcal{D}_n} \Sigma_{\Delta_n} Y = w(X) \cdot \bigcap_{\Delta_n \in \mathcal{D}_n} \bar{Y}(\Delta_n),$$

so that (5.23) holds with

(5.25) $$\bar{Y}_n = \bigcap_{\Delta_n \in \mathcal{D}_n} \bar{Y}(\Delta_n) \subset \nabla Y_X,$$

as each $\bar{Y}(\Delta_n) \subset \nabla Y_X$. Q.E.D.

## 6. Interval integrals.

DEFINITION 6.1. (The interval integral). If $Y$ is an interval function defined on $X = [a, b]$, then the *interval integral of $Y$ over* $[a, b]$ is defined to be the interval

(6.1) $$\int_a^b Y(x)\,dx = \int_X Y(x)\,dx = \bigcap_{n=1}^{\infty} \Sigma_n Y.$$

As usual, $Y$ is said to be *integrable* over $X$ if its interval integral (6.1) is defined.

*Remark* 6.1. By Lemma 5.1, the interval integral (6.1) is a nonempty closed interval, since it is the intersection of a (countable) collection of nested closed intervals.

*Remark* 6.2. An equivalent definition of the interval integral (6.1) is

(6.2) $$\int_X Y(x)\,dx = \left[\underline{\int_X} \underline{y}(x)\,dx, \overline{\int_X} \bar{y}(x)\,dx\right],$$

where, for the sets of step-functions

(6.3) $$\underline{S} = \bigcup_{n=1}^{\infty} \underline{S}_n, \qquad \bar{S} = \bigcup_{n=1}^{\infty} \bar{S}_n,$$

the *lower Darboux integral* of $\underline{y}$ over $X=[a, b]$ is defined to be [3, p. 57]

$$\underline{\int}_X \underline{y}(x)\,dx = \sup_{\underline{s}\in\underline{S}}\left\{\int_X' \underline{s}(x)\,dx\right\}, \tag{6.4}$$

and similarly, the *upper Darboux integral* of $\bar{y}$ over $X=[a, b]$ is

$$\overline{\int}_X \bar{y}(x)\,dx = \inf_{\bar{s}\in\bar{S}}\left\{\int_X' \bar{s}(x)\,dx\right\}. \tag{6.5}$$

The set $\underline{S}$ defined in (6.3) is the set of all step-functions bounded above by $\underline{y}$; similarly, $\bar{S}$ is the set of all step-functions which are greater than or equal to $\bar{y}$ at each point of $X$. As these sets are nonempty (recall the step-functions $\underline{s}\equiv-\infty$ and $\bar{s}\equiv+\infty$), the Darboux integrals (6.4) and (6.5) always exist, no matter how nasty the functions $\underline{y}$, $\bar{y}$ are from the standpoint of ordinary integration theory. This observation furnishes the following result.

THEOREM 6.1. (Theory of interval integration). *If $Y$ is an interval function defined on $X=[a, b]$, then its interval integral* (6.1) *over $[a, b]$ exists.*

In other words, all interval functions are integrable (in the sense of interval integration). The simplicity of this theory is due to the fact that intervals are accepted as values of integrals, including the case that the integrand is degenerate (i.e., a single real function). The requirement that the integral of a real function be a real number rather than a possibly nondegenerate interval leads to numerous difficulties and correspondingly rich theories of integration (as elucidated in [3], for example), which constitute some of the most important chapters of real analysis. By the introduction of interval values for integrals, these difficulties are resolved, and the operation of integration is extended to all functions, interval or real. This is analogous to the way that the introduction of complex numbers extends the operation of root extraction to all numbers, complex or real. However, just as complex analysis does not supersede real analysis, it is to be expected that interval analysis will develop as a complementary, rather than a competitive, discipline to real analysis.

Some implications of the definitions of the interval integral given above, and some basic properties of interval integrals will now be investigated.

*Remark* 6.3. If $\underline{y}$, $\bar{y}$ are Riemann integrable on $[a, b]$, then

$$\int_a^b Y(x)\,dx = \left[(\mathrm{R})\int_a^b \underline{y}(x)\,dx,\ (\mathrm{R})\int_a^b \bar{y}(x)\,dx\right], \tag{6.6}$$

in terms of the Riemann integrals of the lower and upper endpoint functions.

This follows from (6.2) and the definition of a Riemann integrable function [3, p. 57] as one with equal upper and lower Darboux integrals; its Riemann integral is taken to be this common value, so that if $y$ is a Riemann integrable function on $X=[a, b]$, then its Riemann integral is

$$(\mathrm{R})\int_a^b y(x)\,dx = \underline{\int}_X y(x)\,dx = \overline{\int}_X y(x)\,dx. \tag{6.7}$$

*Remark* 6.4. In case $\underline{y}$, $\bar{y}$ are continuous on $[a, b]$, then the construction of the interval integral of $Y$ may be simplified by taking only the *equidistant partitions* $\bar{\Delta}_n$ defined by the points

$$x_k = a+\frac{k}{n}\cdot(b-a), \qquad k=0, 1, \cdots, n, \tag{6.8}$$

for each positive integer $n$, so that $w(X_k)=1/n$, and hence, by (4.20a),

$$\Sigma_{\bar{\Delta}_n} Y = \frac{1}{n} \cdot \sum_{i=1}^{n} \nabla Y_i. \tag{6.9}$$

Here, the formation of the Riemann sums $\Sigma_n Y$ can be skipped, and the interval integral is given by

$$\int_a^b Y(x)\,dx = \bigcap_{n=1}^{\infty} \Sigma_{\bar{\Delta}_n} Y = \left[ (\mathrm{R}) \int_a^b \underline{y}(x)\,dx, (\mathrm{R}) \int_a^b \bar{y}(x)\,dx \right] \tag{6.10}$$

[3, pp. 58–59], as continuous functions are Riemann integrable.

The interval integral (6.10) is the one proposed by R. E. Moore for continuous interval functions [5, Chapt. 8], [6, pp. 50–56], as the endpoint functions of a continuous interval function are necessarily continuous [5, p. 18], [6, p. 33]. Of course, even in the case $\underline{y}$, $\bar{y}$ are continuous, one may be able to find a partition $\Delta_n$ of $[a, b]$ other than $\bar{\Delta}_n$ such that $\Sigma_{\Delta_n} Y$ is properly contained in the Darboux sum (6.9), and hence provides a "more accurate" approximation to the interval integral than given by use of the equidistant partition. Some additional remarks about the numerical approximation of interval integrals will be made later.

Some basic properties of interval integrals come directly from the properties of the corresponding Riemann sums (5.4) which hold under the intersections in (6.1). Thus, from Theorem 5.1, one has the following result.

THEOREM 6.2. *If $Y$, $Z$ are interval functions on $X=[a, b]$ and $k$ is a constant, then*

(6.11)

(a) $$\int_a^b k \cdot Y(x)\,dx = k \cdot \int_a^b Y(x)\,dx;$$

(b) $$\int_a^b (Y(x)+Z(x))\,dx \subset \int_a^b Y(x)\,dx + \int_a^b Z(x)\,dx;$$

(c) *if $Y \subset Z$ on $X$, then*

$$\int_a^b Y(x)\,dx \subset \int_a^b Z(x)\,dx \qquad \text{(inclusion monotonicity).}$$

By taking intersections over all positive integers $n$ of the expressions in (5.15), one gets immediately:

THEOREM 6.3. *If $Y$ is an interval function defined on a finite, nondegenerate interval $X=[a, b]$, and $p$ is such that $a \leqq p \leqq b$, then*

$$\int_a^p Y(x)\,dx + \int_p^b Y(x)\,dx = \int_a^b Y(x)\,dx. \tag{6.12}$$

Similarly, Theorem 5.3 furnishes the following mean interval-value theorem for interval integrals.

THEOREM 6.4. *If $Y$ is defined on a finite, nondegenerate interval $X=[a, b]$, then an interval $\bar{Y} \subset \nabla Y_X$ exists such that*

$$\int_a^b Y(x)\,dx = w([a, b]) \cdot \bar{Y}. \tag{6.13}$$

*Proof.* Taking intersections of both sides of (5.23) over all positive integers $n$ gives (6.13) with

$$\bar{Y} = \bigcap_{n=1}^{\infty} \bar{Y}_n \subset \nabla Y_X. \tag{6.14}$$

Q.E.D.

Theorem 6.4 is useful in connection with properties of indefinite integrals.

DEFINITION 6.2. The interval function

$$I_aY(x) = \int_a^x Y(t)\,dt, \tag{6.15}$$

is called the *indefinite integral* of the interval function $Y$ over $[a, x]$ for $x \geqq a$. ($I^bY(x)$ is similarly defined over $[x, b]$ for $x \leqq b$.)

THEOREM 6.5. *If $Y$ is a bounded interval function on $[a, b]$, then $I_aY(x)$ is a continuous interval function at any $p \in [a, b]$.*

*Proof.* Suppose that $\nabla Y_{[a,b]} = [c, d]$ and take, as usual,

$$|\nabla Y_{[a,b]}| = \max\{|c|, |d|\}, \tag{6.16}$$

which is finite by hypothesis. For $a < p < b$, $a \leqq x \leqq p$, $\bar{Y}_{[x,p]} \subset \nabla Y_{[a,b]}$ exists such that

$$\int_a^p Y(x)\,dx = \int_a^x Y(t)\,dt + w([x, p]) \cdot \bar{Y}_{[x,p]}, \tag{6.17}$$

by Theorems 6.3 and 6.4; likewise, an interval $\bar{Y}_{[p,x]} \subset \nabla Y_{[a,b]}$ exists such that

$$\int_a^x Y(t)\,dt = \int_a^p Y(x)\,dx + w([x, p]) \cdot \bar{Y}_{[p,x]}, \tag{6.18}$$

for $p < x \leqq b$. Given any $\varepsilon > 0$, for $\delta = \varepsilon/|\nabla Y_{[a,b]}|$, the endpoints of $I_aY(p)$ thus differ from the endpoints of $I_aY(x)$ by less than $\varepsilon$ for $|x - p| < \delta$. Continuity of $I_aY(x)$ from the right at $x = a$ and from the left at $x = b$ is obtained from (6.18) and (6.17), respectively, as $I_aY(a) = 0$ by Theorem 6.4. Q.E.D.

Indefinite integrals also exhibit a type of differentiability if the limits

$$I'_-Y(x) = \lim_{p\uparrow x} \frac{1}{w([p, x])} \cdot I_pY(x) = \lim_{p\uparrow x} \bar{Y}_{[p,x]} \tag{6.19}$$

and

$$I'_+Y(x) = \lim_{q\downarrow x} \frac{1}{w([x, q])} \cdot I^qY(x) = \lim_{q\downarrow x} \bar{Y}_{[x,q]} \tag{6.20}$$

exist and are equal, where $\bar{Y}_{[p,x]}$ and $\bar{Y}_{[x,q]}$ are the intervals defined in Theorem 6.4 and $x$ lies interior to the interval of definition $[a, b]$ of $Y$. (One-sided derivatives at $x = a$ and $x = b$ are defined by (6.20) and (6.19), respectively.)

DEFINITION 6.3. If the limits $I'_-Y(x)$ and $I'_+Y(x)$ exist and are equal, then

$$I'_xY = I'_-Y(x) = I'_+Y(x) \tag{6.21}$$

is called the *derivative* of the indefinite integral of $Y$ at $x$.

The following theorem gives a condition under which the derivative of an indefinite interval integral is equal to its integrand.

THEOREM 6.6. *If $Y$ is a continuous interval function on $[a, b]$, then its indefinite integral is differentiable, and*

$$\begin{aligned} I'_xY &= Y(x), \qquad a < x < b, \\ I'_+Y(a) &= Y(a), \qquad I'_-Y(b) = Y(b). \end{aligned} \tag{6.22}$$

*Proof.* Let, for example, $\tilde{Y}_{[p,x]} = [\underline{z}(p), \bar{z}(p)]$ for $p < x$. If the upper endpoint function $\bar{y}$ of $Y$ is considered as an interval function, it follows from (4.31), (5.25), and (6.14) that $\bar{z} \in \nabla \bar{y}_{[p,x]}$. As $\bar{y}$ is continuous if $Y$ is a continuous interval function,

$$\lim_{p\uparrow x} \nabla \bar{y}_{[p,x]} = \bar{y}(x) = \lim_{p\uparrow x} \bar{z}(p). \tag{6.23}$$

Similarly, $\lim_{p\uparrow x} \underline{z}(p) = \underline{y}(x)$, so that $I'_- Y(x)$ exists, and

$$I'_- Y(x) = Y(x), \qquad a < x \leqq b. \tag{6.24}$$

In the same way, one has

$$I'_+ Y(x) = Y(x), \qquad a \leqq x < b, \tag{6.25}$$

which establishes (6.22). Q.E.D.

**7. Relationships between interval, Riemann, and Lebesgue integrals of real functions.** Ordinarily, no distinction will be made between a real function $y$ and the corresponding *degenerate* interval function $[y] = [y, y]$ having equal upper and lower endpoint functions. It is convenient, however, to distinguish between possible integrals of $y$ over an interval $\boldsymbol{X} = [a, b]$. The notation

$$\int_a^b y(x)\,dx, \qquad (\mathrm{L}) \int_a^b y(x)\,dx, \qquad (\mathrm{R}) \int_a^b y(x)\,dx \tag{7.1}$$

will be used to denote respectively the interval integral of $y$ as a degenerate interval function (which integral always exists), the Lebesgue integral of $y$ if $y$ is Lebesgue integrable over $[a, b]$, and finally, the Riemann integral of $y$ if it exists.

*Remark* 7.1. The integral of a degenerate interval function $y$ is a degenerate interval, that is,

$$\int_a^b y(x)\,dx = [r, r], \tag{7.2}$$

if and only if the real function $y$ is Riemann integrable over $[a, b]$, so that

$$r = (\mathrm{R}) \int_a^b y(x)\,dx. \tag{7.3}$$

This follows directly from Remark 6.3 and the definition (6.7) of the Riemann integral.

Thus, one ordinarily expects an interval integration, even of a single function, to result in a nondegenerate interval. For example, if $\chi_\rho$ is the characteristic function of the rationals, that is,

$$\begin{aligned} \chi_\rho(x) &= 1 \quad \text{for } x \text{ rational}, \\ \chi_\rho(x) &= 0 \quad \text{for } x \text{ irrational}, \end{aligned} \tag{7.4}$$

then

$$\int_0^1 \chi_\rho(x)\,dx = [0, 1], \tag{7.5}$$

as is well known. On the other hand, some nondegenerate interval functions have degenerate interval integrals. Consider the function $Y$ defined by

$$Y(x)=\begin{cases} 0, & 0\leqq x<\frac{1}{3}, \\ [0,1], & x=\frac{1}{3}, \\ 1, & \frac{1}{3}<x<\frac{2}{3}, \\ [1,2], & x=\frac{2}{3}, \\ 2, & \frac{2}{3}<x\leqq 1; \end{cases} \tag{7.6}$$

i.e., $Y$ is an *interval step-function*, which includes the "risers" as well as the "treads". For this function

$$\int_0^1 Y(x)\,dx=[1,1], \tag{7.7}$$

as the lower and upper boundary functions of $Y$ have equal (Riemann) integrals.

Any interval function $Y$ may be interpreted, of course, as a *set* of functions, that is,

$$Y=\{y\,|\,\underline{y}(x)\leqq y(x)\leqq \bar{y}(x),\ a\leqq x\leqq b\}. \tag{7.8}$$

If $Y$ is degenerate, then the set (7.8) consists of only the single function $y=\underline{y}=\bar{y}$. Otherwise, $Y$ will contain a number of functions, among which there may be subsets with certain distinguishing properties (continuity, differentiability, monotonicity, etc.). For the discussion of integration, the following subsets of functions will be singled out for special mention.

DEFINITION 7.1. If $Y$ is an interval function on $[a, b]$, then the set of Lebesgue (Riemann) integrable functions $y\in Y$ will be called the *Lebesgue* (*Riemann*) *core* of $Y$, and will be denoted by $C_{\mathrm{L}}(Y)$ $(C_{\mathrm{R}}(Y))$.

One has $C_{\mathrm{R}}(Y)\subset C_{\mathrm{L}}(Y)$ always, but these sets may, of course, be empty. For example, if $M$ is a subset of $[0, 1]$ which is not measurable in the sense of Lebesgue, then its characteristic function $\chi_M$ is a degenerate interval function with an empty Lebesgue (and hence Riemann) core. The characteristic function $\chi_\rho$ of the rationals considered earlier (see (7.4)) provides an example of a degenerate interval function with an empty Riemann core, but a nonempty Lebesgue core (the function $\chi_\rho$ itself).

DEFINITION 7.2. The *value* $v(C_{\mathrm{L}}(Y))$ $(v(C_{\mathrm{R}}(Y)))$ of the Lebesgue (Riemann) core of $Y$ on $[a, b]$ is defined by

$$\begin{aligned} v(C_{\mathrm{L}}(Y))&=\left\{r\,|\,r=(\mathrm{L})\int_a^b y(x)\,dx,\ y\in C_{\mathrm{L}}(Y)\right\}, \\ v(C_{\mathrm{R}}(Y))&=\left\{r\,|\,r=(\mathrm{R})\int_a^b y(x)\,dx,\ y\in C_{\mathrm{R}}(Y)\right\}, \end{aligned} \tag{7.9}$$

respectively, provided that the indicated cores of $Y$ are nonempty.

Each set $v(C_{\mathrm{L}}(Y))$ and $v(C_{\mathrm{R}}(Y))$, when nonempty, is *convex*, that is, if one contains values $r_1, r_2$, with $r_1\leqq r_2$, then it contains the entire interval $[r_1, r_2]$. This is because if $y_1$ has integral $r_1$ and $y_2$ has integral $r_2$, then the functions $y_\theta=y_1+\theta(y_2-y_1)$ are all integrable for $0\leqq\theta\leqq 1$, and have integrals equal to $r_\theta=r_1+\theta(r_2-r_1)$, $0\leqq\theta\leqq 1$, which is just another expression for the interval $[r_1, r_2]$. As a matter of fact, the theory of Lebesgue integration [3] leads to the conclusion that

$$I_{\mathrm{L}}(Y)=v(C_{\mathrm{L}}(Y)), \tag{7.10}$$

if it exists, is a closed interval, which will be called the *Lebesgue subinterval* of the interval integral (6.1) of $Y$ over $[a, b]$. The set $v(C_{\mathrm{R}}(Y))$, on the other hand, is not necessarily closed. This is considered to be a defect of Riemann integration, and led to the construction of the theory of Lebesgue integration. However, as $v(C_{\mathrm{R}}(Y))$ is convex, then its closure,

$$I_{\mathrm{R}}(Y)=\overline{v(C_{\mathrm{R}}(Y))}, \tag{7.11}$$

is a closed interval which, if it exists, will be called the *Riemann subinterval* of the interval integral of $Y$ over $[a, b]$.

The purpose of the introduction of the intervals (7.10) and (7.11) is to provide some quantitative information about the Lebesgue and Riemann cores of an interval function $Y$ which measures its "integrability" in a certain fashion. In the metric topology for intervals [5], [6], the *distance* between intervals $[a, b]$ and $[c, d]$ is defined to be

$$d([a, b], [c, d])=\max\{|a-c|, |b-d|\}. \tag{7.12}$$

(In the extended real number system, rule (2.7x) is used to resolve any indeterminant forms entering into (7.12).)

DEFINITION 7.3. For

$$I(Y)=\int_a^b Y(x)\,dx, \tag{7.13}$$

if the Riemann core $C_{\mathrm{R}}(Y)$ of $Y$ is nonempty, then

$$\rho(Y)=d(I_{\mathrm{R}}(Y), I(Y)) \tag{7.14}$$

is called the *Riemann gap* of the interval function $Y$ on $[a, b]$; similarly, if $C_{\mathrm{L}}(Y)$ is nonempty, then

$$\lambda(Y)=d(I_{\mathrm{L}}(Y), I(Y)) \tag{7.15}$$

is called the *Lebesgue gap* of $Y$ on $[a, b]$.

*Remark* 7.2. One has

$$\lambda(Y)\leqq\rho(Y), \tag{7.16}$$

in case both numbers are defined.

This follows from the inclusion $C_{\mathrm{R}}(Y)\subset C_{\mathrm{L}}(Y)$. If only one of the numbers $\lambda(Y)$, $\rho(Y)$ is defined, it will be $\lambda(Y)$ by the same token. For the example (7.4) of the degenerate interval function $\chi_\rho$, one has $\lambda(\chi_\rho)=1$, and $\rho(\chi_\rho)$ is not defined.

THEOREM 7.1. *If the endpoint functions $\underline{y}$, $\bar{y}$ are Riemann integrable over $[a, b]$, then $\lambda(Y)=0$; if $\lambda(Y)=0$, then $\underline{y}$, $\bar{y}$ are Lebesgue integrable, and*

$$\int_a^b Y(x)\,dx=\left[(\mathrm{L})\int_a^b \underline{y}(x)\,dx,\ (\mathrm{L})\int_a^b \bar{y}(x)\,dx\right]. \tag{7.17}$$

*Proof.* By Remark 6.3, the Riemann integrability of $\underline{y}$, $\bar{y}$ means that $\rho(Y)=0$; hence, $\lambda(Y)=0$ by (7.16). Conversely, if $\lambda(Y)=0$, then the integral $I(Y)$ is finite, and bounded sequences $\{\underline{y}_n\}, \{\bar{y}_n\}\subset Y$ of Lebesgue integrable functions may be found which converge to $\underline{y}$ and $\bar{y}$, respectively. It follows [3, p. 81] that $\underline{y}$ and $\bar{y}$ are Lebesgue integrable on $[a, b]$ and, as

$$\lim_{n\to\infty}(\mathrm{L})\int_a^b \underline{y}_n(x)\,dx=\int_{\underline{X}} \underline{y}(x)\,dx, \tag{7.18}$$

one has that

$$\int_{\underline{X}} \underline{y}(x)\,dx = (\mathrm{L})\int_a^b \underline{y}(x)\,dx, \tag{7.19}$$

and similarly for $\bar{y}$, whence (7.17). Q.E.D.

*Remark* 7.3. If $\underline{y}$ and $\bar{y}$ are Lebesgue integrable on $[a, b]$, then

$$\lambda(Y) = \max\left\{(\mathrm{L})\int_a^b \underline{y}(x)\,dx - \int_{\underline{X}} \underline{y}(x)\,dx,\ \int_X \bar{y}(x)\,dx - (\mathrm{L})\int_a^b \bar{y}(x)\,dx\right\}. \tag{7.20}$$

This is true because $\underline{y}$ is the "smallest" Lebesgue integrable function contained in the interval function $Y$, and $\bar{y}$ the "largest" in the sense that for each function $y \in C_{\mathrm{L}}(Y)$, one has $\underline{y}(x) \leqq y(x) \leqq \bar{y}(x)$, $a \leqq x \leqq b$. Thus,

$$v(C_{\mathrm{L}}(Y)) = [(\mathrm{L})\int_a^b \underline{y}(x)\,dx,\ (\mathrm{L})\int_a^b \bar{y}(x)\,dx], \tag{7.21}$$

from which (7.20) follows by (7.12).

**8. Improper integrals.** In ordinary integration theory, an integral

$$r = \int_a^b y(x)\,dx \tag{8.1}$$

is said to be *improper* if the interval of integration $[a, b]$ is infinite, or if its integrand is unbounded on $X = [a, b]$ in the sense that given any $M > 0$, there exists a nondegenerate subinterval $X_M$ of $X$ such that $|y(x)| \geqq M$ for $x \in X_M$. Supposing that $y$ is unbounded on $X = [a, b]$ in the sense that given any $M > 0$ there exists a nonde-Riemann) on $[\alpha, b]$, one defines the *improper Riemann integral* of $y$ over $[a, b]$ to be

$$(\mathrm{IR})\int_a^b y(x)\,dx = \lim_{\alpha \downarrow a} (\mathrm{R})\int_\alpha^b y(x)\,dx, \tag{8.2}$$

provided this limit exists (in the extended real number system; infinite values will be accepted here for improper integrals). Similarly, if $y$ is Riemann integrable over $[a, b]$ for $b > a$ finite, then

$$(\mathrm{IR})\int_a^\infty y(x)\,dx = \lim_{b \to \infty} (\mathrm{R})\int_a^b y(x)\,dx \tag{8.3}$$

by definition, again if the indicated limit exists.

The definition of interval integrals given in § 6 yields values of certain improper Riemann integrals if the integrand $y$ is interpreted to be the degenerate interval function $[y, y]$, for example,

$$\begin{aligned} &(\mathrm{a}) \quad \int_0^1 x^{-1/3}\,dx = [\tfrac{3}{2}, \infty], \\ &(\mathrm{b}) \quad \int_0^1 x^{-1}\,dx = [\infty, \infty], \\ &(\mathrm{c}) \quad \int_0^\infty (-e^{-x})\,dx = [-\infty, -1]. \end{aligned} \tag{8.4}$$

In the above, the value of the improper Riemann integral appears as the finite endpoint in each of the intervals (8.4a) and (8.4c). The indegenerate interval (8.4b) indicates correctly that the corresponding improper Riemann integral is *divergent.*

DEFINITION 8.1. An interval integral (6.1) is said to be *infinite* if its value is one of the indegenerate intervals $[-\infty, -\infty]$ or $[\infty, \infty]$, *indeterminant* if it is equal to $\boldsymbol{R} = [-\infty, \infty]$, or *improper* if its value is a semi-infinite interval $[a, \infty]$ or $[-\infty, b]$; otherwise, it is said to be *finite*.

The relationship between improper interval and Riemann integrals will now be considered for the cases (8.2) and (8.3), as illustrated by (8.4a) and (8.4c), respectively.

Suppose that $y(x)$ is unbounded above at $x = a$. Thus, every Darboux sum (4.7) will contain a term of the form (after elimination of nondistinct partition points, if necessary)

$$w(X_1) \cdot \nabla y_1 = [w(X_1) \cdot c_1, \infty], \tag{8.5}$$

where $X_1 = [a, x_1]$ and

$$c_1 = \inf_{x \in X_1} \{y(x)\}. \tag{8.6}$$

The interval integral of $y$ will hence be either improper or infinite. The following theorem is illustrated by (8.4a).

THEOREM 8.1. *Suppose that $y$ is Riemann integrable over $[\alpha, b]$ for $a < \alpha < b$, and the indefinite interval integral $I_a y(\alpha)$ satisfies*

$$\lim_{\alpha \downarrow a} I_a y(\alpha) = \lim_{\alpha \downarrow a} \int_a^\alpha y(x)\, dx = [0, \infty]; \tag{8.7}$$

*then the improper Riemann integral* (8.2) *of $y$ over $[a, b]$ exists, and*

$$\int_a^b y(x)\, dx = \left[(\mathrm{IR}) \int_a^b y(x)\, dx, \infty\right]. \tag{8.8}$$

*Proof.* One has

$$\int_a^b y(x)\, dx = \int_a^\alpha y(x)\, dx + \int_\alpha^b y(x)\, dx \tag{8.9}$$

by Theorem 6.3 and, by Remark 6.3,

$$\int_\alpha^b y(x)\, dx = \left[(\mathrm{R}) \int_\alpha^b y(x)\, dx, (\mathrm{R}) \int_\alpha^b y(x)\, dx\right] = (\mathrm{R}) \int_\alpha^b y(x)\, dx \tag{8.10}$$

as degenerate intervals may be identified with the corresponding real numbers. Taking the limit as $\alpha \downarrow a$ of both sides of (8.9) gives (8.8) Q.E.D.

In the case of integration over an infinite interval, say $[a, \infty]$, suppose, for example, that $y$ is negative but that $y(x) \uparrow 0$ as $x \to \infty$, as in (8.4c). Then, each Darboux sum (4.7) will correspond to a partition $\Delta_n$ with $x_{n-1}$ finite, $x_n = +\infty$, and as

$$\nabla y_n = [c, 0], \tag{8.11}$$

where $c = \inf_{x \in X_n} \{y(x)\} < 0$, $w(X_n) = w([x_{n-1}, \infty]) = \infty$, then each will contain a term equal to

$$w(X_n) \cdot \nabla y_n = [-\infty, 0], \tag{8.12}$$

by rules (2.1iii) and (2.1ix). The situation illustrated by the example (8.4c) is a case of the following result.

THEOREM 8.2. *Suppose that $y$ is Riemann integrable over the finite interval $[a, b]$ for each $b > a$, and the indefinite interval integral $I^\infty y(b)$ satisfies*

$$\lim_{b\to\infty} I^\infty y(b) = [-\infty, 0]; \tag{8.13}$$

*then, the improper Riemann integral* (8.3) *of $y$ over $[a, \infty]$ exists, and*

$$\int_a^\infty y(x)\,dx = \left[-\infty, (\mathrm{IR})\int_a^\infty y(x)\,dx\right]. \tag{8.14}$$

*Proof.* This follows exactly in the same way as Theorem 8.1 by writing

$$\int_a^\infty y(x)\,dx = \int_a^b y(x)\,dx + \int_b^\infty y(x)\,dx, \tag{8.15}$$

and noting that

$$\int_a^b y(x)\,dx = (\mathrm{R})\int_a^b y(x)\,dx \tag{8.16}$$

as a degenerate interval. Q.E.D.

Other cases of improper interval and Riemann integrals may be treated in a similar fashion..

**9. Computational implications of the theory.** One purpose of the theory of integration of interval functions developed above is to provide a theoretical framework for the investigation of the numerical solution of linear and nonlinear integral equations such as

$$u(x) = (\mathrm{R})\int_a^b g(x, t, u(x), u(t))\,dt, \tag{9.1}$$

by interval methods. One approach along these lines is to reformulate (9.1) as an interval equation,

$$U = T(U), \tag{9.2}$$

for an interval function $U$ which contains the desired solution $u$ of the integral (9.1). Under certain conditions, the operator $T$ will be a contraction mapping [1], [2], and the iteration process

$$U_{n+1} = T(U_n), \qquad n = 0, 1, 2, \cdots \tag{9.3}$$

will converge to give a solution of (9.2). To implement this for the integral equation (9.1), one forms the interval functions $G_n = [\underline{g}_n, \bar{g}_n]$, $n = 0, 1, 2, \cdots$, where

$$\begin{aligned} \underline{g}_n(x, t) &= \inf\{g(x, t, U_n(x), U_n(t))\}, \\ \bar{g}_n(x, t) &= \sup\{g(x, t, U_n(x), U_n(t))\}, \end{aligned} \tag{9.4}$$

and then (9.3) becomes

$$U_{n+1}(x) = \int_a^b G_n(x, t, U_n(x), U_n(t))\,dt, \tag{9.5}$$

in terms of interval integration. Of course, if $\underline{g}_n(x, t)$ and $\bar{g}_n(x, t)$ are Riemann integrable in $t$, then the endpoint functions $\underline{u}_{n+1}$, $\bar{u}_{n+1}$ of $U_{n+1}$ are obtained by Riemann integration. From a numerical standpoint, in this case approximations $u^*_{n+1} \leqq \underline{u}_{n+1}$, $u^{**}_{n+1} \geqq$

$\bar{u}_{n+1}$ may be obtained to prescribed accuracy by any one of a number of methods, including the use of Darboux sums as defined in § 4 [7], with higher order accuracy being obtainable from integration of Taylor polynomial approximations to the endpoint functions, or by other rules of numerical integration [4], [5], [6], [9], provided, of course, that the endpoint functions are smooth enough.

A particularly simple case occurs if $g$ is *monotone* in the sense that

$$\begin{aligned} \underline{g}_n(x, t) &= g(x, t, \underline{u}_n(x), \underline{u}_n(t)), \\ \bar{g}_n(x, t) &= g(x, t, \bar{u}_n(x), \bar{u}_n(t)), \end{aligned} \tag{9.6}$$

that is, the endpoint functions of $U_n$ transform into the endpoint functions of $G_n$, and if $g$ further transforms Riemann integrable functions into Riemann integrable functions. Here, the iteration (9.5) can be carried out using only the endpoint functions if one starts with an interval $U_0 = [\underline{u}_0, \bar{u}_0]$ which has Riemann integrable endpoint functions. An example of this approach to the solution of a nonlinear integral equation was given by Rall [7], in which step-functions were used as endpoint functions (and $T$ was approximated by a numerical operator $S$ such that $T \subset S$). In many cases, continuous solutions $u$ are sought for integral equations (9.1), which gives rise to the following concept.

DEFINITION 9.1. The *continuous core* $C_C(U)$ of an interval function $U$ on $[a, b]$ is defined to be the set of continuous functions $y$ contained in $U$, that is

$$C_C(U) = \{y \mid y \in U \cap C[a, b]\}. \tag{9.7}$$

Evidently, $C_C(U) \subset C_R(U)$, the Riemann core of $U$ defined earlier.

If $g$ is a continuous function of its arguments, and the interval operator $T$ is such that the continuous function $v$ defined by

$$v(x) = (\mathrm{R}) \int_a^b g(x, t, u(x), u(t))\, dt \tag{9.8}$$

belongs to $T(U)$ for $u \in C_C(U)$, then it follows that each continuous solution $u$ of (9.1) will belong to $C_C(T(U))$ if it belongs to $U$ and hence to $C_C(U)$. Thus, it is tempting to try to compute the sequence (9.3) using only $C_C(U_n)$, where $U_0$ is taken to have continuous endpoint functions. However, in general, the functions $\underline{g}_1(x, t)$ and $\bar{g}_1(x, t)$ obtained from (9.4) will be only *semi-continuous* if $U_0$ is replaced by $C_C(U_0)$, and these so-called $L$- and $U$-functions may not even be Riemann integrable [3]. The theory of interval integration developed in this paper resolves this difficulty by allowing computation with the interval functions $U_n$ directly, regardless of the character of their endpoint functions.

*Remark* 9.1. If $u \in C_C(U_0)$ is a solution of (9.1), then for the sequence (9.3) constructed by the operations (9.4) and the interval integration (9.5), it follows from the condition (9.8) for continuous $g$ that

$$u \in C_C(U_n), \qquad n = 0, 1, 2, \cdots; \tag{9.9}$$

furthermore, for

$$U = \bigcap_{n=1}^{\infty} U_n, \tag{9.10}$$

one has $u \in C_C(U)$.

*Remark* 9.2. In the favorable case that $U_{n+1} \subset U_n$, $n = 0, 1, 2, \cdots$, and

$$\lim_{n\to\infty} \sup_{[a,b]} \{w(U_n(x))\} = 0, \tag{9.11}$$

one has that $U = [u, u] = u$ defined by (9.10) satisfies the integral equation (9.1), since a degenerate interval integral of a degenerate interval function is necessarily a Riemann integral; furthermore, one has error bounds of the form

$$\underline{u}_n(x) \leqq u(x) \leqq \bar{u}_n(x), \qquad a \leqq x \leqq b, \tag{9.12}$$

for $n = 0, 1, 2, \cdots$.

Further applications of interval integration to the solution of integral equations will be investigated in subsequent papers.

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