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Exponential Stability of Interval Dynamical Systems with Quadratic Nonlinearity

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Abstract. This article proposes an approach for investigating the exponential stability of a nonlinear interval dynamical system with the nonlinearity of a quadratic type on the basis of the Lyapunov's direct method. It also constructs an inner estimate of the attraction domain to the origin for the system under consideration.

1. Introduction

There are a considerable number of works devoted to the stability problem of dynamical systems in the state space with interval uncertainty. Omitting a detailed discussion of the advantages, disadvantages and computational complexity of each approach offered before we only discuss here some questions that remain open.

At first sight, of the stability problem for interval dynamical systems given in the state space (in the light of the result of the work [8]) appear to be deceptively simple, even for the linear case. The unsuccessful attempts to generalize the result of the work [8] for the case of interval matrices [3], [4], [7] and the results of the book [9] show that the stability problem for interval matrices is NP-hard.

Unlike the linear case, there are fewer works devoted to the stability problem for the class of nonlinear interval dynamical systems given in the state space, because developing any investigation method for the class of nonlinear interval dynamical systems is obviously more difficult than for the class of linear ones. In our paper an approach developing the Lyapunov's direct method for solving the exponential stability problem for nonlinear interval dynamical systems given in the state space with the quadratic nonlinearity is proposed. The condition obtained using this approach for analyzing the exponential stability of the system under investigation does not require much computational work.

2. Problem Statement

The object of our attention is a dynamical system of which the perturbed motion can be described in the state space using the following nonlinear differential equation with uncertain parameters

$$\dot{x}(t) = (A_c + \Delta A)x(t) + X(t)(B_c + \Delta B)x(t), \quad x(t_0) = x_0, \quad t \in [t_0, \infty), \tag{2.1}$$

where t is the independent variable (time), $x(t) = (x_i(t))$ is the state vector; the components $x_i(t)$ of the state vector $x(t)$ are continuously differentiable functions on $[t_0, \infty)$, i.e. $x_i(t) \in C^1[t_0, \infty)$, $i = 1, 2, \ldots, n$; at the initial time t_0 the value of the state vector is supposed to be known x_0 and belong to some open vicinity $\mathcal{D} \subseteq \mathbb{R}^n$ of the origin. The matrix $X(t)$ is of the block-diagonal form

$$X(t) = \mathrm{Diag}\{\underbrace{x^T(t), x^T(t), \ldots, x^T(t)}_{n}\},$$

i.e. $X(t)$ has the same block-diagonal elements equal to the transposed state vector $x^T(t)$. The constant matrices $A_c \in \mathbb{R}^{n\times n}$ and $B_c \in \mathbb{R}^{n^2\times n}$ are known, the matrix B_c has the following block view

$$B_c = \begin{pmatrix} B_{1c} \\ B_{2c} \\ \vdots \\ B_{nc} \end{pmatrix}, \tag{2.2}$$

where $B_{ic} \in \mathbb{R}^{n\times n}$, $i = 1, 2, \ldots, n$ are the known constant matrices. Parameter uncertainty in the system (2.1) is caused by two unknown constant matrices $\Delta A \in \mathbb{R}^{n\times n}$ and $\Delta B \in \mathbb{R}^{n^2\times n}$. These matrices are supposed to be able to take values from the given interval matrices [1], [10] with known bounds

$$\Delta A \in [-\Delta_A, \Delta_A], \qquad \Delta B \in [-\Delta_B, \Delta_B],$$

where $\Delta_A = |\Delta_A| \in \mathbb{R}^{n\times n}$, $\Delta_B = |\Delta_B| \in \mathbb{R}^{n^2\times n}$ are the given constant matrices, the matrices ΔB and Δ_B having the block view similar to (2.2)

$$\Delta B = \begin{pmatrix} \Delta B_1 \\ \Delta B_2 \\ \vdots \\ \Delta B_n \end{pmatrix}, \qquad \Delta_B = \begin{pmatrix} \Delta_{B_1} \\ \Delta_{B_2} \\ \vdots \\ \Delta_{B_n} \end{pmatrix},$$

where $\Delta B_i, \Delta_{B_i} \in \mathbb{R}^{n\times n}$, $i = 1, 2, \ldots, n$. The operation of taking absolute value $|\cdot|$ concerning matrices and vectors is understood componentwise. We also suppose, that $-A_c \notin [-\Delta_A, \Delta_A]$. The latter condition denotes, that not all elements of the interval matrix $[A_c - \Delta_A,\ A_c + \Delta_A]$ contain zero simultanously.

For any combination of fixed values of the matrices $\Delta A \in [-\Delta_A, \Delta_A]$ and $\Delta B \in [-\Delta_B, \Delta_B]$ the differential equation (2.1) has a unique solution for the given $x_0 \in \mathcal{D} \subseteq \mathbb{R}^n$. For the zero initial value $x_0 = 0$ we have the trivial solution $x(t, t_0, x_0) = x(t, t_0, 0) \equiv 0$, which is an equilibrium position of the system (2.1).

DEFINITION 2.1. The trivial solution $x(t, t_0, 0) \equiv 0$ of the system (2.1) is said to be exponentially stable when $t \to \infty$, if there exist such positive constants N and α, that for any values $\Delta A \in [-\Delta_A, \Delta_A]$ and $\Delta B \in [-\Delta_B, \Delta_B]$ and any solution $x(t, t_0, x_0)$ for $x_0 \in \mathcal{D}$ the following inequality

$$||x(t, t_0, x_0)||_2 \leq N||x(t_0)||_2 \exp\big(-\alpha(t - t_0)\big)$$

holds, where $||\cdot||_2$ is the Euclidean norm.

The system (2.1) can have the exponential stability property not for all initial values $x_0 \in \mathcal{D}$, but for some subset $\mathcal{D}^* \subseteq \mathcal{D}$ containing the origin. We shall call the set of all those initial values x_0 for which the system (2.1) is exponentially stable an *attraction domain* to the origin for the exponential stability property.

Our **task** is to find the attraction domain or its inner estimation and conditions under which the equilibrium position $x(t, t_0, 0) \equiv 0$ of the nonlinear interval dynamical system (2.1) is exponentially stable in the sense of Definition 2.1.

3. Main Result

To find exponential stability conditions we use the Lyapunov's direct method and choose Lyapunov's function from the quadratic forms class

$$V(x) = x^T H x, \tag{3.1}$$

where $H \in \mathbb{R}^{n\times n}$, $H = H^T \rhd 0$ is the symmetric positive definite matrix to be determined from the following matrix equation

$$A_c^T H + HA_c + HH = -Q, \tag{3.2}$$

where $Q \in \mathbb{R}^{n\times n}$, $Q = Q^T \rhd 0$ is some symmetric positive definite matrix.

We enter the interval matrices $\mathbf{B}_i = [B_{ic} - \Delta_{B_i}, B_{ic} + \Delta_{B_i}]$, $i = 1, 2, \ldots, n$ and the following value

$$l = \sum_{i=1}^{n} \left(\max_{B_i \in \mathbf{B}_i} \varrho\left(\frac{1}{2}(B_i + B_i^T)\right)\right)^2$$

into consideration, where $\varrho(\cdot)$ is the spectral radius of a real square matrix. The minimal and maximal eigenvalues of a real square symmetric matrix are denoted as $\lambda(\cdot)$ and $\Lambda(\cdot)$ respectively.

In the space $\mathbb{R}^n$ we construct the closed set

$$\mathcal{E}(\mu) = \{x \in \mathbb{R}^n \mid V(x) \leq \lambda(H)\mu \,/\, l\}, \tag{3.3}$$

where $\mu \in \mathbb{R}$, $\mu > 0$. The set (3.3) is a hyperellipsoid having its center in the origin of $\mathbb{R}^n$. The value $\lambda(H)\mu / l$ is inversely proportional to the square of the major axis length of the hyperellipsoid $V(x) = 1$.

The following theorem gives an exponential stability condition for the system under consideration.

THEOREM 3.1. *Let for the given matrices* $A_c, \Delta_A \in \mathbb{R}^{n\times n}$, $B_c, \Delta_B \in \mathbb{R}^{n^2\times n}$ *and some symmetric positive definite matrix* $Q \in \mathbb{R}^{n\times n}$, $Q = Q^T \rhd 0$ *the matrix equation (3.2) has the symmetric positive definite solution* $H \in \mathbb{R}^{n\times n}$, $H = H^T \rhd 0$, *and the inequality*

$$\lambda(Q) > \varrho(\Delta_A^T|H| + |H|\Delta_A)$$

holds. Then the trivial solution $x(t, t_0, 0) \equiv 0$ *of the system (2.1) is exponentially stable for* $x_0 \in \mathcal{E}(\mu)$, $0 < \mu < \lambda(Q) - \varrho(\Delta_A^T|H| + |H|\Delta_A)$, *and the set (3.3) belongs to the attraction domain to the origin.*

Proof. We denote $A = A_c + \Delta A$ and $B = B_c + \Delta B$ for $\Delta A \in [-\Delta_A, \Delta_A]$ and $\Delta B \in [-\Delta_B, \Delta_B]$. After simple transformations one can obtain the following expression for the time derivative of the function (3.1) on the motion trajectories of the system (2.1)

$$\begin{aligned}
\dot{V}(x) &= \dot{x}^T(t)Hx(t) + x^T(t)H\dot{x}(t) \\
&= \big(Ax(t) + X(t)Bx(t)\big)^T Hx(t) + x^T(t)H\big(Ax(t) + X(t)Bx(t)\big) \\
&= x^T(t)(A^T H + HA)x(t) + x^T(t)\big(B^T X^T(t)H + HX(t)B\big)x(t) \\
&= x^T(t)(A^T H + HA + HH)x(t) - x^T(t)\big(X(t)B - H\big)^T\big(X(t)B - H\big)x(t) \\
&\quad + x^T(t)B^T X^T(t)X(t)Bx(t).
\end{aligned}$$

Since $x^T(t)(X(t)B - H)^T(X(t)B - H)x(t) \geq 0$ we can write

$$\begin{aligned}
\dot{V}(x) &\leq x^T(t)(A^T H + HA + HH)x(t) + x^T(t)B^T X^T(t)X(t)Bx(t) \\
&\leq x^T(t)(A_c^T H + HA_c + HH)x(t) + |x^T(t)|(\Delta_A^T|H| + |H|\Delta_A)|x(t)| \\
&\quad + x^T(t)B^T X^T(t)X(t)Bx(t).
\end{aligned}$$

Now we shall find an upper estimate for the first two items of the last inequality

$$\begin{aligned}
&x^T(t)(A_c^T H + HA_c + HH)x(t) + |x^T(t)|(\Delta_A^T|H| + |H|\Delta_A)|x(t)| \\
&\qquad\qquad \leq -\lambda(Q)x^T(t)x(t) + \varrho(\Delta_A^T|H| + |H|\Delta_A)x^T(t)x(t).
\end{aligned}$$

Further, for positive values μ and $\Delta\mu$ satisfying the inequality $0 < \mu < \mu + \Delta\mu < \lambda(Q) - \varrho(\Delta_A^T|H| + |H|\Delta_A)$ the following expression

$$\begin{aligned}
&-\big(\lambda(Q) - \varrho(\Delta_A^T|H| + |H|\Delta_A)\big)x^T(t)x(t) + x^T(t)B^T X^T(t)X(t)Bx(t) \\
&\qquad\qquad < -(\mu + \Delta\mu)x^T(t)x(t) + x^T(t)B^T X^T(t)X(t)Bx(t)
\end{aligned}$$

holds. Now we shall find a set of such x, that the right-hand side of the last expression will be negative for any $B = B_c + \Delta B$, $\Delta B \in [-\Delta_B, \Delta_B]$. It will be sufficient if we find a set of such x, that for any $B = B_c + \Delta B$, $\Delta B \in [-\Delta_B, \Delta_B]$ the following inequality

$$\mu x^T(t)x(t) - x^T(t)B^T X^T(t)X(t)Bx(t) \geq 0 \tag{3.4}$$

holds. Taking the detailed representation of the expression $x^T(t)B^T X^T(t)X(t)Bx(t)$ into account

$$\begin{aligned}
x^T(t)B^T X^T(t)X(t)Bx(t) &= x^T(t)\begin{pmatrix} B_1^T & B_2^T & \dots & B_n^T \end{pmatrix}\mathrm{Diag}\{\underbrace{x(t), x(t), \dots, x(t)}_{n}\} \\
&\quad \times \mathrm{Diag}\{\underbrace{x^T(t), x^T(t), \dots, x^T(t)}_{n}\}\begin{pmatrix} B_1 \\ B_2 \\ \dots \\ B_n \end{pmatrix} x(t) \\
&= x^T(t)\begin{pmatrix} B_1^T & B_2^T & \dots & B_n^T \end{pmatrix} \\
&\quad \times \mathrm{Diag}\{\underbrace{x(t)x^T(t), x(t)x^T(t), \dots, x(t)x^T(t)}_{n}\}\begin{pmatrix} B_1 \\ B_2 \\ \dots \\ B_n \end{pmatrix} x(t) \\
&= x^T(t)\sum_{i=1}^{n}\left(B_i^T x(t)x^T(t)B_i\right)x(t) \\
&= \sum_{i=1}^{n} x^T(t)B_i^T x(t)x^T(t)B_i x(t) \\
&= \sum_{i=1}^{n}\left(x^T(t)B_i x(t)\right)^2,
\end{aligned}$$

where $B_i = B_{ic} + \Delta B_i$, $\Delta B_i \in [-\Delta_{B_i}, \Delta_{B_i}]$, we can obtain the inequality

$$\begin{aligned}
x^T(t)B^T X^T(t)X(t)Bx(t) &\leq \left(x^T(t)x(t)\right)^2 \sum_{i=1}^{n} \varrho^2\left(\frac{1}{2}(B_i + B_i^T)\right) \\
&\leq \left(x^T(t)x(t)\right)^2 \sum_{i=1}^{n}\left(\max_{B_i \in \mathbf{B}_i} \varrho\left(\frac{1}{2}(B_i + B_i^T)\right)\right)^2 \\
&= \left(x^T(t)x(t)\right)^2 l.
\end{aligned}$$

Now it is easy to see, that for all x satisfying the inequality

$$\mu x^T(t)x(t) \geq \left(x^T(t)x(t)\right)^2 l$$

the relation (3.4) holds. The last inequality can be rewritten in the following equivalent form

$$x^T(t)x(t) = ||x(t)||_2^2 \leq \mu / l,$$

since $l, \mu > 0$ and $x^T(t)x(t) \geq 0$.

On the other hand

$$V(x) = x^T(t)Hx(t) \geq \lambda(H)x^T(t)x(t) = \lambda(H)||x(t)||_2^2.$$

Combining the last two relations one can write the following inequality

$$V(x) \leq \lambda(H)\mu / l.$$

If this inequality holds, then the inequality (3.4) will also hold. In other words, for all x satisfying the last inequality, i.e. for $x \in \mathcal{E}(\mu)$ the inequality (3.4) holds for all $B = B_c + \Delta B$, $\Delta B \in [-\Delta_B, \Delta_B]$. Then in the set $\mathcal{E}(\mu)$ the time derivative of Lyapunov's function (3.1) on the motion trajectories of the system (2.1) satisfies the inequality

$$\dot{V}(x) < -(\mu + \Delta\mu)x^T(t)x(t) + x^T(t)B^TX^T(t)X(t)Bx(t) \leq -\Delta\mu x^T(t)x(t)$$

uniformly on $\Delta A \in [-\Delta_A, \Delta_A]$ and $\Delta B \in [-\Delta_B, \Delta_B]$. Thus, the time derivative of Lyapunov's function being a positive definite quadratic form is limited above with a negative definite quadratic form. Applying some known results, for example [5], [2], we conclude, that the trivial solution $x(t, t_0, 0) \equiv 0$ is exponentially stable for $x_0 \in \mathcal{E}(\mu)$. The set (3.3) belongs to the attraction domain to the origin. The values α and N in Definition 2.1 are determined as [5]

$$\alpha = \lambda(Q) / \bigl(2\Lambda(H)\bigr), \qquad N = \sqrt{\Lambda(H) / \lambda(H)}.$$

The theorem is proved. □

Remark. To construct the set (3.3), which is an inner estimate of the attraction domain to the origin, we must determine the value l. This calculation is a rather difficult task. It is easy to see, that for constructing an inner estimate of the attraction domain to the origin one can use an upper estimate for the value l. Now we shall show how one can calculate upper estimate like this. Using arithmetical operations of the classical interval arithmetic [1], [10] we calculate the interval matrices

$$\mathbf{G}_i = (\mathbf{B}_i + \mathbf{B}_i^T) / 2, \qquad i = 1, 2, \ldots, n.$$

It is easy to note, that the following inequality

$$l = \sum_{i=1}^{n} \left(\max_{B_i \in \mathbf{B}_i} \varrho\left(\frac{1}{2}(B_i + B_i^T)\right)\right)^2 \leq \sum_{i=1}^{n} \left(\max_{G_i = G_i^T \in \mathbf{G}_i} \varrho(G_i)\right)^2$$

holds, since

$$\{B_i + B_i^T \mid B_i \in \mathbf{B}_i\} \subseteq \mathbf{G}_i, \qquad i = 1, 2, \ldots, n,$$

where the inclusion sign is understood in the theoretical-set sense. Calculation of the upper estimate for the value l is convenient in practice applying either Gershgorin's theorem [6] to the interval matrices $\mathbf{G}_i$, or the following formula

$$\max_{G_i = G_i^T \in \mathbf{G}_i} \varrho(G_i) \leq \lambda(G_{ic}) + \varrho(\Delta_{G_i}),$$

where $\mathbf{G}_i = [G_{ic} - \Delta_{G_i}, G_{ic} + \Delta_{G_i}]$. This formula can be easily obtained using the results of [9].

4. Numeric Example

Let us consider the system (2.1) for $n = 3$ and the following matrices

$$A_c = \begin{pmatrix} -1.1 & 1 & 0 \\ -0.95 & -1 & -0.05 \\ 0 & 0.05 & -1 \end{pmatrix}, \qquad \Delta_A = \begin{pmatrix} 0.1 & 0 & 0 \\ 0.05 & 0 & 0.05 \\ 0 & 0.05 & 0 \end{pmatrix},$$

$$B_{1c} = \begin{pmatrix} 0.15 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Delta_{B_1} = \begin{pmatrix} 0.05 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$B_{ic} = \Delta_{B_i} = 0$ for $i = 2, 3$.

For the matrix

$$Q = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the solution of the matrix equation (3.2) is of the view

$$H \simeq \begin{pmatrix} 0.707151 & -0.028814 & -0.009662 \\ -0.028814 & 0.762911 & 0.005555 \\ -0.009662 & 0.005555 & 0.979233 \end{pmatrix}.$$

The matrix H in our case is symmetric positive definite ($\lambda_1(H) \simeq 0.6948$, $\lambda_2(H) \simeq 0.7747$, $\lambda_3(H) \simeq 0.9798$). Now we shall check the second condition of the theorem. To do it we shall calculate the matrix

$$\Delta_A^T|H| + |H|\Delta_A \simeq \begin{pmatrix} 0.144312 & 0.041510 & 0.002685 \\ 0.041510 & 0.000555 & 0.087107 \\ 0.002685 & 0.087107 & 0.000555 \end{pmatrix}.$$

We have

$$\lambda(Q) = 1,$$
$$\varrho(\Delta_A^T|H| + |H|\Delta_A) \simeq 0.1607,$$
$$\lambda(Q) - \varrho(\Delta_A^T|H| + |H|\Delta_A) \simeq 0.8393 > 0.$$

Thus, all the conditions of the theorem are satisfied, therefore the system under investigation is exponentially stable. Using the expressions given in the theorem proof we can calculate the values α and N

$$\alpha \simeq \frac{1}{(2 \cdot 0.9798)} \simeq 0.5103, \qquad N \simeq \sqrt{\frac{0.9798}{0.6948}} \simeq 1.1875.$$

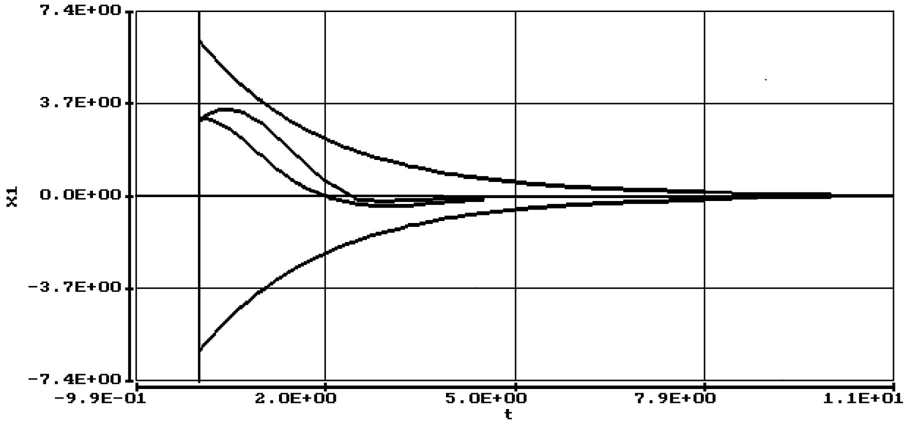

Figure 1. An inner estimate for the first component of x for $x_0 = (3.0, 3.0, 3.0)^T$ and two exponential curves for the found values α and N.

To construct the set (3.3) one needs to find the matrices $\mathbf{G}_i$, $i = 1, 2, 3$. It is easy to check, that the matrices $\mathbf{G}_2$ and $\mathbf{G}_3$ are zero ones, but the matrix

$$\mathbf{G}_1 = \begin{pmatrix} [0.1, 0.2] & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By virtue of the special view of the matrices $\mathbf{G}_i$, $i = 1, 2, 3$ it is not difficult to find the value $l = 0.04$. For the matrix H found above we have $\lambda(H) \simeq 0.6948$. On the base of the proved theorem we infer, that the set (3.3) for the found values l, $\lambda(H)$ and $0 < \mu < \lambda(Q) - \varrho(\Delta_A^T|H| + |H|\Delta_A) \simeq 0.8393$ belongs to the attraction domain to the origin for the system under consideration.

The results of graphical modelling of the system are shown in Figures 1 and 2. Figure 1 shows an inner estimate for the first component of the state vector x of the system under consideration when x_0 belongs the hyperellipsoid. Similarly, Figure 2 shows an inner estimate for the second component of x. These figures also show two exponential curves for the found values α and N. As it can be easily seen from these figures, the graphical results shown in these figures are in full agreement with the theoretical results obtained above.

5. Conclusion

The approach offered in the paper on the base of the Lyapunov's direct method enables the investigation of exponential stability and to construct an inner estimate of the attraction domain to the origin for nonlinear interval dynamical systems with the quadratic nonlinearity. This approach does not require much computational work. It is worth adding, that this approach can be also applied for investigating

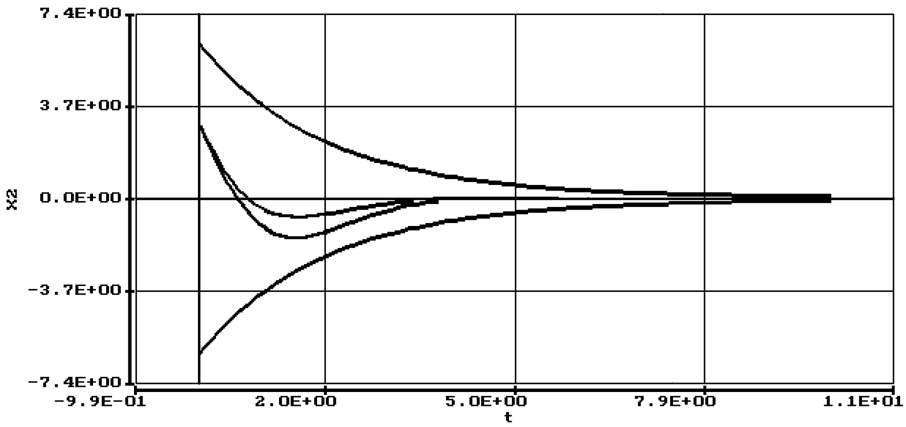

Figure 2. An inner estimate for the second component of x for $x_0 = (3.0, 3.0, 3.0)^T$ and two exponential curves for the found values α and N.

the stability of nonlinear interval dynamical systems given in the state space with nonlinearities of other types.

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