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Interval Analytic Treatment of Convex Programming Problems

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Abstract — Zusammenfassung

Interval Analytic Treatment of Convex Programming Problems. A nonlinear convex programming problem is solved by methods of interval arithmetic which take into account the input errors and the round-off errors. The problem is reduced to the solution of a nonlinear parameter dependent system of equations. Moreover error estimations are developed for special problems with uniformly convex cost functions.

Intervallanalytische Behandlung konvexer Optimierungsaufgaben. Es wird ein nichtlineares konvexes Optimierungsproblem mit Hilfe der Intervallarithmetik gelöst, wobei die Eingangs- und Rundungsfehler berücksichtigt werden. Dieses Problem wird zurückgeführt auf die Lösung eines parameterabhängigen nichtlinearen Gleichungssystems. Außerdem werden Fehlerabschätzungen für spezielle Probleme mit stark konvexen Zielfunktionen angegeben.

1. Introduction

Till now linear programming problems are handled by methods of interval analysis by Machost [18], Krawczyk [15], and Beeck [5, 6]. Beeck gives a summary of the state-of-the-art in [6] and shows that the use of interval analysis for the treatment of linear programming problems has been successful. Nonlinear programming problems have been investigated by Dussel [9], Robinson [24], Mancini and McCormick [19], Oelschlägel and Süße [22, 23].

Data errors were considered in [22, 23] for the special case of quadratic programming problems. Therefore we want to consider a general convex programming problem with errors in data. We assume that the reader is well versed in interval analysis. A complete discussion can be found in Alefeld, Herzberger [1].

2. The Problem

Consider the nonlinear programming problem of the form

$$\hat{z}(a, b_1, \ldots, b_l) = f(\hat{x}(a, b_1, \ldots, b_l); a) = \min_x f(x; a), \quad a \in [a] \in V_m(I(R))$$

subject to

$$g_i(x; b_i) \leq 0, \; b_i \in [b_i] \in V_{k_i}(I(R)), \; i = 1, \ldots, l \qquad x \geq 0, \tag{1}$$

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where $a, b_1, \ldots, b_l$ are vectors of parameters. The elements of $[a]$ and $[b_i]$, $i=1, \ldots, l$ are tolerance intervals, in which the parameters vary independently.

$[a], [b_1], \ldots, [b_l]$ are interval vectors.

All functions are assumed to have the following properties for any

$$a \in [a],\ b_i \in [b_i],\ i=1, \ldots, l.$$

$$f: R^n \to R^1,\ f \in C_1(R^n),\ f \text{ strictly convex in } R^n,$$

$$g_i: R^n \to R^1,\ g_i \in C_1(R^n),\ g_i \text{ convex in } R^n,\ i=1, \ldots, l.$$

Furthermore all functions are to be continuously differentiable to all parameters. These derivations and f are continuous in $R^n \times [b_i]$, $i=1, \ldots, n$, $R^n \times [a]$.

We get the problem (1) from

$$\hat{z} = f(\hat{x}) = \min f(x)$$

subject to

$$g_i(x) \leq 0, \quad i=1, 2, \ldots, l$$

$$x \geq 0,$$

where we have replaced data with errors by parameters.

To formulate the numerical problem, we introduce two sets.

Definition 1:

$$\hat{X} := \{\hat{x}(a, b_1, \ldots, b_l)/a \in [a],\ b_i \in [b_i],\ i=1, \ldots, l\},$$

$$\hat{Z} := \{\hat{z}(a, b_1, \ldots, b_l)/a \in [a],\ b_i \in [b_i],\ i=1, \ldots, l\}.$$

Definition 2: Denote the constraint set by

$$P(b_1, \ldots, b_l) := \{x/g_i(x; b_i) \leq 0,\ x \geq 0,\ i=1, \ldots, l\}.$$

If $\hat{X} \neq \emptyset$, then we want to find including sets for $\hat{X}$ and $\hat{Z}$ by the methods of interval analysis.

The question is now about the following interval vectors.

a) Compute an overhull $[x] \in V_n(I(R))$ of $\hat{X}$, i.e. $[x] \supseteq \hat{X}$.

b) Compute the interval hull $[x]^H \in V_n(I(R))$, i.e. $[x]^H \supseteq \hat{X}$ and $d(X_i^H) = \min!$, $i=1, \ldots, n$.
($d([\underline{a}, \bar{a}]) = \bar{a} - \underline{a}$ means the width of $[\underline{a}, \bar{a}]$.)

c) Compute an interior estimation $[x] \in V_n(I(R))$ of $\hat{X}$, i.e. $[x] \subseteq \hat{X}$.

d) Compute a maximal interior estimation $[x] \in V_n(I(R))$, i.e. $[x] \subseteq \hat{X}$ and $\prod_{i=1}^{n} d(X_i) = \max!$

e) Compute the same intervals for the unidimensional set $\hat{Z}$.

To compute some of these vectors we must know elementary properties of the sets $\hat{X}$ and $\hat{Z}$. The sets should not be neither compact nor connected. But we can guarantee under certain conditions these properties.

Theorem 1: *If the optimal-value map*

$$\hat{z}:([a]^T,[b_1]^T,\ldots,[b_l]^T)^T\subseteq R^{m+\sum_i k_i}\to R^1$$

is continuous, then $\hat{Z}\in I(R)$.

If the optimal-point map

$$\hat{x}:([a]^T,[b_1]^T,\ldots,[b_l]^T)^T\subseteq R^{m+\sum_i k_i}\to R^n$$

is continuous, then $\hat{X}$ is compact, connected and $\hat{Z}\in I(R)$.

Proof: The continuous image of a compact and connected set is compact and connected.

The analysis of parametric programming and sensitivity in nonlinear programming is important for the investigation of continuity of the maps $\hat{x}$ and $\hat{z}$. (See also [7, 8, 10, 12, 13, 14, 16].) We shall now consider the important special case of definite quadratic programming problems:

$$f(x;a)=\tfrac{1}{2}x^T C x+p^T x=\min!\ \ C\in[C],\ p\in[p]$$

subject to

$$A x\le b,\ \ A\in[A],\ b\in[b] \tag{2}$$

(C is positive definite and symmetric, $[C]$ is a symmetric interval matrix).

We can part A as $\begin{pmatrix}A_1\\A_2\end{pmatrix}$ and b as $\begin{pmatrix}b_1\\b_2\end{pmatrix}$ with the conditions $A_1x\le b_1$ and $A_2x\le b_2$. Furthermore we denote the constraint set by $\bar{P}(A,b):=\{x/Ax\le b\}$.

Theorem 2:

(*i*) *$\bar{P}(A,b)$, $A\in[A]$, $b\in[b]$ is nonempty.*

(*ii*) *If there exists an $\bar{A}\in[A]$, $\bar{b}\in[b]$ such that $\bar{A}_2x=\bar{b}_2$ for all $x\in\bar{P}(\bar{A},\bar{b})$, then we assume the following condition for all matrices from an ε-region*[1] *of $\bar{A}_2$* rank $(\bar{A}_2)=$ rank (A_2).

If the conditions (i) and (ii) are satisfied, then the set $\hat{X}$ is compact, connected and $\hat{Z}\in I(R)$ for the special case of definite quadratic programming problems (2).

If there exists an interior point (Slaterpoint) for any $\bar{P}(A,b)$, $A\in[A]$, $b\in[b]$, then the conditions of Theorem 2 are trivially satisfied. Theorem 2 can be immediately proved with the stability theorem of Daniel [8].

We consider now the general problem (1), where the following theorem holds.

Theorem 3: *If there exists an interior point (Slaterpoint) for any $P(b_1,\ldots,b_l)$, $b_1\in[b_1],\ldots,b_l\in[b_l]$ and if there exists a compact set Y with $P(b_1,\ldots,b_l)\subseteq Y$, $b_1\in[b_1],\ldots,b_l\in[b_l]$, then $\hat{X}$ is compact, connected and $\hat{Z}\in I(R)$.*

This theorem is a special case of a general theorem of Bruns [7]. Under the conditions of Theorem 3, we get $\hat{Z}\in I(R)$ also from a theorem of Krabs [14].

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[1] $\{A_2/A_2\in[A_2],\ \|A_2-\bar{A}_2\|\le\varepsilon,\ \varepsilon>0\}$.

For the special case of quadratic programming problems the assumptions of Theorem 2 are not so strict as the assumptions of Theorem 3. If we consider the numerical problems a)—e), then we want to compute the interval hull of $\hat{X}$ and $\hat{Z}$ at first, which means, we have to compute the vectors $\inf \hat{X}$, $\sup \hat{X}$ and the real numbers $\inf \hat{Z}$, $\sup \hat{Z}$.

We shall discuss possibilities to find Z^H. Seeking $[x]^H$ of $\hat{X}$ is a very difficult problem and in the general form (1) is at present unsolved.

We consider the index sets $I_1=\{1, \ldots, \bar{l}\}$, $I_2=\{\bar{l}+1, \ldots, l\}$, $I=\{1, \ldots, l\}$ with $I_1 \cup I_2 = I$.

Theorem 4: *If the following conditions hold:*

(*i*) *$f(x; a)$ is either isoton or antiton in the vector a in $[a]$ for $x \geq 0$,*

(*ii*) *$g_i(x; b_i)$ are isoton in the vector b_i in $[b_i]$ for $x \geq 0$, $i \in I_1$, $g_i(x; b_i)$ are antiton in the vector b_i in $[b_i]$ for $x \geq 0$, $i \in I_2$,*

(*iii*) *$P(\bar{b}_1, \ldots, \bar{b}_{\bar{l}}, \underline{b}_{\bar{l}+1}, \ldots, \underline{b}_l) \neq \emptyset$ and $P(\underline{b}_1, \ldots, \underline{b}_{\bar{l}}, \bar{b}_{\bar{l}+1}, \ldots, \bar{b}_l)$ is compact*[2].

Then:

1. *all problems $f(x; a) = \min!$ $a \in [a]$ subject to*

$$x \in P(b_1, \ldots, b_l),\ b_i \in [b_i],\ i \in I$$

have the unique optimal solution $\hat{x}$,

2. *if $f(x; a)$ is isoton in the vector a, then follows*

$$\inf \hat{Z} = \hat{z}(\underline{a}, \underline{b}_1, \ldots, \underline{b}_{\bar{l}}, \bar{b}_{\bar{l}+1}, \ldots, \bar{b}_l) = \min_x f(x; \underline{a}) \tag{3}$$

subject to

$$x \in P(\underline{b}_1, \ldots, \underline{b}_{\bar{l}}, \bar{b}_{\bar{l}+1}, \ldots, \bar{b}_l),$$

$$\sup \hat{Z} = \hat{z}(\bar{a}, \bar{b}_1, \ldots, \bar{b}_{\bar{l}}, \underline{b}_{\bar{l}+1}, \ldots, \underline{b}_l) = \min_x f(x; a) \tag{4}$$

subject to

$x \in P(\bar{b}_1, \ldots, \bar{b}_{\bar{l}}, \underline{b}_{\bar{l}+1}, \ldots, \underline{b}_l)$,

3. *if $f(x; a)$ is antiton in the vector a, then similar formulas are obtained.*

Without going in details we remark only that Theorem 4 is a generalization of a theorem of Vajda [26] for the case of linear programming problems. If we require in addition to Theorem 4 the existence of a Slaterpoint of $P(\bar{b}_1, \ldots, \bar{b}_{\bar{l}}, \underline{b}_{\bar{l}+1}, \ldots, \underline{b}_l)$, then we get moreover $\hat{Z} \in I(R)$.

Remark: If we would permit, that the parameters do not independently vary in the tolerance intervals, for instance if $a = b_1 = b_2 = \ldots = b_l$, then the statement of Theorem 4 holds only if all functions $f(x; a)$, $g_1(x; a), \ldots, g_l(x; a)$ are either isoton or antiton in a simultaneously.

The problems (3) and (4) differ from (1) and have the decisive advantage, that they are not parametric. But it is a disadvantage, that we are unable to construct

[2] If $[b] \in V_n(I(R))$, then is $\underline{b} = \inf_{b \in [b]} b$, $\bar{b} = \sup_{b \in [b]} b$.

bounds for $\hat{X}$ from the optimal solution of (3) and (4). The assumption of isotony, antitony, respectively, is satisfied for wide classes of programming problems, for instance, for linear and quadratic problems.

3. Bounds for $\hat{\mathrm{X}}$ and $\hat{\mathrm{Z}}$

We assume throughout this paper that the conditions of Theorem 3 are satisfied. For any $a \in [a]$, $b_i \in [b_i]$, $i \in I$ $\hat{x}$ is optimal solution of (1) if and only if there exists $\hat{v} \in R^n$, $\hat{u} \in R^l$, $\hat{w} \in R^l$ such that

$$\begin{aligned}
&\nabla_x f(\hat{x}; a) + [\nabla_x g(\hat{x}; b)] \cdot \hat{u} - \hat{v} = 0^3 \quad a \in [a],\ b_i \in [b_i],\ i \in I \\
&\qquad\qquad g(\hat{x}; b) + \hat{w} = 0 \\
&\hat{u}_i \cdot \hat{w}_i \qquad\qquad = 0 \quad i = 1, \ldots, l \\
&\hat{x}_j \cdot \hat{v}_j \qquad\qquad = 0 \quad j = 1, \ldots, n \\
&\qquad\qquad \hat{x}, \hat{v}, \hat{u}, \hat{w} \geq 0.
\end{aligned} \tag{5}$$

(5) are the Kuhn-Tucker conditions (K-T-C) for the optimal solution of problem (1).

Denote (5) by

$$h(y; a, b_1, \ldots, b_l) = 0, \ y \geq 0 \ \text{and} \ y^T = (x^T, v^T, u^T, w^T).$$

Furthermore we assume throughout this paper that the solution $\hat{y}$ of (5) is unique for any $a \in [a]$, $b_i \in [b_i]$, $i \in I$.

The solution set of (5) is

$$Y_1 := \{\hat{y}/h(\hat{y}; a, b_1, \ldots, b_l) = 0, \hat{y} \geq 0, a \in [a], b_i \in [b_i], i \in I\}.$$

We want to bound Y_1 by an interval vector, but by the methods of interval analysis it is only possible to bound the set

$$Y_2 := \{\hat{y}/h(\hat{y}; a, b_1, \ldots, b_l) = 0, \ a \in [a], \ b_i \in [b_i], \ i \in I\},$$

or only a part of Y_2, respectively, by an interval vector $[y]$, because for $a \in [a]$, $b_i \in [b_i]$, $i \in I$ the solution of $h(y; a, b_1, \ldots, b_l) = 0$ is not necessarily unique.

We assume that the interval vector $[y]$ contains a solution of $h(y; a, b_1, \ldots, b_l) = 0$ for any $a \in [a]$, $b_i \in [b_i]$, $i \in I$.

Theorem 5: *If the conditions*

$$\begin{gathered}
[((0 \in X_i) \wedge (\inf V_i > 0)) \vee ((\inf X_i > 0) \wedge (0 \in V_i))]^4 \\
\wedge \ [((0 \in U_j) \wedge (\inf W_j > 0)) \vee ((\inf U_j > 0) \wedge (0 \in W_j))] \\
i = 1, \ 2, \ldots, n \\
j = 1, 2, \ldots, l
\end{gathered}$$

are satisfied, then $[y]$ *is an overhull of* Y_1.

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3 $\nabla_x g(x; b) = (\nabla_x g_1(x; b_1), \nabla_x g_2(x; b_2), \ldots, \nabla_x g_l(x; b_l))$.

4 $[x] = (X_i)$, $[v] = (V_i)$, $[u] = (U_i)$, $[w] = (W_i)$.

Proof: Assume for instance $(0 \in X_i) \wedge (\inf V_i > 0)$ for some $i \in \{1, 2, \ldots, n\}$.

Because there exists a solution in $[y]$ for $a \in [a]$, $b_i \in [b_i]$, $i \in I$ and according to the complementary slackness $x_i \cdot v_i = 0$, $i = 1, 2, \ldots, n$, is $\hat{x}_i = 0$ and $\hat{v}_i > 0$ for some $i \in \{1, 2, \ldots, n\}$ and for all $a \in [a]$, $b_i \in [b_i]$, $i \in I$. The other cases can be proved in an analogous way.

The assumptions of Theorem 5 imply the strict complementary slackness with the same set of variables for all $a \in [a]$, $b_i \in [b_i]$, $i \in I$.

For fixed $\bar{a} \in [a]$, $\bar{b}_i \in [b_i]$, $i \in I$ we denote the unique solution of the K-T-C by $\bar{y}$. We define the sets of variables:

$$K := \{y_i / \bar{y}_i > 0,\ i = 1, 2, \ldots, 2(n+l)\}$$
$$\bar{K} := \{y_i / \bar{y}_i = 0,\ i = 1, 2, \ldots, 2(n+l)\}.$$

Definition 3: The programming problem (1) is called complementarily stable with respect to the variable set K (or $\bar{K}$), if the conditions are satisfied.

(i) $\operatorname{card}(K) = \operatorname{card}(\bar{K})$ [5],

(ii) the set K (or $\bar{K}$) is independent from the selection of $a \in [a]$, $b_i \in [b_i]$, $i \in I$.

This idea is a generalization of basis stable linear programming problems. (See Beeck [6].)

If the conditions of Theorem 5 are satisfied, then the programming problem (1) is complementarily stable. We can find estimations for $\hat{X}$ only for complementarily stable programming problems by using the K-T-C.

The partition of (1) in two problems stated in Theorem 4 is independent from this fact.

Denote by y_K a vector with components in K. If we compute an interval vector fullfilling the assumptions of Theorem 5, then we have to bound the solution set of a high dimensional nonlinear parameter-depending system of equations with the dimension $2(n+l)$. But the assumptions of Theorem 5 mean, that strictly $(n+l)$ variables are exactly zero and so we can diminish the computational expense, if we know the set K (or $\bar{K}$) of variables.

In practice we propose the following procedure. We arbitrarily choose fixed $\tilde{a} \in [a]$, $\tilde{b}_i \in [b_i]$, $i \in I$, in order to solve the K-T-C directly. If not possible, we consider the problem

$$f(x; \tilde{a}) = \min! \tag{6}$$

subject to

$$g_i(x; \tilde{b}_i) \leq 0,\ i \in I$$
$$x \geq 0.$$

We can find a solution of (6) by conventional numerical methods of nonlinear programming. Therefore we get a solution $\tilde{y}$ of K-T-C or a solution $\tilde{x}$ of (6). If we can only compute $\tilde{x}$ of (6), then it is possible to compute $\hat{y}$ using $\tilde{x}$. It is generally

[5] By card $(\cdot)$ we denote the number of elements of the set.

impossible to compute $\tilde{y}$ exactly. Using $\tilde{y}$ we determine the set K (or $\bar{K}$), but this set can differ from the actual K (or $\bar{K}$). In the equation $h(y; a, b_1, \ldots, b_l)=0$ we put in for all variables exactly zero with $y_i \in \bar{K}$ and we obtain therefore the function vector

$$\bar{h}(y_K; a, b_1, \ldots, b_l).$$

We consider the solution set

$$Y_K := \{\hat{y}_K / \bar{h}(\hat{y}_K; a, b_1, \ldots, b_l)=0, \hat{y}_K \geq 0, a \in [a], b_i \in [b_i], i \in I\}.$$

Assumed, $[y]_K$ is an interval vector, which contains solutions $\hat{y}_K$ satisfying the equation

$$h(y_K; a, b_1, \ldots, b_l)=0$$

for any $a \in [a], b_i \in [b_i], i \in I$.

Then we have the following theorem.

Theorem 6: *If $\underline{y}_K \geq 0$, then follows $[y]_K \supseteq Y_K$.*
If $\underline{y}_K > 0$, then (1) is a complementarily stable programming problem.

If the conditions of Theorem 6 are satisfied, then the x part of $[y]_K$ with the values of variables contained in $\bar{K}$, is an overhull of $\hat{X}$. That means, we have solved the numerical problem a).

Remark 1: By computers the interval vector $[y]_K$ has to be computed by methods of interval arithmetic. Therefore it is also possible to take into account automatically the round-off errors. In the following part we do not consider the solution methods of system of nonlinear equations. We only remark, every parameter has to occur only once. (See also [1].) But the K-T-C do not realize this assumption. Therefore we can get with interval analytic methods only an overhull of $\hat{X}$.

Remark 2: If we compute $[y]_K$ instead of $[y]$, then we have a decisive advantage. The dimension $2(n+l)$ of the system which is to be solved, is reduced to $n+l$. Therefore the numerical expense could be reduced dramatically. On the other hand we must know, that for the class of definite quadratic programming problems the nonlinear system of equations is reduced to a linear system of equations with data errors. In contrast to nonlinear systems these systems can be solved "easily". Furthermore we can get very good estimations of $\hat{X}$, if some assumptions are satisfied. (See [3], [4].)

Remark 3: There are three reasons, because of which the condition of Theorem 6 might not be valid.

1. The problem (1) was not complementary stable.
2. The designation of the set K (or $\bar{K}$) was not correct.
3. The intervalarithmetic estimation had yield a pessimistic overestimation $[y]_K$ of Y_K.

From the vector $[y]_K$ we can get an interval vector $[x] \supseteq \hat{X}$. Then follows $\hat{Z} \subseteq Z$ with

$$Z := F([x]; [a])^{6}$$

and Z is an overhull of $\hat{Z}$.

[6] We assume that there exists a natural interval extension F of f.

Often Z is a pessimistic overestimation of $\hat{Z}$.

If the conditions of Theorem 4 are satisfied, then the numerical problems b) and d) are solved. In practice the problems (3) and (4) can be solved simultaneously, where $[x]_1$ contains the optimal solution $\hat{x}_1$ of (3) and $[x]_2$ contains the optimal solution $\hat{x}_2$ of (4). It is to compute

$$Z_1 := F([x]_1; \underline{a}),\ Z_2 := F([x]_2; \bar{a}),\ \bar{Z} := [\inf Z_1, \sup Z_2].$$

Generally is $\bar{Z}$ not so pessimistic as Z.

Theorem 7: *If* $\sup Z_1 \leq \inf Z_2$, *then*

$$Z := [\sup Z_1, \inf Z_2]$$

is an interior estimation of $\hat{Z}$, *i.e.* $Z \subseteq \hat{Z}$.

Special nonlinear programming problems have been considered in [9, 19, 23] without using K-T-C. We will now consider a special problem also without using the K-T-C.

Definition 4: A function $f(x; a)$ with continuous gradient will be called an uniformly convex function in a convex set G, if the condition

$$[\nabla_x f(x; a) - \nabla_x f(y; a)]^T (x-y) \geq c(a) \; \| x-y \|^2,$$[7]

$$c(a) > 0,\ x, y \in G$$

is satisfied.

We consider the following programming problem:

$$f(x; a) = \min!\ a \in [a] \in V_m(I(R))$$

subject to

$$\begin{aligned} A\,x = b,\ A \in [A] \in M_{l,n}(I(R)) \\ b \in [b] \in V_l(I(R)). \end{aligned} \tag{7}$$

Denote the constraint set by $P^*(A, b) := \{x / A\,x = b\}$.

We assume that $P^*(A, b)$ is nonempty for any $A \in [A]$, $b \in [b]$ and that for any $a \in [a]$ f is a uniformly convex function in the convex set G with $P^*(A, b) \subseteq G$, $A \in [A], b \in [b]$.

Furthermore we assume that $f(x; a)$ has a Lipschitz continuous gradient

$$\| \nabla_x f(x; a) - \nabla_x f(y; a) \| \leq L(a) \; \| x-y \|,$$

$x, y \in H$, H defined by the convex hull of all projection finite lines from a point $\tilde{x} \in R^n$ to any constraint set $P^*(A, b)$, $A \in [A], b \in [b]$.

We will need the following assumptions:

$$e^T = (1, 1, \ldots, 1),$$

$$M = \inf_{a \in [a]} c(a) > 0, \quad \bar{L} = \sup_{a \in [a]} L(a) < +\infty,$$

[7] See also Göpfert [11].

$A(l,n)$ with $l<n$ and $\operatorname{rank}(A)=l$ for all $A\in[A]$, $Y\approx[\tilde{A}\cdot\tilde{A}^T]^{-1}$ for a special $\tilde{A}\in[A]$, $\tilde{x}\in R^n$ computed on a computer is an approximate solution of problem (7),

$$[d]=[A]\cdot\tilde{x}-[b]$$
$$[R]=E-Y([A]\cdot[A]^T).$$

With the above properties we get the

Theorem 8: *If* $\|[R]\|<1$[8] *and*

$$\varrho=\frac{\|[A]^T\|\cdot\|Y\|}{1-\|[R]\|}\cdot\|[d]\|\left(1+\frac{\bar{L}}{M}\right)+\frac{1}{M}\|\nabla_x F(\tilde{x};[a])\|, \tag{8}$$

then we obtain

$$\hat{X}\subseteq[\tilde{x}-\varrho\cdot e;\ \tilde{x}+\varrho\, e].$$

Proof: Let $\tilde{x}_0$ be the projection from $\tilde{x}$ to $P^*(A,b)$. Then the inequality

$$(\tilde{x}_0-\hat{x})^T\cdot\nabla_x f(\hat{x};a)\geq 0$$

holds. From the definition of the uniform convexity (Definition 4), we have

$$\|\tilde{x}_0-\hat{x}\|\leq\frac{1}{M}\cdot\|\nabla_x f(\tilde{x}_0;a)\|.$$

An estimation of the distance $\|\tilde{x}-\tilde{x}_0\|$ completes the proof.

Conclusion 1: With

$$\varrho=\frac{1}{M}\cdot\|\nabla_x F(\tilde{x};[a])\|$$

we obtain estimations of $\hat{X}$ for the following problems

$$f(x;a)=\min!\quad a\in[a] \tag{9}$$

subject to

$$x\in G,$$

where G is a closed convex set and the relation $\tilde{x}\in G$ is satisfied. Furthermore we do not need a Lipschitz continuous gradient.

Conclusion 2:

$$f(x;a)=\tfrac{1}{2}x^T C\,x+p^T x,\ C\in[C],\ p\in[p]$$

(C-symmetric and positive definite) is uniformly convex in R^n. Then $c(a)=\lambda_c$ is the smallest eigenvalue of C and therefore $M=\inf_{C\in[C]}\lambda_c$.

Trivially we obtain $\bar{L}=\|[C]\|$.

In this special case we can easily compute bounds for M.

[8] $\|[R]\|=\sup_{R\in[R]}\|R\|$, $\|[d]\|=\sup_{d\in[d]}\|d\|$, $\|\cdot\|$-norm of column sums or Euclidian norm.

Either we use a theorem of Gerschgorin [25] or we have

$$M \geq \frac{1 - \| [R] \|}{\| Y \|}, \quad \text{if} \quad \| [R] \| < 1, \tag{10}$$

where $Y \approx \tilde{C}^{-1}$ for a special $\tilde{C} \in [C]$ and $[R] = E - Y[C]$.

By considering the special case of quadratic programming problems of problem (9) and with the inequality (10) we obtain

$$\varrho = \frac{\| Y \| \cdot \| [C] \cdot \tilde{x} + [p] \|}{1 - \| [R] \|}, \tag{11}$$

(11) is a well-known estimation (see Krawczyk [15]) for linear interval systems.

Under certain assumptions the estimation declared by Theorem 8 yields a pessimistic overestimation. Our goal then is to improve the bounds by iterative methods.

Conclusion 3: The special problem of (9)

$$\tfrac{1}{2} x^T C x + p^T x = \min! \; C \in [C], \; p \in [p] \tag{12}$$

subject to

$$x \in [x] \in V_n(I(R))$$

can be solved by an iterative method developed by Oelschlägel-Süße [23].

With the help of this method we can solve problems of the form

$$\tfrac{1}{2} x^T C x + p^T x = \min \; C \in [C], \; p \in [p]$$

subject to $\quad x \geq 0$.

These problems occur in various publications, for instance of Hildreth and d'Esopo [17].

But it is difficult to determine a start region $[x]^0$ with $[x]^0 \supseteq \hat{X}$. If $\tilde{x} = 0$, then we obtain with

$$\varrho = \frac{1}{M} \cdot \| [p] \|, \quad \varrho = \frac{\| Y \| \cdot \| [p] \|}{1 - \| [R] \|},$$

respectively, and

$$\| [R] \| < 1$$

the start region

$$[x]^0 = [0, \varrho \cdot e].$$

4. Numerical Example

Our calculations were done using sixteen decimal digit interval arithmetic on the R 21 computer (at the research centre TH Leuna-Merseburg). We have used the programming language PL/1.

To find an interval vector $[x]$ containing the solution set of a nonlinear system, we have applied the well-known interval version of Newton's method to parameter depending systems of equations. (See also [1], [20], [21].)

Example:

$$f(x;a)=x_1^2-a\,x_1-x_2=\min!\quad a\in[1.99,2.001]$$

subject to

$$b_1\,x_1^2+b_2\,x_2^2-6\le 0,\quad b_1\in[1.99,2.001]$$
$$x_1,x_2\ge 0,\quad b_2\in[2.999,3.01].$$

We obtain the parameter depending K-T-C:

$$\begin{aligned}
2\,x_1-a+u_1\cdot 2\,b_1\,x_1-v_1&=0\\
-1+u_1\cdot 2\,b_2\,x_2-v_2&=0\\
b_1\,x_1^2+b_2\,x_2^2-6+w_1\quad&=0\\
u_1\,w_1&=0\\
x_1\,v_1&=0\\
x_2\,v_2&=0,\quad x,u,v,w\ge 0.
\end{aligned}$$

In contrast to the original problem we have various occurrences of the parameters.

We have obtained the following approximate solution $\tilde{y}$ by some noninterval method:

$$\tilde{x}_1=0.8,\ \tilde{x}_2=1.26,\ \tilde{u}_1=0.13,\ \tilde{w}_1=0,\ \tilde{v}_1=0,\ \tilde{v}_2=0.$$

We determine the sets K, $\bar{K}$, respectively.

$$K:=\{x_1,x_2,u_1\},\quad \bar{K}:=\{w_1,v_1,v_2\}.$$

Therefore we can obtain the reduced system:

$$\begin{array}{llll}
2\,x_1 & +2\,b_1\,x_1\,u_1-a=0 & & a\in[1.99,2.001]\\
2\,b_2\,x_2\,x_3 & & -1=0 & b_1\in[1.99,2.001]\\
b_1\,x_1^2 & +\ \ b_2\,x_2^2 & -6=0 & b_2\in[2.999,3.01].
\end{array}$$

At first we have determined a start region. Five interval Newton operations were performed.

$$\begin{aligned}
\hat{x}_1&\in[0.7861180160926976,\ 0.7927157709884446]\\
\hat{x}_2&\in[1.254973946581317,\ \ 1.261428015992586]\\
\hat{u}_1&\in[0.1316239598278200,\ 0.1329037678477008]
\end{aligned}$$

Furthermore the interval

$$Z=[-2.229671631247111,\ -2.190949600000112]$$

was computed.

Therefore the original problem is complementarily stable. Trivially the assumptions of the Theorems 3 and 4 are satisfied, where the function f is antiton in a and the function g is isoton in b.

Applying the Theorem 4, we can solve the problems:

$$\inf\hat{Z}=x_1^2-2.001\,x_1-x_2=\min!$$

subject to

$$1.99\,x_1^2+2.999\,x_2^2-6\leq 0,\ x\geq 0,$$

$$\sup \hat{Z}=x_1^2-1.99\,x_1-x_2=\min!$$
$$\text{subject to } 2.001\,x_1^2+3.01\,x_2^2-6\leq 0,\ x\geq 0.$$

We have computed both solutions with eight significant digits.

$$\inf \hat{Z}\in[-2.216275400000011,\ -2.216273800010003]$$
$$\sup \hat{Z}\in[-2.204382000002101,\ -2.204380400002110]$$

The interval

$$[\inf Z_1,\ \sup Z_2]=[-2.216275400000011,\ -2.204380400002110]$$

is a better estimation of $\hat{Z}$ than Z.

Since the assumptions of Theorem 7 are satisfied, we have

$$Z^*=[-2.216273800010003,\ -2.004382000002101]\subseteq\hat{Z},$$

i.e. Z^* is an interior estimation of $\hat{Z}$.

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