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An Interval Computational Method for Approximating Controllability Sets

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Abstract — Zusammenfassung

An Interval Computational Method for Approximating Controllability Sets. An interval computational method is proposed by means of which the null-controllability set of autonomous nonlinear systems can be approximated.

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Ein Intervallverfahren zur Approximation von Steuerbarkeitsbereichen. Es wird eine Intervallmethode vorgeschlagen, mit deren Hilfe der Null-Steuerbarkeitsbereich von autonomen nichtlinearen Systemen approximiert werden kann.

1. Introduction

Controllability problems are undoubtedly of great practical importance in control theory. That also includes the determination of the appropriate controllability sets (CS). Approximations are here inevitable because the exact computation of these ranges is complicated and linked up to great difficulties especially for nonlinear systems. The references in this direction (see [1]–[7]) contain almost only theoretical aspects. The aim of this paper is to show a way for the numerical realization of such methods demonstrated at the variant proposed in [6]. Therefore, it is described once more in Section 2. Then the numerical realization of essential parts of that method will be shown in the following two sections. The results achieved by it will be illustrated by examples given in Section 5.

2. Controllability and Controllability Sets

Consider the nonlinear autonomous system

$$\dot{x}(t) = f(x(t), u(t)), \qquad t \in [0, \infty), \tag{2.1}$$

where $f\colon X \times U \to \mathbb{R}^n$ is continuous, $X \subseteq \mathbb{R}^n$, $0 \in \operatorname{int}(X)$, $U \subseteq \mathbb{R}^m$, $m \leq n$ and U is compact. We consider as admissible controls piecewise continuous functions u:

$[0, \infty) \to U$ and assume further that for each admissible control under a given initial condition $x(0) = x^0 \in \mathbb{R}^n$ the system (2.1) has a unique solution $x(t) = x(t; 0, x^0, u)$, $t \geq 0$.

Definition 1: (2.1) is called null-controllable from $x^0 \in X$ iff there exist a time $T \geq 0$ and an admissible control u: $[0, T] \to U$ such that $x(T; 0, x^0, u) = 0$. The set of all these points $x^0 \in X$ from which (2.1) is null-controllable is called the null-controllability set 0-CS of system (2.1). A system is said to be local null-controllable iff $0 \in \operatorname{int}(0\text{-CS})$. □

The method for estimating the 0-CS of system (2.1) from the inside and the outside proposed in [6] is based on the use of LYAPUNOV-functions which are functions of the following kind:

Definition 2: A function V is called LYAPUNOV-function iff

(a) V: $\mathbb{R}^n \to \mathbb{R}$ is continuous and in $\mathbb{R}^n \backslash \{0\}$ continuous differentiable;
(b) $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$;
(c) $D(c) := \{x \in \mathbb{R}^n | V(x) < c\}$ (2.2)

is bounded for all $c \in \mathbb{R}_+ := \{z \in \mathbb{R} | z > 0\}$.

Let $\mathscr{L}$ be the set of all LYAPUNOV-functions. □

We remark that the condition (c) holds if $\lim_{\|x\| \to \infty} V(x) = \infty$, where $\|.\|$ is the Euclidian norm in $\mathbb{R}^n$ (see [8]). Besides, we consider such $c \in \mathbb{R}_+$ for which $\overline{D(c)} \subseteq X$, where X, of course, in practical computations is bounded, too.

Moreover, the following notations are used:

$$W(x, u) := \sum_{i=1}^{n} \frac{\partial V(x)}{\partial x_i} f_i(x, u) = \left\langle \frac{\partial V(x)}{\partial x}, f(x, u) \right\rangle, \tag{2.3}$$

$$M := \left\{ x \in X \,\middle|\, \min_{u \in U} W(x, u) \leq 0 \right\}, \tag{2.4}$$

$$E := \left\{ x \in X \,\middle|\, \min_{u \in U} W(x, u) = 0 \right\}, \tag{2.5}$$

$$B(E, \eta) := \left\{ x \in X \,\middle|\, \inf_{y \in E} \|x - y\| < \eta \right\}, \tag{2.6}$$

$$D(c, \varepsilon) := D(c) \backslash S(\varepsilon), \tag{2.7}$$

$$S(\varepsilon) := \{x \in \mathbb{R}^n |\, \|x\| < \varepsilon\}, \tag{2.8}$$

$$G(c, E, \varepsilon, \eta) := D(c, \varepsilon) \cap B(E, \eta) \tag{2.9}$$

and finally

$$d(c) := \{x \in \mathbb{R}^n | V(x) = c\}. \tag{2.10}$$

As a possible access for estimating the 0-CS of autonomous nonlinear systems from the inside and the outside, the following theorems are proved in [6]:

Theorem 1: *Suppose for for a local null-controllable system* (2.1) *that there are a number $c \in \mathbb{R}_+$, a LYAPUNOV-function $V \in \mathscr{L}$ and a continuous differentiable function $F: X \to \mathbb{R}$ such that the following conditions hold:*

(i) $\min_{u \in U} W(x, u) \leq 0$ *for all* $x \in D(c) \subseteq X$ *(in other words,* $D(c) \subseteq M$*);*
(ii) *for each $\varepsilon \in \mathbb{R}_+$ there exist numbers η, $\zeta \in \mathbb{R}_+$ and a partition of $G(c, E, \varepsilon, \eta)$ into G_1 and G_2 with positive distance such that for each $\bar{x} \in G(c, E, \varepsilon, \eta)$ there is a $\bar{u} \in U$ satisfying the inequalities*

$$\left\langle \frac{\partial F(\bar{x})}{\partial x}, f(\bar{x}, \bar{u}) \right\rangle > \zeta \quad \text{for } \bar{x} \in G_1$$

$W(\bar{x}, \bar{u}) \leq 0$ *and*

$$\left\langle \frac{\partial F(\bar{x})}{\partial x}, f(\bar{x}, \bar{u}) \right\rangle < -\zeta \quad \text{for } \bar{x} \in G_2 .$$

Then the 0-*CS of system* (2.1) *contains the set* $D(c)$. □

Theorem 2: *If there are $c \in \mathbb{R}_+$ and $V \in \mathscr{L}$ such that $W(x, u) > 0$ for all $(x, u) \in d(c) \times U$, then no point of the set $\mathbb{R}^n \backslash D(c)$ belongs to the* 0-*CS of system* (2.1). □

In [6, 7], examples demonstrate the applicability of these theorems but at the same time they also show the problems to verify the appropriate conditions. Especially, the computation of the ranges M and $D(c) \subseteq M$ involves considerable difficulties already for simple nonlinear systems.

Knowing E, a function F often can be chosen in a simple way, so that the condition (ii) which at first sight appears much more complicated is not so difficult to verify. Furthermore, in general it does not influence the largeness of the approximations such that in the following we only look at the computation of the ranges M and $D(c)$ as well.

3. An Interval Method to Compute Enclosures of *M*

In order to determine

$$M = \left\{ x \in X \,\middle|\, \min_{u \in U} W(x, u) \leq 0 \right\},$$

numerical methods are suitable working with sets. Such sets should be simply representable and it should be possible that arithmetical operations and determination of values of more complicated expressions are computable without difficulties. Therefore, closed intervals are used as such sets.

An n-dimensional closed interval $Y \subseteq \mathbb{R}^n$ is represented by

$$Y = \begin{pmatrix} [\underline{y}_1, \bar{y}_1] \\ \vdots \\ [\underline{y}_n, \bar{y}_n] \end{pmatrix},$$

where $\underline{y}_1 \le \bar{y}_1, \ldots, \underline{y}_n \le \bar{y}_n$, and Y is the set of all such $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ satisfying $\underline{y}_i \le y_i \le \bar{y}_i$ for each $i = 1(1)n$. On this manner the infinite set Y is given by $2n$ real numbers $\underline{y}_1, \ldots, \bar{y}_n$. The arithmetical operations $+, -, \cdot, /$ are defined for one-dimensional intervals $Y = [\underline{y}, \bar{y}]$, $Z = [\underline{z}, \bar{z}]$ in such a way that

$$Y * Z = \{y * z | y \in Y, z \in Z\}, \qquad * \in \{+, -, \cdot, /\}$$

holds. Then the endpoints of $Y * Z$ are given in a simple manner by the endpoints of Y and Z, and this remains valid also if the operations will be extended to vectors and matrices (see [9]).

In an analogous manner the domain of the standard functions $\sqrt{\ }$, sin, cos, ... can be extended such that for each function f, its extension f_I and each interval $Z \subseteq \mathrm{dom}(f)$ the following relations hold ($I(\mathbb{R})$: set of all closed intervals of real numbers):

$$f_I(Z) \in I(\mathbb{R}) \tag{3.1}$$

$$\{f(z) | z \in Z\} \subseteq f_I(Z) \tag{3.2}$$

$$z \in \mathrm{dom}(f) \Rightarrow \{f(z)\} = f_I([z, z]). \tag{3.3}$$

As an example let be $f(z_1, z_2) := z_1(1 - z_2)$. Then it can be set up $f_I(Z_1, Z_2) := Z_1(1 - Z_2)$, and the properties (3.1), (3.2), (3.3) hold with equality under (3.2). Setting $Z_1 = [-1, 2]$ and $Z_2 = [0, 3]$, we get

$$\{f(z_1, z_2) | z_1 \in Z_1, z_2 \in Z_2\} = [-1, 2](1 - [0, 3]) = [-1, 2][-2, 1] = [-4, 2].$$

In general under (3.2) only "$\subseteq$" is valid, and an interval function f_I having the properties (3.1), (3.2) and (3.3) is called an interval extension of f. The property (3.2) remains valid if during the computation of the value of $f_I(Z)$ always outwardly directed rounding will be done. In [9] several possibilities are given to get interval extensions f_I of real functions f.

The interval extensions f_I used in this paper are stated without difficulties in such a way that

$$Y, Z \in \mathrm{dom}(f_I) \wedge Y \subseteq Z \Rightarrow f_I(Y) \subseteq f_I(Z) \tag{3.4}$$

(inclusion isotony of f_I) holds too.

Now we return to the essential request of this section, namely to compute the above M. For the sake of simplicity we directly assume without loss of generality that $X \in I(\mathbb{R})^n$ and $U \in I(\mathbb{R})^m$. Because of the continuity of W the set M can also be described in the following manner:

$$M = \{x \in X | \exists u (u \in U \wedge W(x, u) \le 0)\}. \tag{3.5}$$

To enclose M in the algorithm below, partitions of U into subintervals $U^{(j)}$ will be used in order to be able to decide from subintervals $X^{(i)}$ of X whether

$$X^{(i)} \subseteq M \tag{3.6}$$

or

$$X^{(i)} \cap M = \varnothing \tag{3.7}$$

hold or whether nothing can be said about the correctness of (3.6) and (3.7), respectively. (3.6) resp. (3.7) can be confirmed by the strength of (3.2) in the following manner: let $(U^{(j)})_{j\in Q}$ be a family of subintervals of U with

$$\bigcup_{j\in Q} U^{(j)} = U, \tag{3.8}$$

$X^{(i)}$ any subinterval of X and let W_I be an inclusion isotone interval extension of W then

$$\exists j(j \in Q \wedge W_I(X^{(i)}, U^{(j)}) \leq 0) \Rightarrow X^{(i)} \subseteq M; \tag{3.9}$$

$$\forall j(j \in Q \Rightarrow W_I(X^{(i)}, U^{(j)}) > 0) \Rightarrow X^{(i)} \cap M = \varnothing. \tag{3.10}$$

If neither of the two premises stated in (3.9) resp. (3.10) hold, i.e., if for each $j \in Q$ the right hand endpoint $r(W_I(X^{(i)}, U^{(j)}))$ of $W_I(X^{(i)}, U^{(j)})$ is greater than zero and there is a $j \in Q$ such that the left hand endpoint $l(W_I(X^{(i)}, U^{(j)}))$ of $W_I(X^{(i)}, U^{(j)})$ is smaller or equal than zero, then one has to try by means of finer partitions of U and possibly of $X^{(i)}$ to verify (3.6) resp. (3.7). On the strength of (3.4) the inequality $W_I(X^{(i)}, \tilde{U}) \leq 0$ is correct at most for such subintervals $\tilde{U}$ of U, for which $\tilde{U} \subseteq U^{(j)}$ and $l(W_I(X^{(i)}, U^{(j)})) \leq 0$ for any j. Consequently for a further investigation to $X^{(i)}$ only such $U^{(j)}$ should be subdivided. But in order to limit the organizational expense we compute enclosures of M by means of the following algorithm:

It will be built up four lists $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ and $\mathfrak{U}$, where the lists $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ contain only subintervals of X and the list $\mathfrak{U}$ is always a splitting of U into subintervals $U^{(j)}$. At the beginning $\mathfrak{X}$ only contains X and $\mathfrak{U}$ only U, $\mathfrak{Y}$ and $\mathfrak{Z}$ are empty. The variables P, Q, R, S will have as values the actual numbers of members of $\mathfrak{X}$, $\mathfrak{U}$, $\mathfrak{Y}$ and $\mathfrak{Z}$, respectively, so that at the beginning $P = Q = 1$ and $R = S = 0$. Moreover, let $\bar{P}$, $\bar{Q}$, $\bar{R}$ and $\bar{S}$ be defined as maximum values of P, Q, R and S, respectively. Let the elements of $\mathfrak{X}$, $\mathfrak{U}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ be numbered from $X^{(1)}$ to $X^{(P)}$, $U^{(1)}$ to $U^{(Q)}$, $Y^{(1)}$ to $Y^{(R)}$ and $Z^{(1)}$ to $Z^{(S)}$, respectively. The list $\mathfrak{Z}$ contains only such subintervals $X^{(i)}$ of X for which $X^{(i)} \subseteq M$ is proven and analogously $\mathfrak{Y}$ only such $X^{(i)}$ with $X^{(i)} \cap M = \varnothing$ (the list $\mathfrak{Y}$ will be used at first time in the next section).

After the above-mentioned initialization the algorithm may work as follows:

while $P \leq \bar{P}$ *and* $Q \leq \bar{Q}$ *and* $R \leq \bar{R}$ *and* $S \leq \bar{S}$ *do*
  begin $i := 1$;
  repeat $H := 0$; $j := 1$;
    repeat if $r(W_I(X^{(i)}, U^{(j)})) \leq 0$ *then*
    begin $S := S + 1$; $Z^{(S)} := X^{(i)}$; $\mathfrak{X} := \mathfrak{X}\backslash\{X^{(i)}\}$; $\mathfrak{Z} := \mathfrak{Z} \cup \{Z^{(S)}\}$;
      if $P = 1$ *then* STOP *else*
      begin $X^{(i)} := X^{(P)}$; $P := P - 1$; $j := 1$; $H := 0$ *end*
    end else
    begin if $l(W_I(X^{(i)}, U^{(j)})) > 0$ *then* $H := H + 1$; $j := j + 1$ *end*
    until $j > Q$;
    if $H = Q$ *then*
    begin $R := R + 1$; $Y^{(R)} := X^{(i)}$; $\mathfrak{Y} := \mathfrak{Y} \cup \{Y^{(R)}\}$; $\mathfrak{X} := \mathfrak{X}\backslash\{X^{(i)}\}$;
      if $P = 1$ *then* STOP
        else begin $X^{(i)} := X^{(P)}$; $P := P - 1$ *end*
      end else $i := i + 1$;

until $i > P$;
subdivide some of the $X^{(i)} \in \mathfrak{X}$, $U^{(j)} \in \mathfrak{U}$ and replace $X^{(i)}$ resp.
$U^{(j)}$ by subintervals;
renew the values of P and Q
end.

The variable H counts for each fixed $i \in \{1, \ldots, P\}$ the number of those $U^{(j)}$ satisfying $l(W_I(X^{(i)}, U^{(j)})) > 0$. If and only if $H = Q$ it is proven that $X^{(i)} \cap M = \emptyset$, and $X^{(i)}$ will be taken over from the list $\mathfrak{X}$ to the list $\mathfrak{Y}$.

There are several strategies to get splittings (partitions) of the $X^{(i)}$ and $U^{(j)}$, respectively. For instance, one can bisect those $X^{(i)}$ resp. $U^{(j)}$ which have maximum span, or one bisects some or each of the $X^{(i)}$ resp. $U^{(j)}$ completely getting 2^n resp. 2^m subintervals. The span of $Y \in I(\mathbb{R})^n$ is given by

$$\operatorname{span} Y := \max_{1 \le i \le n} \bar{y}_i - \underline{y}_i . \tag{3.11}$$

If the algorithm will be broken up at any time then always

$$\bigcup_{i=1}^{S} Z^{(i)} \subseteq M \subseteq \bigcup_{i=1}^{S} Z^{(i)} \cup \bigcup_{i=1}^{P} X^{(i)} \tag{3.12}$$

for the actual lists $\mathfrak{X}$ and $\mathfrak{Z}$. If one of the STOP's will be reached then the list $\mathfrak{X}$ is empty and

$$M = \bigcup_{i=1}^{S} Z^{(i)} . \tag{3.13}$$

Under very general assumptions concerning W and W_I it can be proved that as a result of splitting the elements of $\mathfrak{X}$ and $\mathfrak{U}$ finer and finer the set M will be enclosed by (3.12) as close as one wishes. Such an assumption says that for each interval $Y \subseteq X$

$$\text{if} \quad \operatorname{span} Y \to 0 \qquad \text{then} \quad \operatorname{span} W_I(Y) \to 0 . \tag{3.14}$$

4. Computing the Set $D(c)$

If

$$c := \inf\{V(x) | x \in X \backslash M\} \tag{4.1}$$

then for each $x \in X$

$$V(x) \le c \Rightarrow x \in M , \tag{4.2}$$

and for each $c_1 > c$ there are elements $x \in \mathbb{R}^n \backslash M$ satisfying

$$c < V(x) \le c_1 .$$

By the following algorithm we want to compute an interval $[a2, b2]$ with span $[a2, b2] < \varepsilon$ and $c \in [a2, b2]$, where $\varepsilon > 0$ is given. It will be used that for the lists $\mathfrak{X}$ and $\mathfrak{Y}$ computed in Section 3

$$\bigcup_{i=1}^{R} Y^{(i)} \subseteq X \backslash M \subseteq \bigcup_{i=1}^{P} X^{(i)} \cup \bigcup_{i=1}^{R} Y^{(i)}, \tag{4.3}$$

the boundary points of M excepted.

We assume $\max\{P, R\} \geq 1$ (in the other case we must replace X by a larger interval).

Now the algorithm will be given where UP is a subroutine enclosing the minimum of V on Y into an interval $[a1, b1]$ satisfying $b1 - a1 < \varepsilon$.

if $P \geq 1$ *then*
begin $b2 := r(V_I(X^{(1)}))$; $a2 := b2$;
  for $i := 1$ *to* P *do*
  if $b2 > l(V_I(X^{(i)}))$ *then*
    begin $Y := X^{(i)}$; UP; $a2 := \min\{a1, a2\}$;
      $b2 := \min\{b1, b2\}$
    end
end else
begin $b2 := r(V_I(Y^{(1)}))$; $a2 := b2$ *end*;
if $R \geq 1$ *then*
for $i := 1$ *to* R *do*
  if $b2 > l(V_I(Y^{(i)})$ *then*
  begin $Y := Y^{(i)}$; UP; $a2 := \min\{a1, a2\}$;
    $b2 := \min\{b1, b2\}$
  end;
write $a2, b2$.

After termination of the algorithm it is $c \in [a2, b2]$ and $b2 - a2 < \varepsilon$. Therefore for $c := a2$

$$\{x \in X \,|\, V(x) \leq c\} \subseteq M. \tag{4.4}$$

Algorithm UP:

$k := 1$; $j := 1$; $W^{(1)} := Y$;
repeat bisect $W^{(1)}$ in one coordinate direction j into the subintervals Y_1, Y_2;
  $W^{(1)} := Y_1$; $k := k + 1$; $W^{(k)} := Y_2$;
  $a1 := \min_{1 \leq l \leq k} l(V_I(W^{(l)}))$; $b1 := \min_{1 \leq l \leq k} r(V_I(W^{(l)}))$;
  remove all those $W^{(j)}$ from the list $\mathfrak{W}$, for which $l(V_I(W^{(j)})) \geq b1$ holds; give k its new value;
  arrange the remaining $W^{(j)}$ monotone increasing with respect to the $l(V_I(W^{(j)}))$;
  if $j < n$ *then* $j := j + 1$ *else* $j := 1$;
until $b1 - a1 < \varepsilon$.

Here, a list $\mathfrak{W}$ will be built up of actual length k, where immediately such $W^{(i)}$ will be removed from $\mathfrak{W}$ on which V cannot take its minimum value. All those $W^{(i)}$ will be partitioned which are earliest candidates for V taking its minimum value of $W^{(i)}$.

After computation of c finally an algorithm will be given to compute the set

$$D(c) := \{x \in M \,|\, V(x) \leq c\}. \tag{4.5}$$

It will be used that

$$\bigcup_{i=1}^{S} Z^{(i)} \subseteq M \tag{4.6}$$

and intervals will be searched under the $Z^{(i)}$ or as subintervals of them such that the maximum value of V and $Z^{(i)}$ is smaller or equal than c. Each found interval will be taken into a list $\mathfrak{W}$ which is at first empty and whose actual length is given by the value of k. At each stage of the computation the enclosure

$$\bigcup_{i=1}^{k} W^{(i)} \subseteq D(c) \subseteq \bigcup_{i=1}^{k} W^{(i)} \cup \bigcup_{i=1}^{S} Z^{(i)} \tag{4.7}$$

holds.

The algorithm:

$k := 0$; $l := 0$; $T := S$;
for $i := 1$ *to* S *do*
  if $l(V_I(Z^{(i)})) > c$ *then*
    begin $Z^{(i)} := Z^{(S)}$; $S := S - 1$; $l := l + 1$ *end*
    else
    if $r(V_I(Z^{(i)})) \leq c$ *then*
      begin $k := k + 1$; $W^{(k)} := Z^{(i)}$; $Z^{(i)} := Z^{(S)}$;
        $S := S - 1$ *end*;
if $l + k = T$ *then* STOP *else*
  begin arrange $Z^{(1)}, \ldots, Z^{(S)}$ in such a way that
    span $Z^{(1)} \geq \cdots \geq$ span $Z^{(S)}$;
    repeat bisect $Z^{(1)}$ in coordinate direction of maximum width and name the two
      subintervals by X_1 and X_2; $Z^{(1)} := X_1$; $S := S + 1$; $Z^{(S)} := X_2$;
      arrange $Z^{(1)}, \ldots, Z^{(S)}$ in such a way that span $Z^{(1)} \geq \cdots \geq$ span $Z^{(S)}$;
      if $r(V_I(Z^{(1)})) \leq c$ *then*
      begin $k := k + 1$; $W^{(k)} := Z^{(1)}$;
        for $i := 1$ *to* $S - 1$ *do* $Z^{(i)} := Z^{(i+1)}$;
          $S := S - 1$ *end*;
    until span $Z^{(1)} < \varepsilon$
  end.

If during the computation the STOP will be reached then

$$D(c) = \bigcup_{i=1}^{k} W^{(i)}.$$

Since the list $\mathfrak{Y}$ will be used only once, namely in order to compute c, it is not necessary to built it up explicitly. One can compute recursively the minimum value of V on the intervals $Y^{(i)}$.

5. Examples

In [6, 7] five second order examples are considered. Some of them were to demonstrate the applicability of the proved theorems without carrying out the approximations in detail.

For examples developed from whose we shall illustrate the above described computational method whereby the first linear system is computed in order to compare the achieved result with those in [6].

Example 1:
Consider the system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_1(t) + x_2(t) + u(t) \end{aligned} \tag{5.1}$$

with $U = [-1, 1]$. Using a LYAPUNOV-function V of the form

$$V(x) = x_1^2 + x_2^2$$

we obtain

$$W(x, u) = x_2(x_2 + u).$$

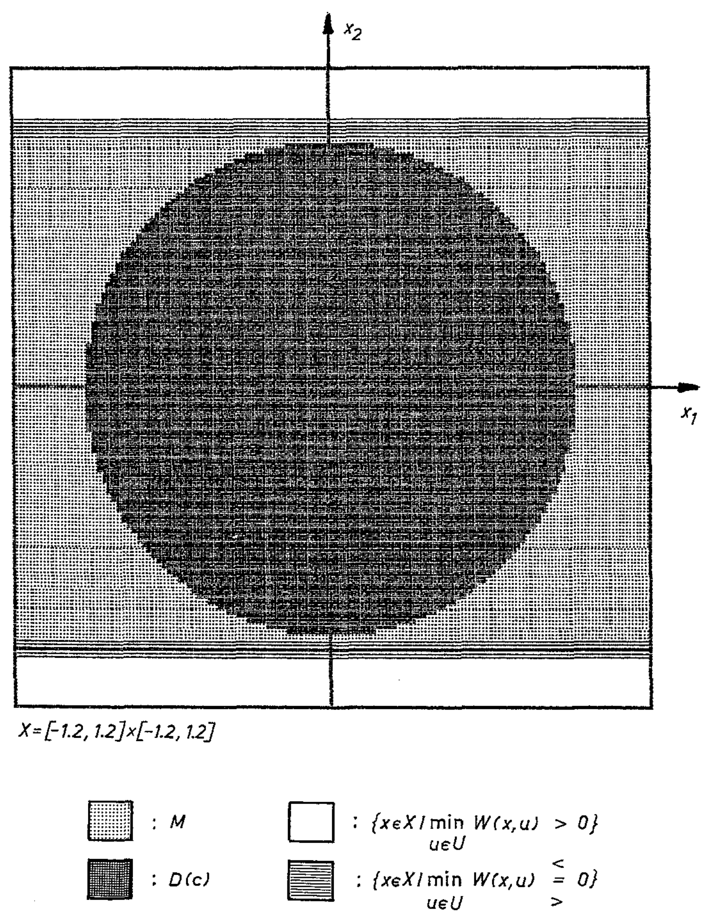

Figure 1

The interval extensions V_I and W_I used in this and the following examples are obtained in such a way that real variables in V and W are interpreted as variables for intervals.

The appropriate ranges M and $D(c)$ are shown in Fig. 1. They agree with those in [6] which are determined analytically. In the same paper it is also shown that the condition (ii) of Theorem 1 is satisfied.

Example 2:
A nonlinear system derived from Example 2 in [6] is the following:

$$\begin{aligned} \dot{x}_1(t) &= 2x_1(t) + u(t) \\ \dot{x}_2(t) &= x_2(t)[1 + \exp(x_1(t)x_2(t))] + u(t) \end{aligned} \tag{5.2}$$

where again $U = [-1, 1]$.

The LYAPUNOV-function V with

$$V(x) = 0.5(x_1^2/4 + x_2^2/9)$$

leads to the sets in Fig. 2. Here, one can see that no range $D(c) \subseteq M$ is determined for a sensible number c. Simultaneously it becomes clear that on the other hand the

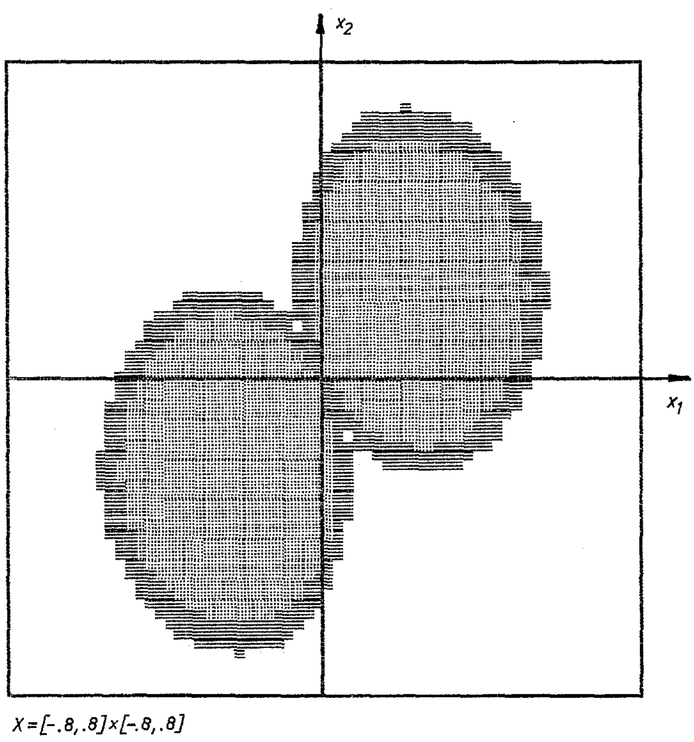

Figure 2

assumptions of Theorem 2 are verifyable numerically using the algorithm elaborated in Section 4 with appropriate modifications.

Example 3:
Finally we consider another nonlinear system

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\exp(x_2(t)) + u(t)\\ \dot{x}_2(t) &= x_1^3(t)\end{aligned} \tag{5.3}$$

with $U = [-1, 1]$.

It is derived from Example 2 in [7] where the validity of the appropriate assumptions of the theorem are proved without determination of the controllability set in detail.

Choosing this time a function V of the form

$$V(x) = \int_0^{x_1} z^3\,dz + 0.5x_2^2 = 0.25x_1^4 + 0.5x_2^2$$

we obtain the sets M and $D(c)$ in Fig. 3.

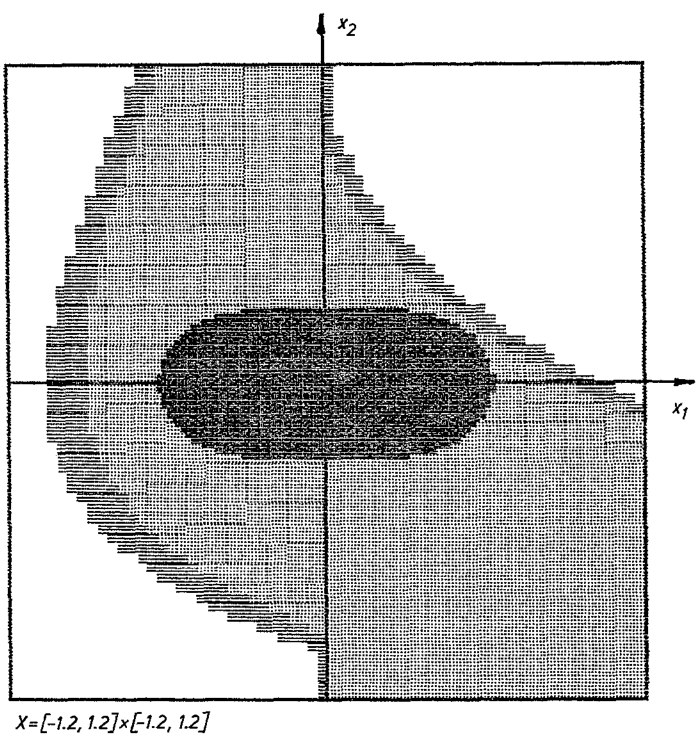

Figure 3

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