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INTEGRATION OF INTERVAL FUNCTIONS II. THE FINITE CASE*

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Abstract. Caprani, Madsen and Rall [Siam J. Math. Anal., 12 (1981), pp. 321–341] have shown previously that the use of interval values leads to a simple theory of integration in which all functions, interval and real, are integrable. Here, a simplified construction of the interval integral is given for the case that the integrand and interval of integration are finite; the interval integral is shown to be the intersection of the interval Darboux sums corresponding to the partitions of the interval of integration into subintervals of equal length. A rate of convergence of these interval Darboux sums to the interval integral is given for Lipschitz continuous integrands. An alternate approach to interval integration in the unbounded case via finite interval integrals is presented. The results give theoretical support to interval methods for the solution of integral equations and finding extreme values of functionals defined in terms of integrals.

1. Introduction. The construction of the interval integral, given in the general case in [1], can be simplified drastically in the case that the interval of integration is finite and the integrand is a bounded interval function. (Definitions of the necessary concepts will be given below.) In particular, the use of the extended real number system is not required, so all computations can be done by ordinary interval arithmetic [3], [4]. Furthermore, it is not necessary to consider all partitions of the interval of integration into subintervals as the partition into subintervals with equal lengths will be shown to suffice. This eliminates an inherently nonconstructive portion of the definition of the interval interval, the formation of the so-called interval Riemann sums.

In addition to the simplification of the construction of the interval interval in this case, rates of convergence of the Darboux sums based on the equipartition of the interval of integration to the interval integral will be derived for sufficiently smooth integrands. Another approach to improper interval integrals will also be given.

2. Interval functions. Following the definitions in [1], an *interval function* Y defined on an interval $X=[a,b]$ assigns the interval value

$$Y(x)=[\underline{y}(x),\bar{y}(x)] \tag{2.1}$$

to each real number $x\in X$, where $\underline{y},\bar{y}$ are real functions called respectively the *lower* and *upper boundary functions* (or *endpoint functions*) of Y.

The *vertical extent* of Y on X is defined to be the interval

$$\nabla Y(X)=\left[\inf_{x\in X}\{\underline{y}(x)\},\ \sup_{x\in X}\{\bar{y}(x)\}\right]. \tag{2.2}$$

In this paper only intervals of integration with finite *width* $w(X)=b-a$ and *bounded* interval functions such that $w(\nabla Y(X))<+\infty$ will be considered. This is the *finite case*.

The notation $Y=[\underline{y},\bar{y}]$ will also be used for interval functions. Real functions y may be identified with the interval functions $y=[y,y]$ with equal endpoint functions, which are called *degenerate* interval functions [1].

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3. Interval integrals. In general, the *interval integral* of an interval function Y over the interval $X=[a,b]$ is the interval

$$(3.1)\qquad \int_X Y(x)\,dx=\int_a^b Y(x)\,dx=\left[\underline{\int_X}\underline{y}(x)\,dx,\ \overline{\int_X}\bar{y}(x)\,dx\right],$$

where $\underline{\int_X}\underline{y}(x)\,dx$ denotes the lower Darboux integral of the lower endpoint function $\underline{y}$ over the interval X and $\overline{\int_X}\bar{y}(x)\,dx$ gives the upper Darboux integral of the upper endpoint function $\bar{y}$ over X [2]. As these Darboux integrals always exist in the extended real number system, it follows that all interval (and hence all real) functions are integrable in this sense. The definite and indefinite interval integrals have many properties similar to those of the Riemann integral [1].

The construction of the interval integral, carried out in [1] in the spirit of interval analysis, is done in three steps. The first step consists of partition of the interval X into subintervals $X_i=[x_{i-1},x_i]$, $i=1,2,\cdots,n$ by means of points

$$(3.2)\qquad a=x_0\leq x_1\leq\cdots\leq x_{i-1}\leq x_i\leq\cdots\leq x_{n-1}\leq x_n=b$$

to obtain the *partition*

$$(3.3)\qquad \Delta_n=(X_1,X_2,\cdots,X_n)$$

of X and the corresponding *interval Darboux sum*

$$(3.4)\qquad \sum_{\Delta_n} Y(X)=\sum_{i=1}^{n} w(X_i)\cdot\nabla Y(X_i).$$

Next, for each positive integer n, let $\mathfrak{D}_n$ denote the set of all partitions (3.3). The *interval Riemann sum of order n* is then defined to be

$$(3.5)\qquad \sum_n Y(X)=\bigcap_{\Delta_n\in\mathfrak{D}_n}\sum_{\Delta_n} Y(X).$$

Finally, the interval integral of Y over X is given by

$$(3.6)\qquad \int_a^b Y(x)\,dx=\bigcap_{n=1}^{\infty}\sum_n Y(X),$$

which is nonempty, as the interval Riemann sums form a decreasing sequence of nonempty closed sets [1], and agrees with (3.1). This construction will be simplified in the finite case.

4. The finite case. The interval integral (3.6) will be said to be *finitely defined* if the integrand Y is a bounded interval function and the interval of integration $X=[a,b]$ is finite. The *equipartition* $\bar{\Delta}_n$ of X is defined by the points

$$(4.1)\qquad x_i=a+ih,\qquad h=\frac{b-a}{n},\qquad i=0,1,\cdots,n,$$

so that

$$(4.2)\qquad w(X_i)=x_i-x_{i-1}=\frac{b-a}{n}=\frac{w(X)}{n}=h,\qquad i=1,2,\cdots,n.$$

The corresponding interval Darboux sum is

$$(4.3)\qquad \sum_{\bar{\Delta}_n} Y(X)=\bar{\sum_n} Y(X)=\frac{w(X)}{n}\sum_{i=1}^{n}\nabla Y(X_i).$$

THEOREM 4.1. *In the finite case,*

$$\int_a^b Y(x)\,dx = \bigcap_{n=1}^{\infty} \overline{\sum_n} Y(X). \tag{4.4}$$

Thus, this construction requires only the formation of the single interval Darboux sum (4.3) for each positive integer n and skips the (nonconstructive) calculation of interval Riemann sums (3.5) entirely. Furthermore, (4.4) agrees with the definition of the interval integral given by R. E. Moore [2], [3], in the case that the endpoint functions $y, \bar{y}$ of Y are assumed to be continuous. Theorem 4.1 will be proved in §6 based on results on subintervals established in the next section.

5. Two lemmas on subintervals. The first lemma simplifies the proof of the mean interval-value theorem for interval integrals over a finite interval of integration.

LEMMA 5.1. *If $Z_i = [c_i, d_i] \subset Z = [c, d]$ are finite intervals, $i = 1, 2, \cdots, n$, and $\alpha_i \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$, then*

$$\sum_{i=1}^{n} \alpha_i Z_i \subset Z. \tag{5.1}$$

Proof. This follows at once from the elementary inequalities

$$a \leq \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n \leq \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n \leq b \tag{5.2}$$

for convex combinations of real numbers. Q.E.D.

On the assumption that Theorem 4.1 holds this gives the mean interval-value theorem [1] for the interval integral (4.4) as

$$\frac{1}{n} \sum_{i=1}^{n} \nabla Y(x_i) = \overline{Y}_n \subset \nabla Y(X) \tag{5.3}$$

by Lemma 5.1 and from (4.4)

$$\int_a^b Y(x)\,dx = w(X) \cdot \bigcap_{n=1}^{\infty} \overline{Y}_n = w(X) \cdot \overline{Y}, \tag{5.4}$$

where $\overline{Y} \subset \nabla Y(X)$.

The *excess width* of an interval $Z = [c, d]$ over a subinterval $Z' = [c', d'] \subset Z$ is defined to be

$$e(Z, Z') = \max\{c' - c, d - d'\}. \tag{5.5}$$

It is evident that

$$e(Z, Z') \leq w(Z) = d - c. \tag{5.6}$$

A *symmetric interval* is an interval S of the form $S = [-s, s]$, where $s \geq 0$.

LEMMA 5.2. *If $Z' \subset Z$, then for each symmetric interval $S = [-s, s]$ with $s \geq e(Z, Z')$, one has*

$$Z \subset Z' + S. \tag{5.7}$$

In particular,

$$Z \subset Z' + [-w(Z), w(Z)]. \tag{5.8}$$

Proof. The inclusion (5.7) follows from the definition (5.5) and the definition of interval addition [2], [3]; (5.8) follows immediately by (5.6). Q.E.D.

6. Proof of Theorem 4.1. It is to be shown that definitions (3.6) and (4.4) of the interval integral agree in the finite case. Set

$$I=\bigcap_{n=1}^{\infty} \bar{\sum_{n}} Y(X). \tag{6.1}$$

As the interval integral (3.6) is contained in each Darboux sum $\Sigma_{\Delta_n} Y(X)$, it follows that

$$\int_a^b Y(x)\,dx \subset \bar{\sum_{n}} Y(X), \qquad n=1,2,\cdots, \tag{6.2}$$

and thus

$$\int_a^b Y(x)\,dx \subset I=\bigcap_{n=1}^{\infty} \bar{\sum_{n}} Y(X). \tag{6.3}$$

Suppose that a partition point p, $x_{i-1}\le p\le x_i$ is introduced into an interval X_i. By Lemma 5.1, one has

$$w[x_{i-1},p]\cdot \nabla Y([x_{i-1},p])+w[p,x_i]\cdot \nabla Y([p,x_i])\subset w(X_i)\cdot \nabla Y(X_i). \tag{6.4}$$

Consider an arbitrary interval Darboux sum $\Sigma_{\Delta_m} Y(X)$ for some positive integer m. For each $n\ge m$, the partition points $x_1, x_2,\cdots,x_{m-1}$ of the interval Darboux sum are interior to at most $m-1$ subintervals of $\bar{\Sigma}_n Y(X)$, with total length not exceeding $((m-1)/n)w(X)$. After deletion of these subintervals from $\bar{\Delta}_n$, the remaining partition points of $\bar{\Delta}_n$ will belong to the subintervals of Δ_m. By (6.4) and Lemma 5.2,

$$\bar{\sum_{n}} Y(X)\subset \sum_{\Delta_m} Y(X)+\frac{(m-1)w(X)}{n}[-w(\nabla Y(X)),w(\nabla Y(X))]. \tag{6.5}$$

As (6.5) holds for each partition Δ_m and positive integer $n\ge m$, from (3.5),

$$\bar{\sum_{n}} Y(X)\subset \sum_{m} Y(X)+\frac{(m-1)w(X)}{n} w(\nabla Y(X))\cdot[-1,1]. \tag{6.6}$$

As $w(\nabla Y(X))<+\infty$, taking the intersectioin of both sides of (6.6) with respect to n gives

$$I\subset \sum_{m} Y(X)+[0,0]=\sum_{m} Y(X). \tag{6.7}$$

From (6.7) it follows that

$$I\subset \bigcap_{m=1}^{\infty} \sum_{m} Y(X)=\int_a^b Y(x)\,dx. \tag{6.8}$$

Comparison of (6.3) and (6.8) yields (4.4). Q.E.D.

This result can also be established using the relationships expressed in terms of elementary integrals of step functions as upper and lower limits of the interval Darboux sums [1] as in [2, pp. 54–56].

7. A rate of convergence for smooth integrands. As in ordinary interval analysis, an interval function $Y=[\underline{y},\bar{y}]$ is *continuous* if the real functions $\underline{y},\bar{y}$ are continuous. Similarly, Y is *Lipschitz continuous* if a Lipschitz constant $L>0$ exists for both $\underline{y}$ and $\bar{y}$, that is, $|\underline{y}(x)-\underline{y}(z)|\le L|x-z|$ and $|\bar{y}(x)-\bar{y}(z)|\le L|x-z|$ for $x,z\in X$.

Interval integrals of continuous interval functions can be expressed in terms of the Riemann (R) integrals of their endpoint functions [1]:

$$\int_a^b Y(x)\,dx=\left[(\mathrm{R})\int_a^b \underline{y}(x)\,dx,\ (\mathrm{R})\int_a^b \bar{y}(x)\,dx\right]. \tag{7.1}$$

One may also write

$$(\mathrm{R})\int_a^b y(x)\,dx=\sum_{i=1}^{n}(\mathrm{R})\int_{x_{i-1}}^{x_i} y(x)\,dx \tag{7.2}$$

for a Riemann integrable function y and a given partition Δ_n of $X=[a,b]$. If y is continuous, then on each subinterval X_i, $i=1,2,\cdots,n$,

$$(\mathrm{R})\int_{x_{i-1}}^{x_i} y(x)\,dx=y(\xi_i)(x_i-x_{i-1})=y(\xi_i)w(X_i), \qquad \xi_i\in X_i \tag{7.3}$$

[2, p. 209]. Furthermore,

$$\nabla y(X_i)=[c_i,d_i]=[y(\eta_i),y(\zeta_i)], \qquad \eta_i,\zeta_i\in X_i. \tag{7.4}$$

Thus, if y is Lipschitz continuous, then

$$d_i\cdot w(X_i)-(\mathrm{R})\int_a^b y(x)\,dx=[y(\zeta_i)-y(\xi_i)]\cdot w(X_i)\le L\cdot w(X_i)^2 \tag{7.5}$$

and

$$(\mathrm{R})\int_a^b y(x)\,dx-c_i\cdot w(X_i)=[y(\xi_i)-y(\eta_i)]\cdot w(X_i)\le L\cdot w(X_i)^2. \tag{7.6}$$

Applying (7.5) and (7.6) to $\bar{y}$ and $\underline{y}$ respectively for the equipartition with $w(X_i)=w(X)/n$ gives the following inequality for the excess width of $\bar{\Sigma}_n Y(X)$ over the interval integral (7.1) of Y.

THEOREM 7.1. *If Y is a Lipschitz continuous interval function, then*

$$e\left(\bar{\sum_n} Y(X),\ \int_a^b Y(x)\,dx\right)\le\frac{L\cdot w(X)}{n}. \tag{7.7}$$

The use of interval Darboux sums as approximations to the interval integral is an extension of the crude method of upper and lower Riemann sums [5] for the approximation of the integral of a Riemann integrable real function. The Darboux sums are generally easy to inclose and give rigorous upper and lower bounds for the value of the integral, but the rate of convergence as given by (7.7) is slow. Of course, the use of partitions other than the equipartition may be of benefit in some cases, but for smooth functions, the improvement may be marginal. For example, for

$$Y(x)=[0,3x^2], \qquad \int_0^1[0,3x^2]\,dx=[0,1], \tag{7.8}$$

the equipartition for $n=2$ gives

$$\bar{\sum_2} Y([0,1])=\tfrac{1}{2}\left[0,\frac{3}{4}\right]+\tfrac{1}{2}[0,3]=[0,\tfrac{15}{8}]=[0,1.875]. \tag{7.9}$$

The interval Riemann sum in this case corresponds to the use of the partition point $x_1=1/\sqrt{3}$ and has the value

$$\sum_2 Y([0,1])=\frac{[0,1]}{\sqrt{3}}+[0,3]\cdot\left(1-\frac{1}{\sqrt{3}}\right)=\left[0,3-\frac{2}{\sqrt{3}}\right]\subset[0,1.846]. \tag{7.10}$$

Although this is better than (7.9), extra labor was required to determine the optimal partition, and this additional effort increases rapidly with n.

8. Inner improper interval integrals. In [1] an interval integral was said to be unbounded if its value is an infinite interval. These unbounded interval integrals arise if the integrand or the interval of integration is unbounded. Relationships were developed in [1] between the value of the finite endpoint of a semi-infinite, or *improper* interval integral and the improper Riemann integral of the corresponding endpoint function of the integrand. Here, an approach to improper interval integration will be made via finitely defined interval integrals.

Case I. $Y(x)$ is an unbounded interval function on a finite interval of integration $X=[a,b]$; that is, $\nabla Y(X)=+\infty$, $w(X)<+\infty$. Here, the functions

$$Y_N(x)=Y(x)\cap[-N,N] \tag{8.1}$$

are defined for each positive integer N. The corresponding finitely defined interval integrals

$$I_N Y(X)=\int_a^b Y_N(x)\,dx, \qquad N=1,2,3,\cdots, \tag{8.2}$$

are finite and may be obtained from (4.4). For $M>N$,

$$I_N Y(X)\subset I_M Y(X), \tag{8.3}$$

because the interval integral is inclusion monotone, and $Y_N(X)\subset Y_M(X)$ for $M>N$ [1]. The *inner improper interval integral* in this case is defined to be

$$(\mathrm{I})\int_a^b Y(x)\,dx=\lim_{N\to\infty} I_N Y(X)\subset\int_a^b Y(x)\,dx, \tag{8.4}$$

the inclusion following again from $Y_N(X)\subset Y(X)$ and inclusion monotonicity of the interval integral. It follows that the inner improper interval integral exists (in the extended real number system) if the interval of integration is finite. The following examples are taken from [1].

(a) $Y(x)=x^{-1/3}$, a real function, $X=[0,1]$.

$$I_N Y([0,1])=\int_0^{N^{-3}} N\,dx+\int_{N^{-3}}^1 x^{-1/3}\,dx=\tfrac{1}{2}[3-N^{-2},3-N^{-2}]. \tag{8.5}$$

Thus,

$$(\mathrm{I})\int_0^1 x^{-1/3}\,dx=\left[\frac{3}{2},\frac{3}{2}\right]\subset\int_0^1 x^{-1/3}\,dx=\left[\frac{3}{2},\infty\right]. \tag{8.6}$$

(b) $Y(x)=x^{-1}$, $X=[0,1]$. Here,

$$I_n Y([0,1])=\int_0^{N^{-1}} N\,dx+\int_{N^{-1}}^1 x^{-1}\,dx=[1+\ln N,\ 1+\ln N] \tag{8.7}$$

and

$$(\mathrm{I})\int_0^1 x^{-1}\,dx=[\infty,\infty]=\int_0^1 x^{-1}\,dx, \tag{8.8}$$

an infinite integral. The standard definition of the improper Riemann (IR) integral of real functions over a finite interval ([2, p. 88]) gives the following result.

THEOREM 8.1. *If the endpoint functions* $\underline{y}, \bar{y}$ *of* Y *have improper Riemann* (IR) *integrals over the finite interval* $X=[a,b]$, *then*

$$(8.9)\qquad (\mathrm{I})\int_a^b Y(x)\,dx=\left[(\mathrm{IR})\int_a^b \underline{y}(x)\,dx,\ (\mathrm{IR})\int_a^b \bar{y}(x)\,dx\right].$$

Of course, in case Y is bounded, or the real function y is bounded and Riemann (R) integrable, one may take

$$(8.10)\qquad (\mathrm{I})\int_a^b Y(x)\,dx=\int_a^b Y(x)\,dx,\qquad (\mathrm{IR})\int_a^b y(x)\,dx=(\mathrm{R})\int_a^b y(x)\,dx,$$

respectively.

Finitely defined interval integrals may also be used to construct an improper integral over infinite intervals of integration. For simplicity of notation, take $Y(x)=[0,0]$ outside X and the interval of integration to be the real line $R=[-\infty,\infty]$.

DEFINITION 8.1. *If*

$$(8.11)\qquad I_+Y=\lim_{N\to\infty}(\mathrm{I})\int_0^N Y(x)\,dx,\qquad I_-Y=\lim_{N\to-\infty}(\mathrm{I})\int_N^0 Y(x)\,dx$$

exist, then the *improper interval integral* of Y over $R=[-\infty,\infty]$ is defined to be

$$(8.12)\qquad (\mathrm{I})\int_{-\infty}^{\infty} Y(x)\,dx=I_+Y+I_-Y.$$

Justification. By use of the rules for extended interval arithmetic given in [1], the interval (8.12) is well defined if the limits (8.11) exist, as the formulas $[\infty-\infty,\cdot]=[-\infty,\cdot]$, $[\cdot,\infty-\infty]=[\cdot,\infty]$ resolve any "indeterminant forms" which may arise. The actual interval of integration may be indicated in (8.12) if different from R.

The following example is also taken from [1].

(c) $Y(x)=-e^{-x}$, $X=[0,\infty]$. Here,

$$(8.13)\qquad (\mathrm{I})\int_0^N(-e^{-x})\,dx=\int_0^N(-e^{-x})\,dx=[-1+e^{-N},-1+e^{-N}],$$

and, since $I_-Y=[0,0]$,

$$(8.14)\qquad (\mathrm{I})\int_0^{\infty}(-e^{-x})\,dx=I_+Y=[-1,-1],$$

a finite interval, while the value of the interval integral [1] is the infinite interval

$$(8.15)\qquad \int_0^{\infty}(-e^{-x})\,dx=[-\infty,-1].$$

Finally, the definition of the improper Riemann integral over an infinite interval of integration ([2, p. 94]) gives the following result.

THEOREM 8.2. *If the endpoint functions* $\underline{y}, \bar{y}$ *of* Y *have improper Riemann integrals over* $R=[-\infty,\infty]$, *then*

$$(8.16)\qquad (\mathrm{I})\int_{-\infty}^{\infty} Y(x)\,dx=\left[(\mathrm{IR})\int_{-\infty}^{\infty}\underline{y}(x)\,dx,\ (\mathrm{IR})\int_{-\infty}^{\infty}\bar{y}(x)\,dx\right].$$

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