Computing 44, 47–57 (1990)

Computing

Multicriterion Optimization Using Interval Analysis

K. Ichida and **Y. Fujii**, Kyoto

Received July 31, 1989

Abstract — Zusammenfassung

Multicriterion Optimization Using Interval Analysis. Interval Analysis methods have been applied for obtaining the global optimum of the multimodal multivariable functions. We discuss here the multicriterion optimization problem, where several objective functions must be optimized in conflicting situations.

AMS Subject Classifications: 65G10, 65K10

Key words: Interval analysis, multicriterion optimization, multiobjective optimization.

Gleichzeitige Optimierung von mehreren Kriterien mittels Intervallanalysis. Bei der Bestimmung des globalen Optimums werden in der Vektoroptimierung intervallanalytische Methoden angewendet. Hier diskutieren wir Optimierungsprobleme, bei denen mehrere Zielfunktionen in Konfliktsituationen optimiert werden müssen.

1. Introduction

Global optimization problems are difficult to solve since there are several or many local extrema [1], [2]. Interval analysis methods have been successfully applied to these problems [3], [4], [5]. In interval analysis we extend the variables, constants and functions as intervals [6]. The original domain of variables is divided into subregions successively, and the lower and the upper bounds of the objective function are estimated on each subregions. By discarding subregions where the global solution cannot exist, one can always find the solution with a rigorous error bound. Interval form of the Newton method can be used to get rapid convergence and high accuracy [7], [8].

In engineering and management science optimization problems there often exist several criteria which must be considered in conflicting situations [9], [10]. This situation is formulated as a multicriterion optimization (multiperformance, multiobjective or vector optimization) problem where not a single objective function but several functions are to be minimized or maximized simultaneously.

A multicriterion optimization problem is formulated as follows.

Find the vector of decision variables $x = [x_1, x_2, \ldots, x_n]^T$ which optimizes the vector function

$$\mathfrak{f}(\mathfrak{x}) = [f_1(\mathfrak{x}), f_2(\mathfrak{x}), \ldots, f_k(\mathfrak{x})]^T \tag{1}$$

subject to the constraints

$$\mathfrak{x} \in \mathfrak{X} = \{\mathfrak{x} \in E^n | \mathfrak{g}(\mathfrak{x}) \leq 0, \mathfrak{h}(\mathfrak{x}) = 0\} \tag{2}$$

where

$$\mathfrak{g}(\mathfrak{x}) = [g_1(\mathfrak{x}), g_2(\mathfrak{x}), \ldots, g_r(\mathfrak{x})]^T \tag{3}$$

and

$$\mathfrak{h}(\mathfrak{x}) = [h_1(\mathfrak{x}), h_2(\mathfrak{x}), \ldots, h_t(\mathfrak{x})]^T \tag{4}$$

are r-dimensional and t-dimensional constraint functions, respectively. For the sake of convenience we shall assume that all the objective functions are to be minimized. Functions to be maximized can be converted into the form which allows their minimization by

$$\max f_i(\mathfrak{x}) = \min(-f_i(\mathfrak{x})). \tag{5}$$

Since in general there exists no complete optimal solution $\mathfrak{x}^*$ which satisfies

$$\mathfrak{f}(\mathfrak{x}^*) \leq \mathfrak{f}(\mathfrak{x}) \tag{6}$$

for all $\mathfrak{x} \in \mathfrak{X}$, we shall consider to determine a Pareto optimal solution. This optimum gives a set of non-inferior solutions, i.e., solutions for which there is no way of improving any criterion without worsening at least one other criterion [9]. The main approaches to global optimization problems are as follows.

(i) Weighting Method

This method changes the multicriterion optimization problem to a scalar optimization problem by

$$\underset{\mathfrak{x} \in \mathfrak{X}}{\text{minimize}}\, f(\mathfrak{x}) = \underset{\mathfrak{x} \in \mathfrak{X}}{\text{minimize}} \sum_{i=1}^{k} w_i f_i(\mathfrak{x}) \tag{7}$$

where $w_i \geq 0$ are the weighting coefficients representing the relative importance of criteria. We usually assume that

$$\sum_{i=1}^{k} w_i = 1. \tag{8}$$

Instead of (7), it may be used

$$\underset{\mathfrak{x} \in \mathfrak{X}}{\text{minimize}} \sum_{i=1}^{k} w_i c_i f_i(\mathfrak{x}), \tag{9}$$

where c_i $(i = 1, 2, \ldots, k)$ are constant multipliers chosen by decision-makers or tacitly assumed as $c_i = \dfrac{1}{f_i^0}$, where

$$f_i^0 = \min_{\mathfrak{x} \in \mathfrak{X}} f_i(\mathfrak{x}) \tag{10}$$

denotes the minimum value of the ith objective function. Or, using the utility functions U_i $(i = 1, 2, \ldots, k)$, we transform (7) to the form

$$\underset{\mathfrak{x}\in\mathfrak{X}}{\text{minimize}} \sum_{i=1}^{k} U_i(f_i(\mathfrak{x})). \tag{11}$$

(ii) ε-Constraint Method [11]

This method minimizes one of the objective functions considering the others as constraints, i.e.,

$$\underset{\mathfrak{x}\in\mathfrak{X}}{\text{minimize}}\, f_r(\mathfrak{x}) \tag{12}$$

subject to

$$f_i(\mathfrak{x}) \leq \varepsilon_i \quad (i = 1, 2, \ldots, k, i \neq r). \tag{13}$$

(iii) Minimax Method [9]

The desirable solution is the one which gives the smallest value of the maximum values of all the objective functions, i.e.,

$$\underset{\mathfrak{x}\in\mathfrak{X}}{\text{minimize}} \max_{1\leq i\leq k} w_i f_i(\mathfrak{x}), \tag{14}$$

or equivalently,

$$\underset{\mathfrak{x}\in\mathfrak{X}}{\text{minimize}}\, z \tag{15}$$

subject to

$$w_i f_i(\mathfrak{x}) \leq z \quad (i = 1, 2, \ldots, k). \tag{16}$$

(iv) L_p-Norm Method

Without loss of generality we assume $f_i(\mathfrak{x}) \geq 0$ $(i = 1, 2, \ldots, k)$ for all $\mathfrak{x} \in \mathfrak{X}$. This method utilizes the weighted L_p-norm as

$$\underset{\mathfrak{x}\in\mathfrak{X}}{\text{minimize}} \|\mathfrak{f}(\mathfrak{x})\|_p^w = \underset{\mathfrak{x}\in\mathfrak{X}}{\text{minimize}} \left[\sum_{i=1}^{k} \{w_i f_i(\mathfrak{x})\}^p\right]^{1/p} \quad (1 \leq p \leq \infty). \tag{17}$$

Equation (17) reduces to the weighting method for $p = 1$ and to the minimax method for $p = \infty$.

(v) Goal Programming

If a decision-maker has goals $\bar{f}_i$ for each of the objective functions $f_i(\mathfrak{x})$ $(i = 1, 2, \ldots, k)$, we can formulate (17) as

$$\underset{\mathfrak{x}\in\mathfrak{X}}{\text{minimize}} \|\mathfrak{f}(\mathfrak{x}) - \bar{\mathfrak{f}}\|_p^w = \underset{\mathfrak{x}\in\mathfrak{X}}{\text{minimize}} \left[\sum_{i=1}^{k} w_i |f_i(\mathfrak{x}) - \bar{f}_i|^p\right]^{1/p} \quad (1 \leq p \leq \infty), \tag{18}$$

or

$$\underset{\mathfrak{x}\in\mathfrak{X}}{\text{minimize}} \left\|\frac{\mathfrak{f}(\mathfrak{x}) - \bar{\mathfrak{f}}}{\bar{\mathfrak{f}}}\right\|_p^w = \underset{\mathfrak{x}\in\mathfrak{X}}{\text{minimize}} \left[\sum_{i=1}^{k} w_i \left|\frac{f_i(\mathfrak{x}) - \bar{f}_i}{\bar{f}_i}\right|^p\right]^{1/p} \quad (1 \leq p \leq \infty) \tag{19}$$

when $\bar{\mathfrak{f}} \neq 0$. In particular, $\bar{\mathfrak{f}} = (\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_k)^T$ can be taken to be the minimum value $f_i^0 = (f_1^0, f_2^0, \ldots, f_k^0)$ of each of the objective functions (see equation (10)).

2. Weighting Method

In this section an interval analysis algorithm is described to obtain the global minimum of equation (7). This algorithm is based on the following two principles: (i) descarding subregions where no global minimum can exist, and (ii) calculating relative minima with interval Newton method.

[i] Case of No Constraints

Consider a hyperrectangle $\mathfrak{A} = [a_1, b_1] \otimes [a_2, b_2] \otimes \cdots \otimes [a_n, b_n]$ which contains the global solution as an original region. We calculate the global minimum point of the function

$$f(\mathfrak{x}) = \sum_{i=1}^{k} w_i f_i(\mathfrak{x}). \tag{20}$$

In what follows capital letters $A, F, \ldots$ denote the interval extentions of $a, f, \ldots$. We suppose $f(\mathfrak{x}) \in C^2$ and its first and second derivatives are

$$\nabla f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right], \tag{21}$$

and

$$j = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \dfrac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix} \tag{22}$$

respectively, and

$$d = \det(j). \tag{23}$$

[Algorithm]

(1) The original region is divided successively. A remaining subregion having the largest side is bisected at the center of that side. We give the following test (2) $\sim$ (4) for the remaining subregions.

(2) Let two of the remaining subregions be $\mathfrak{A}_i$ and $\mathfrak{A}_j$, and their interval function values be $F_i = F(\mathfrak{A}_i) = [\underline{f}_i, \bar{f}_i]$ and $F_j = F(\mathfrak{A}_j) = [\underline{f}_j, \bar{f}_j]$, respectively [3]. The letters with underscores and overscores mean that they are the lower bound and the upper bound of the interval values. If $\underline{f}_i \geq \bar{f}_j$, it is clear that $\mathfrak{A}_i$ can not have the minimum point and this subregion can be discarded. The case of $\underline{f}_j \geq \bar{f}_i$ is treated in a similar way. Otherwise both $\mathfrak{A}_i$ and $\mathfrak{A}_j$ are retained (Fig. 1).

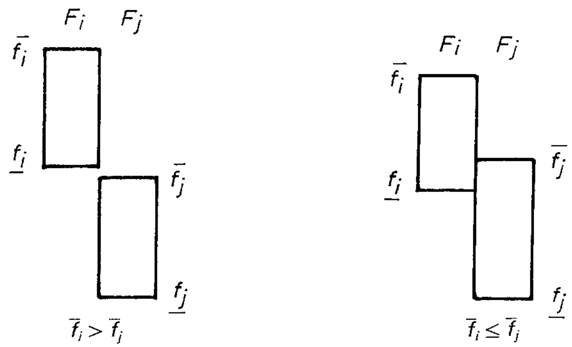

Figure 1. Comparison of interval function values (weighting method).

(3) The original domain $\mathfrak{A}$ can be selected so that the global minimum coincides with one of relative minima. This means that a subregion $\mathfrak{A}_i$ can be discarded if $\nabla F \not\ni 0$ in $\mathfrak{A}_i$.

(4) If $D \not\ni 0$ in $\mathfrak{A}_i$ (D is the interval extension of d), we apply the following interval Newton formula [7]:

$$\left.\begin{aligned} &\mathfrak{N}(\mathfrak{X}_r) = \mathfrak{m}(\mathfrak{X}_r) + \mathfrak{H}_r, \\ &\mathfrak{X}_{r+1} = \mathfrak{X}_r \cap \mathfrak{N}(\mathfrak{X}_r), \\ &J\mathfrak{H}_r = -\nabla F, \\ &\quad (r = 0, 1, 2, \ldots; \mathfrak{X}_0 = \mathfrak{A}_i), \end{aligned}\right\} \tag{24}$$

and save the result when it converges in $\mathfrak{A}_i$.

(5) In this way we apply the tests (2) $\sim$ (4) until the largest side of the remaining subregions becomes less than a prescribed bound. When $f(\mathfrak{x})$ has more than one global minimum points, all of them can be obtained.

[ii] Case of Equality Constraints

Find the global minimum of (20) under the equality constraints

$$h_i(\mathfrak{x}) = 0 \quad (i = 1, 2, \ldots, t). \tag{25}$$

To solve this problem the Lagrangean function

$$L = f(\mathfrak{x}) + \sum_{i=1}^{t} \lambda_i h_i(\mathfrak{x}) \tag{26}$$

is introduced. Differentiating L with respect to x_j ($j = 1, 2, \ldots, n$) and λ_i ($i = 1, 2, \ldots, t$) and setting the results equal to zero, we have the following equations

$$\left.\begin{aligned} &\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} + \sum_{i=1}^{t} \lambda_i \frac{\partial h_i}{\partial x_j} = 0 \quad (j = 1, 2, \ldots, n), \\ &\frac{\partial L}{\partial \lambda_i} = h_i(\underline{x}) = 0 \quad (i = 1, 2, \ldots, t). \end{aligned}\right\} \tag{27}$$

All of the real roots of these simultaneous nonlinear equations can be obtained by the algorithm [i], and the global minimum point is determined from the solutions of (27).

[iii] Case of Inequality Constraints

The problem to minimize $f(\mathfrak{x})$ in (20) subject to

$$g_i(\mathfrak{x}) \le 0 \quad (i = 1, 2, \ldots, r) \tag{28}$$

is transformed to seek the stationary points of the Lagrangean function

$$L = f(\mathfrak{x}) + \sum_{i=1}^{r} \lambda_i (g_i(\mathfrak{x}) + x_{n+i}^2), \tag{29}$$

where x_{n+i} $(i = 1, 2, \ldots, r)$ are the slack variables. Differentiating L with respect to x_j $(j = 1, 2, \ldots, n + r)$, λ_i $(i = 1, 2, \ldots, r)$ and setting the results equal to zero, we have

$$\left.\begin{aligned} &\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} + \sum_{i=1}^{r} \lambda_i \frac{\partial g_i}{\partial x_j} = 0 \quad (j = 1, 2, \ldots, n), \\ &\frac{\partial L}{\partial x_{n+i}} = 2\lambda_i x_{n+i} = 0 \quad (i = 1, 2, \ldots, r), \\ &\frac{\partial L}{\partial \lambda_i} = g_i(\underline{x}) + x_{n+i}^2 = 0 \quad (i = 1, 2, \ldots, r). \end{aligned}\right\} \tag{30}$$

Using the algorithm in [i], the global minimum can be obtained by solving (30) similarly as in [ii].

3. Minimax Method

In this section we transform the multiobjective minimization problem as the following minimax problem:

$$\left.\begin{aligned} &\underset{\mathfrak{x} \in \mathfrak{X}}{\text{minimize}} \max_{1 \le i \le k} f_i(\mathfrak{x}), \\ &\text{subject to} \\ &\mathfrak{x} \in \mathfrak{X} = \{\mathfrak{x} \in E^n \mid \mathfrak{g}(\mathfrak{x}) \le 0, \mathfrak{h}(\mathfrak{x}) = 0\}. \end{aligned}\right\} \tag{31}$$

In this case the comparison of interval function values is modified as follows (see section 2[i] (2)):

Let the interval value of each of the objective function in subregions $\mathfrak{A}_\alpha$ and $\mathfrak{A}_\beta$ be $F_i = [\underline{f}_i, \overline{f}_i]$ $(i = 1, 2, \ldots, k)$ (Fig. 2). In this figure

$$F_\alpha = [\underline{f}_\alpha, \overline{f}_\alpha] = \left[\max_{1 \le i \le k} \underline{f}_i, \max_{1 \le i \le k} \overline{f}_i\right] \tag{32}$$

denotes the maximum values of the lower and the upper bounds of the objective functions in $\mathfrak{A}_\alpha$. F_β in $\mathfrak{A}_\beta$ is given in a similar way. If $\overline{f}_\beta < \underline{f}_\alpha$, then it is clear that $\mathfrak{A}_\alpha$

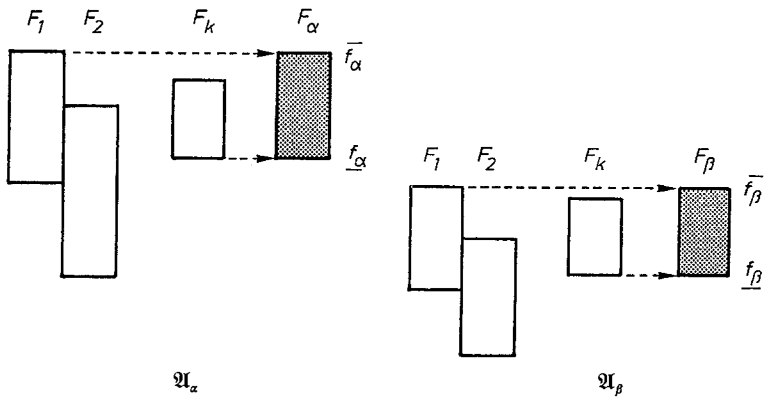

Figure 2. Comparison of interval values (minimax method).

cannot have the optimal point and is discarded. In this way the global minimax solution is determined.

However, this algorithm of dividing subregions and estimating the interval values of functions takes a great deal of computation time. To circumvent this difficulty, the Lagrangean function and the interval Newton method are used.

We rewrite (31) as follows:

$$\left.\begin{aligned} &\text{minimize } z, \\ &\text{subject to} \\ &f_i(\mathbf{x}) - z \le 0 \quad (i = 1,2,\ldots,k), \\ &\quad g_i(\mathbf{x}) \le 0 \quad (i = 1,2,\ldots,r), \\ &\quad h_i(\mathbf{x}) = 0 \quad (i = 1,2,\ldots,t). \end{aligned}\right\} \tag{33}$$

The Lagrangean function corresponding to (33) is

$$L = z + \sum_{i=1}^{k} \lambda_i(f_i(\mathbf{x}) - z + x_{n+1}^2) + \sum_{i=1}^{r} \lambda_{k+i}(g_i(\mathbf{x}) + x_{n+k+i}^2) + \sum_{i=1}^{t} \lambda_{k+r+i} h_i(\mathbf{x}). \tag{34}$$

Differentiating L with respect to $z, x_i\,(i=1,2,\ldots,n+k+r), \lambda_i\,(i=1,2,\ldots,k+r+t)$ and setting the result equal to zero, we have

$$\frac{\partial L}{\partial z} = 1 - \sum_{i=1}^{k} \lambda_i = 0,$$

$$\frac{\partial L}{\partial x_j} = \sum_{i=1}^{k} \lambda_i \frac{\partial f_i}{\partial x_j} + \sum_{i=1}^{r} \lambda_{k+i} \frac{\partial g_i}{\partial x_j} + \sum_{i=1}^{t} \lambda_{k+r+i} \frac{\partial h_i}{\partial x_j} = 0 \quad (j = 1,2,\ldots,n),$$

$$\frac{\partial L}{\partial x_{n+i}} = 2\lambda_i x_{n+i} = 0 \quad (i = 1,2,\ldots,k),$$

$$\frac{\partial L}{\partial x_{n+k+i}} = 2\lambda_{k+i} x_{n+k+i} = 0 \quad (i = 1, 2, \ldots, r),$$

$$\frac{\partial L}{\partial \lambda_i} = f_i - z + x_{n+i}^2 = 0 \quad (i = 1, 2, \ldots, k),$$

$$\frac{\partial L}{\partial \lambda_{k+i}} = g_i + x_{n+k+i}^2 = 0 \quad (i = 1, 2, \ldots, r),$$

$$\frac{\partial L}{\partial \lambda_{k+r+i}} = h_i = 0 \quad (i = 1, 2, \ldots, t). \tag{35}$$

Solving the solution of (35), the optimal solution can be obtained.

4. Numerical Examples

The following Rosenbrock functions are used as the objective functions.

$$\begin{gathered} f_1(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 + 1, \\ f_2(x_1, x_2) = 100(x_2 - x_1^2)^2 + (2 - x_1)^2 + 1, \\ 0 \le x_1 \le 3, \qquad 0 \le x_2 \le 5. \end{gathered} \tag{36}$$

The calculation is terminated when the length of the largest side of the remaining subregions is less than $\varepsilon = 10^{-10}$.

(1) Weighting Method

Changing the weighting factor w_i $(i = 1, 2)$, we calculate the minimum of the function

$$f(x_1, x_2) = w_1 f_1(x_1, x_2) + w_2 f_2(x_1, x_2). \tag{37}$$

(i) $w_1 = 1, w_2 = 0$

$$X_1 = [0.99999\,99999\,84890, \quad 1.00000\,00000\,15132],$$

$$X_2 = [0.99999\,99999\,69768, \quad 1.00000\,00000\,30278],$$

$$F = [1.00000\,00000\,00000, \quad 1.00000\,00000\,00000].$$

The figures in [,] denote the lower and the upper bound of the solution, respectively. The value of F corresponds to f_1^0 in (10).

(ii) $w_1 = \frac{3}{4}, w_2 = \frac{1}{4}$

$$X_1 = [1.24999\,99999\,96855, \quad 1.25000\,00000\,03145],$$

$$X_2 = [1{,}56250\,00000\,00000, \quad 1.56250\,00000\,07873],$$

$$F = [1.18749\,99999\,97640, \quad 1.18750\,00000\,02359].$$

(iii) $w_1 = w_2 = \frac{1}{2}$

$$X_1 = [1.50000\,00000\,00000, \quad 1.50000\,00000\,00218],$$
$$X_2 = [2.24999\,99999\,99506, \quad 2.25000\,00000\,00655],$$
$$F = [1.24999\,99999\,99890, \quad 1.25000\,00000\,00110].$$

(iv) $w_1 = \frac{1}{4}, w_2 = \frac{3}{4}$

$$X_1 = [1.74999\,99999\,91579, \quad 1.75000\,00000\,08405],$$
$$X_2 = [3.06249\,99999\,70524, \quad 3.06250\,00000\,29245],$$
$$F = [1.18749\,99999\,93689, \quad 1.18750\,00000\,06311].$$

(v) $w_1 = 0, w_2 = 1$

$$X_1 = [1.99999\,99999\,99998, \quad 2.00000\,00000\,00001],$$
$$X_2 = [3.99999\,99999\,99994, \quad 4.00000\,00000\,00005],$$
$$F = [1.00000\,00000\,00000, \quad 1.00000\,00000\,00000].$$

The value of F corresponds to f_2^0 in (10).

(2) Minimax Method

(i) By dividing subregions and estimating the function values for the following minimax problem

$$\underset{x \in \mathfrak{X}}{\text{minimize}} \max \left\{ \frac{f_1(x_1, x_2) - f_1^0}{f_1^0}, \frac{f_2(x_1, x_2) - f_2^0}{f_2^0} \right\}, \tag{38}$$

we have

$$X_1 = [1.49999\,99998\,25377, \quad 1.50000\,00001\,74623],$$
$$X_2 = [2.24999\,90652\,43173, \quad 2.25000\,09347\,27722],$$
$$F_1 = [1.24999\,99998\,25377, \quad 1.25000\,00002\,62100],$$
$$F_2 = [1.24999\,99998\,25376, \quad 1.25000\,00002\,62099],$$

after dividing subregions 193199 times.

(ii) Equation (38) can be rewritten as follows:

$$\left.\begin{aligned} &\text{minimize } z \\ &\text{subject to} \\ &\frac{f_1(x_1, x_2) - f_1^0}{f_1^0} - z \le 0, \\ &\frac{f_2(x_1, x_2) - f_2^0}{f_2^0} - z \le 0. \end{aligned}\right\} \tag{39}$$

The Lagrangean function for (39) is

$$L = z + \lambda_1\left(\frac{f_1(x_1, x_2) - f_1^0}{f_1^0} - z + x_3^2\right) + \lambda_2\left(\frac{f_2(x_1, x_2) - f_2^0}{f_2^0} - z + x_4^2\right). \quad (40)$$

By applying the interval Newton method, we obtain the following results:

$$\begin{aligned}
X_1 &= [1.50000\,00000\,00000, \quad 1.50000\,00000\,00000],\\
X_2 &= [2.25000\,00000\,00000, \quad 2.25000\,00000\,00000],\\
X_3 &= [0.0, \quad 0.22539\,92852\,19884 \times 10^{-19}],\\
X_4 &= [0.0, \quad 0.0],\\
\Lambda_1 &= [0.49999\,99999\,99999, \quad 0.50000\,00000\,00000],\\
\Lambda_2 &= [0.49999\,99999\,99999, \quad 0.50000\,00000\,00000],\\
Z &= [0.24999\,99999\,99999, \quad 0.25000\,00000\,00000],\\
F_1 &= [1.25000\,00000\,00000, \quad 1.25000\,00000\,00001],\\
F_2 &= [1.24999\,99999\,99999, \quad 1.25000\,00000\,00000].
\end{aligned}$$

The true values are

$$x_1 = \frac{3}{2}, \quad x_2 = \frac{9}{4}, \quad x_3 = 0, \quad x_4 = 0,$$

$$\lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{1}{2}, \quad z = \frac{1}{4}.$$

5. Conclusion

An interval analysis method is described for finding the global optimal solution of the multiobjective minimization problems. Interval analysis is applicable for the global optimization problems which contain a number of local extrema. It guarantees the true global optimum and the rigorous error bound; however, it takes more computation time and memory capacity than other methods. Further, these quantities increase rapidly when increasing the number of variables, objective functions and constraints. A further study is necessary for developing methods that reduce the amount of computation.

References

[1] Dixon, L. C. W., Szegö, G. P.: Towards Global Optimisation. North-Holland/American Elsevier, Amsterdam 1975.

[2] Dixon, L. C. W., Szegö, G. P.: Towards Global Optimisation 2. North-Holland/American Elsevier, Amsterdam 1978.

[3] Ichida, K., Fujii, Y.: An interval arithmetic method for global optimization, Computing, *23*, 85–97 (1979).

[4] Hansen, E. R.: Global optimization using interval analysis—The one-dimensional case, J. Optim. Theory Appl., *29*, 331–344 (1979).
[5] Hansen, E. R.: Global optimization using interval analysis—The multi-dimensional case, Numer. Math. *34*, 247–270 (1980).
[6] Moore, R. E.: Interval Analysis. Englewood Cliffs, N.J., Prentice-Hall 1966.
[7] Hansen, E. R.: Interval forms of Newton's method, Computing, *20*, 153–163 (1978).
[8] Fuji, Y., Ichida, K., Ozasa, M.: Maximization of multivariable functions using interval analysis, Interval Mathematics 1985 (K. Nickel, ed.), Springer-Verlag, Lecture Note, Comp. Sci. 212, 17–26 (1986).
[9] Osyczka, A.: Multicriterion Optimization in Engineering. Ellis Horwood, Chichester Limited 1984.
[10] Stadler, W. (Ed.): Multicriteria Optimization in Engineering and in the Sciences. Plenum Press/ New York and London 1988.
[11] Haimes, Y. Y., Hall, W. A.: Multiobjectives in water resources systems analysis: The surrougate worth tread-off method, Water Resources Res., *10*, 614–624 (1974).

Kozo Ichida
Faculty of Business Administration
Kyoto Sangyo University
Kamigamo
Kyoto 603
Japan

Yasuo Fujii
Educational Center for
Information Processing
Kyoto University
Kyoto 606
Japan