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Interval-Affine Gaussian Algorithm for Constrained Systems

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Abstract. The paper presents interval-affine Gaussian algorithm for the interval linear systems $\boldsymbol{A}x = \boldsymbol{b}$ subject to some constraints on real matrices A from the interval matrix $\boldsymbol{A}$. The interval-affine method is based on the so-called interval-affine arithmetic that allows to take the constraints into account during the computation of interval enclosures of the united solution set of the system $\boldsymbol{A}x = \boldsymbol{b}$, and to make the estimates more accurate.

1. Introduction

Let us be given an interval matrix $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and an interval vector $\boldsymbol{b} \in \mathbb{IR}^n$. We denote

$$\Xi_{\text{uni}}(\boldsymbol{A}, \boldsymbol{b}) = \{x \in \mathbb{R}^n \mid (\exists A \in \boldsymbol{A})\ (\exists b \in \boldsymbol{b})\ (Ax = b)\},$$

a set that will be referred to as *united solution set* of the interval linear system $\boldsymbol{A}x = \boldsymbol{b}$.

Various problems can be stated with respect to $\Xi_{\text{uni}}(\boldsymbol{A}, \boldsymbol{b})$. In the present paper, we are going to deal with the problem of its *outer interval estimation*:

$$\text{find a box } \boldsymbol{U} \subset \mathbb{R}^n, \text{ such that } \Xi_{\text{uni}}(\boldsymbol{A}, \boldsymbol{b}) \subseteq \boldsymbol{U}.$$

By the box, we mean a subset of $\mathbb{R}^n$ which is cartesian product of n intervals. We are interested in $\boldsymbol{U}$ which estimates Ξ_{uni} from the outside most closely in some sense.

Suppose now that we have some constraints (ties) on coefficients of real point systems whose solutions form the set Ξ_{uni}. For example, imposing the restriction that all the matrices $A \in \boldsymbol{A}$ must be symmetric we get the so-called *symmetric interval linear system*. The united solution set to such systems can be defined in the form

$$\Xi_{\text{sym}}(\boldsymbol{A}, \boldsymbol{b}) = \{x \in \mathbb{R}^n \mid (\exists A \in \boldsymbol{A})\ (\exists b \in \boldsymbol{b})\ (A = A^\top \text{ and } Ax = b)\}.$$

The condition $A = A^\top$ is a special case of linear ties on the elements of the matrix. Similarly, one can define systems with the other constraint types. Quite a lot of

attention has been paid to interval linear systems with constraints (ties) in the recent years, see e.g. [1], [2], [8].

Let $\Xi_{\text{tie}}(\boldsymbol{A},\boldsymbol{b})$ be the united solution set of a constrained (tied) interval system. It is obvious that $\Xi_{\text{tie}} \subseteq \Xi_{\text{uni}}$. If the inclusion is strict, one should expect that $\square\Xi_{\text{tie}} \subset \square\Xi_{\text{uni}}$, where "$\square$" is the interval hull of set. Indeed, there are examples (e.g., [1]) of the solution sets of symmetric systems where the above is true. The following question naturally arises:

> How can we take into consideration the constraints on coefficients of the system to make an outer estimate of Ξ_{tie} more exact?

The *interval Gaussian algorithm* (see, e.g., [6]) is widely applied to the classical formulation of the outer interval estimation problem without constraints. The interval version of Gaussian algorithm is a straightforward generalization of the usual point Gaussian algorithm to the interval case. It substantially relies on the classical interval arithmetic, with all its drawbacks and deficiencies. It particular, the classical interval arithmetic is known to suffer from the so-called "dependency problem," which is due to the assumption that the interval quantities under operation are independent from each other. So, traditional interval Gaussian algorithm is hardly able to take any dependence between the coefficients into account, even such trivial like $a_{ij} = a_{ji}$ for symmetric matrices. The equality $\boldsymbol{a}_{ij} = \boldsymbol{a}_{ji}$ does not imply $a_{ij} = a_{ji}$ for all $a_{ij} \in \boldsymbol{a}_{ij}$, $a_{ji} \in \boldsymbol{a}_{ji}$.

A good idea would be to reconstruct the method on a basis of another arithmetic, which is less subject to the dependency problem and which is able to take into account the constraints on coefficients of the system. We are going to describe such a method in the next sections.

2. Basic Concepts and Notation

We distinguish between the notions of "*variable*" and "*value of variable*." We denote variables by lower-case symbols in mathsf style ($\mathsf{x}, \mathsf{y}, \mathsf{z}, \mathsf{u}, \ldots$), while particular values of these variables are denoted by ordinary symbols ($x, y, z, u, \ldots$). Interval objects are traditionally denoted by symbols $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A}, \ldots$ (see [5]).

A set of all values a variable x can take under the current circumstances will be called the *range of values of the variable* x.

The range of values of a variable can be assigned by various ways. For example, when we say "a variable x runs over a set M" or "a variable x varies within M," we thereby inform one that M is the range of values of x. The range of values can also be determined by a set of conditions the variable has to meet.

If the range of values of a variable is contained in $\mathbb{R}$, then such a variable will be called an *one-dimensional (simple) real quantity*. If the range of values of a variable is contained in $\mathbb{R}^n$ for some natural $n > 1$, then the variable will be called an *n-dimensional real quantity* or a *multidimensional real quantity*. In cases when the dimension is clear from the context or does not matter, one-dimensional real quantities or multidimensional real quantities will be also called just *quantities*.

Let us have n one-dimensional quantities $\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n$ with the ranges of values $M_1, M_2, \ldots, M_n$, respectively. There can exist ties and relations between the variables. Let $D \subset \mathbb{R}^n$ be a set of all possible values the tuple $\langle \mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n \rangle$ can take under all the interrelations between the quantities $\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n$. We will call this set the *joint range of values of the quantities* $\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n$ or the *range of values of the tuple* $\langle \mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n \rangle$. A particular value of the tuple $\langle \mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n \rangle$ will be denoted by $\langle x_1, x_2, \ldots, x_n \rangle$. Thus, the statement

$$(\forall \langle x_1, x_2, \ldots, x_n \rangle)\ (\langle x_1, x_2, \ldots, x_n \rangle \in D)$$

is always true.

Remark 2.1. Sometimes it is senseless or inexpedient to consider some variables *jointly*. For example, variables can be related to the matters that are different and disconnected with each other, can be tied by quantifiers, summation signs, etc. In such cases, we will say that the joint range of values of the variables is *undefined* or *unassigned*. In our case, we suppose the range of values D of the variables $\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n$ to be *assigned*.

Before going further, we need to introduce the following notation. Let $\alpha = \langle a_1, a_2, \ldots, a_s \rangle$ be a tuple, of the length n, of arbitrary objects and $1 \le i_1 < i_2 < \cdots < i_k \le s$. We denote

$$\mathrm{pr}_{i_1, i_2, \ldots, i_k} \alpha = \langle a_{i_1}, a_{i_2}, \ldots, a_{i_k} \rangle$$

—a *projection of the tuple* α *to the axes* $i_1, i_2, \ldots, i_k$. Let A be a set of tuples of the length s. We denote

$$\mathrm{pr}_{i_1, i_2, \ldots, i_k} A = \{\mathrm{pr}_{i_1, i_2, \ldots, i_k} \alpha \mid \alpha \in A\}$$

—a *projection of the set A to the axes* $i_1, i_2, \ldots, i_k$.

Let D be the range of values of a tuple $\langle \mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n \rangle$ and $1 \le i_1 < i_2 < \cdots < i_k \le n$. The range of values of the tuple $\langle \mathsf{x}_{i_1}, \mathsf{x}_{i_2}, \ldots, \mathsf{x}_{i_k} \rangle$ will take on the form $D' = \mathrm{pr}_{i_1, i_2, \ldots, i_k} D$. In particular, the range of values M_i of each variable x_i will be of the form $M_i = \mathrm{pr}_i D$.

DEFINITION 2.1. Let $D \subset \mathbb{R}^n$ be the joint range of values of one-dimensional quantities $\mathsf{x}_1, \ldots, \mathsf{x}_n$ and a function $f: \mathbb{R}^n \to \mathbb{R}^m$ be defined on D. We define the *range of the function* $f(\mathsf{x}_1, \ldots, \mathsf{x}_n)$ as a set in $\mathbb{R}^m$ of the form

$$\mathrm{ran}\, f(\mathsf{x}_1, \ldots, \mathsf{x}_n) \stackrel{\mathrm{df}}{=} \{y \mid (\exists \langle x_1, \ldots, x_n \rangle)\ (y = f(x_1, \ldots, x_n))\}. \tag{2.1}$$

Remark 2.2. The designation "$(\exists \langle x_1, \ldots, x_n \rangle)$" means "$(\exists \langle x_1, \ldots, x_n \rangle \in D)$", since D is the range of values of the tuple $\langle \mathsf{x}_1, \ldots, \mathsf{x}_n \rangle$. This designation, in general, is not equivalent to "$(\exists x_1)(\exists x_2)\ldots(\exists x_n)$", which implicitly presumes the independent choice of values of each x_i from the corresponding sets.

A particular value of a function $f(\mathsf{x}_1, \ldots, \mathsf{x}_n)$ will be denoted by $f(x_1, \ldots, x_n)$. Here, the tuple $\langle x_1, \ldots, x_n \rangle$ must be a particular value of the tuple $\langle \mathsf{x}_1, \ldots, \mathsf{x}_n \rangle$. Thus, the statement

$$f(x_1, \ldots, x_n) \in \operatorname{ran} f(\mathsf{x}_1, \ldots, \mathsf{x}_n)$$

is always true.

If, in (2.1), we take $f(\mathsf{x}_1, \ldots, \mathsf{x}_n) \stackrel{\text{df}}{=} \langle \mathsf{x}_1, \ldots, \mathsf{x}_n \rangle$ for f, this results in $\operatorname{ran}\langle \mathsf{x}_1, \ldots, \mathsf{x}_n \rangle = D$.

Let $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Then, it also follows from (2.1) that

$$\operatorname{ran}\langle \mathsf{x}_{i_1}, \mathsf{x}_{i_2}, \ldots, \mathsf{x}_{i_k} \rangle = \operatorname{pr}_{i_1, i_2, \ldots, i_k} \operatorname{ran}\langle \mathsf{x}_1, \ldots, \mathsf{x}_n \rangle. \tag{2.2}$$

One can understand a tuple of quantities $\langle \mathsf{x}_1, \ldots, \mathsf{x}_n \rangle$ as an n-dimensional quantity x with the range of values $\operatorname{ran}\langle \mathsf{x}_1, \ldots, \mathsf{x}_n \rangle \subset \mathbb{R}^n$. Consequently, (2.2) can be rewritten as follows:

$$\operatorname{ran} \operatorname{pr}_{i_1, i_2, \ldots, i_k} \mathsf{x} = \operatorname{pr}_{i_1, i_2, \ldots, i_k} \operatorname{ran} \mathsf{x}.$$

By using the construction "ran", it is possible to denote the range of values of a quantity or the joint range of values of quantities simply by $\operatorname{ran} \mathsf{x}$ or $\operatorname{ran}\langle \mathsf{x}_1, \mathsf{x}_2, \ldots \rangle$. Before doing so, the corresponding sets must be assigned.

In the notation "$\operatorname{ran} f(\mathsf{x})$" we have two parameters, namely the function f and the variable x. If, instead of x, we take another variable z with different range of values, then, generally speaking, the set $\operatorname{ran} f(\mathsf{x})$ differs from the set $\operatorname{ran} f(\mathsf{z})$.

Let an n-dimensional quantity $\mathsf{x} = \langle \mathsf{x}_1, \ldots, \mathsf{x}_n \rangle$ and a k-dimensional quantity $\mathsf{y} = \langle \mathsf{y}_1, \ldots, \mathsf{y}_k \rangle$ be given. From the quantities x and y, we can made up the tuple $\langle \mathsf{x}, \mathsf{y} \rangle$. We suppose, by definition, that

$$\langle \mathsf{x}, \mathsf{y} \rangle \stackrel{\text{df}}{=} \langle \mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{y}_1, \ldots, \mathsf{y}_k \rangle.$$

Therefore,

$$\operatorname{ran}\langle \mathsf{x}, \mathsf{y} \rangle = \operatorname{ran}\langle \mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{y}_1, \ldots, \mathsf{y}_k \rangle \subset \mathbb{R}^{n+k}.$$

For arbitrary quantities x and y, the inclusion $\operatorname{ran}\langle \mathsf{x}, \mathsf{y} \rangle \subseteq \operatorname{ran} \mathsf{x} \times \operatorname{ran} \mathsf{y}$ is always true, where "$\times$" is Cartesian product of sets.

DEFINITION 2.2. We say that two quantities (simple or multidimensional) x and y are *independent from each other* if there is the equality:

$$\operatorname{ran}\langle \mathsf{x}, \mathsf{y} \rangle = \operatorname{ran} \mathsf{x} \times \operatorname{ran} \mathsf{y}.$$

Otherwise, x and y will be called *dependent from each other* or *tied with each other* (see [8]).

For the quantities x and y that are independent from each other, the symbolical statements "$(\exists\langle x, y\rangle)$" and "$(\exists x)(\exists y)$" are equivalent. Additionally, projections of x and y onto any axes are also independent from each other, i.e.

$$\begin{aligned}\operatorname{ran}\langle \operatorname{pr}_{i_1,\ldots,i_k}\mathsf{x},\ \operatorname{pr}_{j_1,\ldots,j_s}\mathsf{y}\rangle &= \operatorname{ran}\operatorname{pr}_{i_1,\ldots,i_k}\mathsf{x}\times\operatorname{ran}\operatorname{pr}_{j_1,\ldots,j_s}\mathsf{y}\\ &= \operatorname{pr}_{i_1,\ldots,i_k}\operatorname{ran}\mathsf{x}\times\operatorname{pr}_{j_1,\ldots,j_s}\operatorname{ran}\mathsf{y}.\end{aligned}$$

Sometimes, the terms "dependence" and "tie", as applied to variables, are used in the irrelative sense. For example, they say "an independent variable x" or "a dependent variable y." We are going to use such phrases too.

DEFINITION 2.3. An *independent variable* is a variable whose range of values is assigned *a priori* and not determined by dependencies (ties) the variable has with any other variables. Otherwise, a variable will be called a *dependent variable*.

If we have, for example, $\mathsf{y} = f(\mathsf{x})$ and $\operatorname{ran}\mathsf{x} = [-2, 4]$, then x is an independent variable, and y is dependent. But if assigned are $\mathsf{y} = f(\mathsf{x})$ and $\operatorname{ran}\mathsf{y} = \{-3, 6, 8\}$, then y will be an independent variable. In both cases, x and y are dependent from each other in case f is not a constant over $\operatorname{ran}\mathsf{x}$.

Remark 2.3. From Definition 2.3, it follows that two independent variables x and y are independent from each other. Hence, an independent variable can not have two different names, i.e. we cannot say "independent variable x and independent variable y and $\mathsf{x} = \mathsf{y}$."

We will say that quantities x and y of the same dimensions are *equivalent in functional sense* or simply *equivalent*, if $\operatorname{ran}(\mathsf{x} - \mathsf{y}) = \{\bar{0}\}$, where $\bar{0}$ denotes the zero in the corresponding space.

We will say that quantities x and y are *equivalent in global sense*, if $\operatorname{ran}\mathsf{x} = \operatorname{ran}\mathsf{y}$. The equality $\operatorname{ran}\mathsf{x} = \operatorname{ran}\mathsf{y}$ implies that x and y have the same dimensions, i.e. quantities of different dimensions can not be equivalent in global sense.

EXAMPLE 2.1. Assume that one-dimensional quantities $\mathsf{x}, \mathsf{y}, \mathsf{z}$ are given, $\operatorname{ran}\mathsf{x} = [0, 1] \subset \mathbb{R}$, and the relations $\mathsf{y} = \mathsf{x}^2$ and $\mathsf{z} = \mathsf{x}^3$ are valid. Then $\operatorname{ran}\mathsf{y} = \operatorname{ran}\mathsf{z} = [0, 1]$, but $\operatorname{ran}(\mathsf{y} - \mathsf{z}) = \operatorname{ran}\mathsf{x}^2(1 - \mathsf{x}) \neq \{0\}$, since, for example, $0.5 \in \operatorname{ran}\mathsf{x}^2(1 - \mathsf{x})$. Thus, y and z are equivalent in global sense, but not equivalent in functional sense.

DEFINITION 2.4. A quantity y will be referred to as an *outer estimate* for a quantity x, if $\operatorname{ran}\mathsf{x} \subseteq \operatorname{ran}\mathsf{y}$. We also say that y *estimates* x *from the outside*.

Remark 2.4. The term "outer estimate" is usually applied to sets. They say that a set A is an outer estimate for a set B, if $B \subseteq A$. We arrive at this case if we put $\operatorname{ran}\mathsf{y} = A$ and $\operatorname{ran}\mathsf{x} = B$. On the other hand, under certain conditions we can, for example, say that $f(\mathsf{x}) + g(\mathsf{x}, \mathsf{y})$ is an outer estimate for $h(\mathsf{x}, \mathsf{y})$ for some quantities x, y and functions f, g, h. It is impossible in case of sets.

Let $\operatorname{ran}\mathsf{x} \subseteq \operatorname{ran}\mathsf{y}$, x and y have the dimension $n > 1$ and $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Then $\operatorname{ran}\operatorname{pr}_{i_1, i_2, \ldots, i_k}\mathsf{x} \subseteq \operatorname{ran}\operatorname{pr}_{i_1, i_2, \ldots, i_k}\mathsf{y}$. This follows from the

fact that $\operatorname{ran} \operatorname{pr}_{i_1, i_2, \ldots, i_k} \mathsf{x} = \operatorname{pr}_{i_1, i_2, \ldots, i_k} \operatorname{ran} \mathsf{x}$ and $\operatorname{ran} \operatorname{pr}_{i_1, i_2, \ldots, i_k} \mathsf{y} = \operatorname{pr}_{i_1, i_2, \ldots, i_k} \operatorname{ran} \mathsf{y}$, and from the properties of projections. Thus, an estimate for a quantity automatically generates an estimate for any its projection.

It is obvious, if $\operatorname{ran} \mathsf{x} \subseteq \operatorname{ran} \mathsf{y}$, then $\operatorname{ran} f(\mathsf{x}) \subseteq \operatorname{ran} f(\mathsf{y})$ for any function f which is defined over $\operatorname{ran} \mathsf{y}$.

3. Interval Quantities

To construct of outer estimates of the objects of interest, we are going to use special variables, the simplest of them being the so-called "interval quantities."

DEFINITION 3.1. An *interval quantity* is an independent quantity with the range of values being a box in the corresponding space.

Therefore, if x is an n-dimensional interval quantity, then $\operatorname{ran} \mathsf{x} = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ for some intervals $[a_i, b_i] \subset \mathbb{R}$.

The independence of interval quantities implies that the range of values of an interval quantity is a specific determined set, and there is a simple way to describe it. Namely, we first assign a box and then associate, with it, a variable that varies within the box. If the range of values of a quantity is built in another way, we shall not consider such a quantity as an interval quantity. For example, let x be an one-dimensional interval quantity, i.e. $\operatorname{ran} \mathsf{x}$ is an interval in $\mathbb{R}$, and a function $f\colon \mathbb{R} \to \mathbb{R}$ be defined and continuous over $\operatorname{ran} \mathsf{x}$. Then the range of values of the quantity $\mathsf{y} = f(\mathsf{x})$ will also be an interval, but, in our understanding, y is not an interval quantity at all.

All the components of a multidimensional interval quantity are independent from each other. The range of values of the projection of an interval quantity to any axes is a box. The joint range of values of several different interval quantities is a box, too (see Remark 2.3).

For any positive integer n and for any box $B \in \mathbb{R}^n$, we suppose that there is a possibility to introduce a *new* interval quantity var, such that $\operatorname{ran} \mathsf{var} = B$. In the latter, "$\mathsf{var}$" is a certain new name that has not been used previously in our consideration.

Let an arbitrary quantity x be given. An interval quantity $\tilde{\mathsf{x}}$ that is an outer estimate of x will be called an *interval estimate of the quantity* x.

Let $\mathsf{x} = \langle \mathsf{x}_1, \ldots, \mathsf{x}_n \rangle$ and $\operatorname{ran} \mathsf{x}$ be a connected compact set in $\mathbb{R}^n$ for some n. Then $\operatorname{ran} \mathsf{x}_i$ for every i will be an interval. An interval estimate $\tilde{\mathsf{x}}'$ of the quantity x, such that $\operatorname{ran} \tilde{\mathsf{x}}' = \operatorname{ran} \mathsf{x}_1 \times \cdots \times \operatorname{ran} \mathsf{x}_n$, will be called the *best interval estimate for the quantity* x. The quantity $\tilde{\mathsf{x}}'$ is also called the *interval hull of* x.

4. Affine Quantities

DEFINITION 4.1. An *one-dimensional affine quantity* is a quantity x of the form

$$\mathsf{x} = a_0 + a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_s\varepsilon_s,$$

where $a_i \in \mathbb{R}$, ε_i is a simple interval quantity, $\operatorname{ran}\varepsilon_i = [-1, 1] \subset \mathbb{R}$ for all i and $a_i \neq 0$ for $i \neq 0$.

A tuple of n one-dimensional affine quantities will be called a *n-dimensional affine quantity*.

The quantities x and ε_i are dependent from each other since the coefficient a_i in front of ε_i in x is nonzero. By the definition of an interval quantity, ε_i is an independent quantity. Therefore, we can also say that x *depends on* ε_i. The set of interval quantities that a quantity x depends on will be called *dependency set of the affine quantity* x.

Any n-dimensional affine quantity x can be represented in the form $\mathsf{x} = L(\varepsilon)$, where $\operatorname{ran}\varepsilon = [-1, 1] \times \cdots \times [-1, 1] \subset \mathbb{R}^k$ for some natural k and L is a linear mapping from $\mathbb{R}^k$ to $\mathbb{R}^n$. The range of values of x is a polyhedron in $\mathbb{R}^n$ named a *zonotope*. The range of values of a projection of x onto any axis is a zonotope too.

Let an affine quantity x depend on $\varepsilon = \langle \varepsilon_1, \ldots, \varepsilon_k \rangle$ and the i-th component of x have the form

$$\mathsf{x}_i = a_{i0} + a_{i1}\varepsilon_{j_{i1}} + a_{i2}\varepsilon_{j_{i2}} + \cdots + a_{ik_i}\varepsilon_{j_{ik_i}},$$

where $\varepsilon_{j_{i1}}, \ldots, \varepsilon_{j_{ik_i}}$ is a subsequence of the sequence $\varepsilon_1, \ldots, \varepsilon_k$, depending on i. Then, for the range of values of x_i, we will have

$$\operatorname{ran}\mathsf{x}_i = \left[a_{i0} - \sum_{t=1}^{k_i} |a_{it}|,\ a_{i0} + \sum_{t=1}^{k_i} |a_{it}|\right].$$

The latter allows us to easily construct the best interval estimate for the affine quantity x.

On the contrary, for any interval quantity y, it is always possible to build such an affine quantity x that $\operatorname{ran}\mathsf{x} = \operatorname{ran}\mathsf{y}$. To do that, one can, for example, associate the affine quantity $\mathsf{x}_i = \operatorname{mid}(\operatorname{ran}\mathsf{y}_i) + \operatorname{rad}(\operatorname{ran}\mathsf{y}_i)\varepsilon_i$ with every interval component $\mathsf{y}_i = \operatorname{pr}_i\mathsf{y}$, where $\operatorname{mid}(\cdot)$ and $\operatorname{rad}(\cdot)$ denote the midpoint and the radius of interval respectively.

An affine quantity $\hat{\mathsf{x}}$ which estimates an arbitrary quantity x from outside will be called an affine estimate of the quantity x.

5. Construction of Outer Estimates

The classical problem of *outer interval estimation* (of range of values, of a solution set, etc.) can be reformulated in our terminology as follows:

> **Given** are a quantity x, an outer interval estimate $\tilde{\mathsf{x}}$ for x and a function f defined over $\operatorname{ran}\mathsf{x}$. (5.1)
>
> **Find** an outer interval estimate $\tilde{\mathsf{y}}$ for the quantity $\mathsf{y} = f(\mathsf{x})$.

In problem (5.1), we are interested in the most exact, ideally best, interval estimate we can compute under the available computational capability.

We can understand the accuracy of the outer estimate in two senses: in global (set-theoretic) sense and in functional sense. Let h be the Hausdorff metric in $\mathbb{R}^n$. By definition, for any two compact sets $A, B \subset \mathbb{R}^n$,

$$h(A,B) \stackrel{\text{df}}{=} \max\Big\{ \max_{a\in A} \min_{b\in B} \|a-b\|,\ \max_{b\in B} \min_{a\in A} \|a-b\| \Big\},$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. The metric h generates the Hausdorff metric (more precisely, *multimetric*) ρ_H on the set of all quantities. For any quantities u, v of the same dimension

$$\begin{aligned}\rho_H(\mathsf{u},\mathsf{v}) &\stackrel{\text{df}}{=} h(\operatorname{ran}\mathsf{u}, \operatorname{ran}\mathsf{v}) \\ &= \max\Big\{ \max_{\mathsf{u}} \min_{\mathsf{v}} \|u-v\|,\ \max_{\mathsf{v}} \min_{\mathsf{u}} \|u-v\| \Big\}.\end{aligned}$$

We remind that u and v are understood as particular values of the variables u and v, and the designation of the form "$\max_{\mathsf{u}} \ldots$" means "$\max_{u\in\operatorname{ran}\mathsf{u}} \ldots$". For ρ_H, the axiom

$$\rho_H(\mathsf{u},\mathsf{v}) = 0 \Rightarrow \mathsf{u} = \mathsf{v}$$

is not true. The metric ρ_H characterizes the closeness of the quantities in global sense, i.e. the closeness of the ranges of values. The closeness of the quantities in functional sense is characterized by the metric

$$\rho(\mathsf{u},\mathsf{v}) \stackrel{\text{df}}{=} \max \operatorname{ran} \|\mathsf{u}-\mathsf{v}\| = \max_{\langle \mathsf{u},\mathsf{v}\rangle} \|u-v\|$$

—the *uniform metric*. For simplicity, we suppose that the sets $\operatorname{ran}\mathsf{u}$, $\operatorname{ran}\mathsf{v}$ and $\operatorname{ran}\langle \mathsf{u},\mathsf{v}\rangle$ are closed, i.e. the metrics ρ_H and ρ are correctly defined.

When searching an interval estimate $\tilde{\mathsf{y}}$ for y in problem (5.1), one can aim to minimize the distance $\rho_H(\mathsf{y},\tilde{\mathsf{y}})$ or the distance $\rho(\mathsf{y},\tilde{\mathsf{y}})$. These distances will be called the *Hausdorff* and *functional errors* of an estimate respectively. We note that, since $\operatorname{ran}\mathsf{y} \subseteq \operatorname{ran}\tilde{\mathsf{y}}$, the distance $\rho_H(\mathsf{y},\tilde{\mathsf{y}})$ has a simpler form:

$$\rho_H(\mathsf{y},\tilde{\mathsf{y}}) = \max_{\tilde{\mathsf{y}}} \min_{\mathsf{y}} \|\tilde{y}-y\|.$$

Insofar as the interval quantity $\tilde{\mathsf{y}}$ is independent from y, it follows that

$$\rho(\mathsf{y},\tilde{\mathsf{y}}) = \max_{\langle \mathsf{y},\tilde{\mathsf{y}}\rangle} \|y-\tilde{y}\| \geq \operatorname{diam}(\operatorname{ran}\mathsf{y}), \tag{5.2}$$

where $\operatorname{diam}(\cdot)$ denotes diameter of the set.

In classical interval analysis, one usually tries to minimize the distance ρ_H. Property (5.2) implies that we cannot really minimize the error $\rho(\mathsf{y},\tilde{\mathsf{y}})$ of the estimate for sufficiently wide ranges by using only interval estimates. As we will see, it is often more profitable to minimize exactly the functional error $\rho(\mathsf{y},\tilde{\mathsf{y}})$. To do that, we will have to use another class of estimating quantities.

For rational function f, the simplest way to solve problem (5.1) is to use a special-purpose "estimative" arithmetic. Namely, we get an overall estimate of the result by constructing the estimates consecutively for all intermediate quantities.

5.1. Interval Arithmetic

Assume that we have an interval estimate $\tilde{\mathsf{u}}$ for a quantity u. The problem of outer interval estimation for rational functions will be solved if, based on an estimate $\tilde{\mathsf{u}}$ for u, we are able to compute an interval estimate for the quantity $\langle \mathsf{u}, \mathsf{x} \star \mathsf{y} \rangle$, where x, y are one-dimensional components of u and "$\star$" is a arithmetical operation from $\{+, -, *, /\}$.

Let $\langle \mathsf{x}, \mathsf{y} \rangle = \mathrm{pr}_{i,j}\mathsf{u}$ for the axes i, j and the corresponding components $\langle \tilde{\mathsf{x}}, \tilde{\mathsf{y}} \rangle = \mathrm{pr}_{i,j}\tilde{\mathsf{u}}$ from the estimate $\tilde{\mathsf{u}}$ are chosen. Assume that the operation "$\star$" is defined over $\mathrm{ran}\langle \tilde{\mathsf{x}}, \tilde{\mathsf{y}} \rangle = \mathrm{ran}\,\tilde{\mathsf{x}} \times \mathrm{ran}\,\tilde{\mathsf{y}}$. It is always true, except for the case when "$\star$" is the division operation "/" and $0 \in \mathrm{ran}\,\tilde{\mathsf{y}}$. The range of values of $\tilde{\mathsf{x}} \star \tilde{\mathsf{y}}$ is an interval:

$$\mathrm{ran}\,\tilde{\mathsf{x}} \star \tilde{\mathsf{y}} = \left[\min_{\langle \tilde{\mathsf{x}}, \tilde{\mathsf{y}} \rangle} \tilde{x} \star \tilde{y},\ \max_{\langle \tilde{\mathsf{x}}, \tilde{\mathsf{y}} \rangle} \tilde{x} \star \tilde{y} \right].$$

We choose such a new interval quantity $\tilde{\mathsf{z}}$ that $\mathrm{ran}\,\tilde{\mathsf{z}} = \mathrm{ran}\,\tilde{\mathsf{x}} \star \tilde{\mathsf{y}}$. Since $\tilde{\mathsf{u}}$ estimates u, and $\tilde{\mathsf{z}}$ is independent, it follows that

$$\mathrm{ran}\langle \mathsf{u}, \mathsf{x} \star \mathsf{y} \rangle \subseteq \mathrm{ran}\langle \tilde{\mathsf{u}}, \tilde{\mathsf{x}} \star \tilde{\mathsf{y}} \rangle \subseteq \mathrm{ran}\,\tilde{\mathsf{u}} \times \mathrm{ran}\,\tilde{\mathsf{x}} \star \tilde{\mathsf{y}} = \mathrm{ran}\langle \tilde{\mathsf{u}}, \tilde{\mathsf{z}} \rangle.$$

We have thus built an *outer interval estimate* $\langle \tilde{\mathsf{u}}, \tilde{\mathsf{z}} \rangle$ for the quantity $\langle \mathsf{u}, \mathsf{x} \star \mathsf{y} \rangle$.

Notice that, in computing $\tilde{\mathsf{z}}$, we used only the quantities $\tilde{\mathsf{x}}, \tilde{\mathsf{y}}$ and the operation "$\star$", and one can perceive the procedure for the computation of $\tilde{\mathsf{z}}$ as a binary operation over the interval quantities $\tilde{\mathsf{x}}$ and $\tilde{\mathsf{y}}$.

DEFINITION 5.1. An *interval arithmetical operation* corresponding to a real arithmetical operation "$\star$" is such a binary operation "$\boxtimes$" that, applied to any two one-dimensional interval quantities v, w, it produces a *new* interval quantity r satisfying

$$\mathrm{ran}\,\mathsf{r} = \left[\min_{\langle \mathsf{v}, \mathsf{w} \rangle} v \star w,\ \max_{\langle \mathsf{v}, \mathsf{w} \rangle} v \star w \right].$$

We write briefly $\mathsf{r} := \mathsf{v} \boxtimes \mathsf{w}$.

Remark 5.1. In Definition 5.1, we use the assignment operator ":=" instead of the equality sign "=", which is due to the fact that the operation "$\boxtimes$" always generates a new quantity. Therefore, the operation "$\boxtimes$" is not single-valued. If we sequentially compute $\mathsf{r}_1 := \mathsf{v} \boxtimes \mathsf{w}$, $\mathsf{r}_2 := \mathsf{v} \boxtimes \mathsf{w}$, then, in general, $\mathsf{r}_1 \neq \mathsf{r}_2$. This is equivalent to $\mathrm{ran}(\mathsf{r}_1 - \mathsf{r}_2) \neq \{0\}$.

For a function of interval quantities $f(\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n)$, one can write its interval analogue. To do that, one needs to rewrite all the operations from f "putting then

into the box $\square$." If, afterward, we perform all the interval operations in compliance with Definition 5.1, we will get an outer interval estimate of $f(\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n)$.

EXAMPLE 5.1. Let $f\colon \mathbb{R}^4 \to \mathbb{R}^2$ and

$$f(\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3, \mathsf{x}_4) = \langle (\mathsf{x}_1 + \mathsf{x}_2 + \mathsf{x}_4) \,/\, (\mathsf{x}_3 - \mathsf{x}_1),\ \mathsf{x}_4 * (\mathsf{x}_2 - \mathsf{x}_1 \,/\, \mathsf{x}_3) \rangle,$$

where all of x_i are interval quantities. Then

$$\mathsf{y} := \langle (\mathsf{x}_1 \boxplus \mathsf{x}_2 \boxplus \mathsf{x}_4) \,/\, (\mathsf{x}_3 \boxminus \mathsf{x}_1),\ \mathsf{x}_4 \boxtimes (\mathsf{x}_2 \boxminus \mathsf{x}_1 \boxslash \mathsf{x}_3) \rangle$$

will be an outer interval estimate for $f(\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3, \mathsf{x}_4)$.

By construction, the interval quantity $\tilde{\mathsf{z}} := \tilde{\mathsf{x}} \boxtimes \tilde{\mathsf{y}}$ is an exact outer estimate of $\tilde{\mathsf{x}} \star \tilde{\mathsf{y}}$ in the sense of the error ρ_H, since $\rho_H(\tilde{\mathsf{z}}, \tilde{\mathsf{x}} \star \tilde{\mathsf{y}}) = 0$. If we take into account the overall environment and suppose that the quantities $\tilde{\mathsf{x}}, \tilde{\mathsf{y}}$ are further used in the computational process jointly with $\tilde{\mathsf{z}}$, we can conclude that the error

$$\sigma = \rho_H(\langle \tilde{\mathsf{x}}, \tilde{\mathsf{y}}, \tilde{\mathsf{z}} \rangle, \langle \tilde{\mathsf{x}}, \tilde{\mathsf{y}}, \tilde{\mathsf{x}} \star \tilde{\mathsf{y}} \rangle)$$

is more appropriate in the situation. This error may be not equal to 0. For example, for the operations "+" or "$-$", one can compute that

$$\sigma = \frac{\operatorname{wid}(\operatorname{ran} \tilde{\mathsf{x}}) + \operatorname{wid}(\operatorname{ran} \tilde{\mathsf{y}})}{\sqrt{3}},$$

where $\operatorname{wid}(\cdot)$ denotes the wide of an interval. For the operations "$*$", "/", it is possible to get similar formulas for the error σ.

Let us compute the error $\rho(\tilde{\mathsf{z}}, \tilde{\mathsf{x}} \star \tilde{\mathsf{y}})$. Inasmuch as $\operatorname{ran} \tilde{\mathsf{z}} = \operatorname{ran} \tilde{\mathsf{x}} \star \tilde{\mathsf{y}}$ and $\tilde{\mathsf{z}}$ is independent from $\tilde{\mathsf{x}} \star \tilde{\mathsf{y}}$, we get

$$\gamma = \rho(\tilde{\mathsf{z}}, \tilde{\mathsf{x}} \star \tilde{\mathsf{y}}) = \max \operatorname{ran} |\tilde{\mathsf{z}} - \tilde{\mathsf{x}} \star \tilde{\mathsf{y}}| = \operatorname{wid} \operatorname{ran} \tilde{\mathsf{z}} = \operatorname{wid}(\operatorname{ran} \tilde{\mathsf{x}} \star \tilde{\mathsf{y}}).$$

The error γ is closely tied with σ. For example, it is possible to show that

$$\operatorname{wid}(\operatorname{ran} \tilde{\mathsf{x}} \pm \tilde{\mathsf{y}}) = \operatorname{wid}(\operatorname{ran} \tilde{\mathsf{x}}) + \operatorname{wid}(\operatorname{ran} \tilde{\mathsf{y}}).$$

That is, for the operations "+" and "$-$", $\sigma = \gamma \,/\, \sqrt{3}$. In most cases, $\sigma > \gamma \,/\, \sqrt{3}$ for the operations "$*$", "/". If we decrease the functional error $\rho(\tilde{\mathsf{z}}, \tilde{\mathsf{x}} \star \tilde{\mathsf{y}})$, we therefore decrease the Hausdorff error

$$\rho_H(\langle \tilde{\mathsf{x}}, \tilde{\mathsf{y}}, \tilde{\mathsf{z}} \rangle, \langle \tilde{\mathsf{x}}, \tilde{\mathsf{y}}, \tilde{\mathsf{x}} \star \tilde{\mathsf{y}} \rangle).$$

5.2. Affine Arithmetic

To construct outer estimates of ranges, we can use affine quantities. Assume that we have an affine estimate $\hat{\mathsf{u}}$ for a quantity u. Let x, y be one-dimensional components of u and "$\star$" be an arithmetical operation from $\{+, -, *, /\}$.

Let us, like in Section 5.1, be given $\langle \mathsf{x}, \mathsf{y} \rangle = \mathrm{pr}_{i,j}\mathsf{u}$, $\langle \hat{\mathsf{x}}, \hat{\mathsf{y}} \rangle = \mathrm{pr}_{i,j}\hat{\mathsf{u}}$ for some axes i, j and the operation "$\star$" be defined over $\mathrm{ran}\langle \hat{\mathsf{x}}, \hat{\mathsf{y}} \rangle$.

Note that for any real numbers a, b, c, the quantity $p(\hat{\mathsf{x}}, \hat{\mathsf{y}}) = a\hat{\mathsf{x}} + b\hat{\mathsf{y}} + c$ is an affine quantity. Denote $\mathsf{t} = \hat{\mathsf{x}} \star \hat{\mathsf{y}} - p(\hat{\mathsf{x}}, \hat{\mathsf{y}})$ and $\delta = \max \mathrm{ran}|\mathsf{t}|$. We introduce a new interval quantity ε, such that $\mathrm{ran}\,\varepsilon = [-1, 1]$. Then $\mathrm{ran}\,\mathsf{t} \subseteq \mathrm{ran}\,\delta\varepsilon = [-\delta, \delta]$, and we have

$$\begin{aligned} \mathrm{ran}\langle \mathsf{u}, \mathsf{x} \star \mathsf{y} \rangle &\subseteq \mathrm{ran}\langle \hat{\mathsf{u}}, \hat{\mathsf{x}} \star \hat{\mathsf{y}} \rangle = \mathrm{ran}\langle \hat{\mathsf{u}}, p(\hat{\mathsf{x}}, \hat{\mathsf{y}}) + \mathsf{t} \rangle \\ &\subseteq \mathrm{ran}\langle \hat{\mathsf{u}}, p(\hat{\mathsf{x}}, \hat{\mathsf{y}}) + \delta\varepsilon \rangle \end{aligned}$$

from the independence of ε. The quantity

$$\hat{\mathsf{z}} = p(\hat{\mathsf{x}}, \hat{\mathsf{y}}) + \delta\varepsilon = a\hat{\mathsf{x}} + b\hat{\mathsf{y}} + c + \delta\varepsilon$$

is an affine quantity, and, therefore, $\langle \hat{\mathsf{u}}, \hat{\mathsf{z}} \rangle$ is an *outer affine estimate* of $\langle \mathsf{u}, \mathsf{x} \star y \rangle$.

We can compute the functional error of the estimate $\hat{\mathsf{z}}$:

$$\begin{aligned} \rho(\hat{\mathsf{z}}, \hat{\mathsf{x}} \star \hat{\mathsf{y}}) &= \max \mathrm{ran}\,|a\hat{\mathsf{x}} + b\hat{\mathsf{y}} + c + \delta\varepsilon - \hat{\mathsf{x}} \star \hat{\mathsf{y}}| \\ &= \max \mathrm{ran}\,|\delta\varepsilon - \mathsf{t}| = \mathrm{wid}(\mathrm{ran}\,\delta\varepsilon) = 2\delta. \end{aligned}$$

Since $\delta = \max \mathrm{ran}|\mathsf{t}|$, δ may be rewritten as

$$\delta = \max_{\langle \hat{\mathsf{x}}, \hat{\mathsf{y}} \rangle} |\hat{x} \star \hat{y} - (a\hat{x} + b\hat{y} + c)|.$$

Choosing the numbers a, b, c in a proper way, we can decrease δ. By doing that, we will decrease the error $\rho(\hat{\mathsf{z}}, \hat{\mathsf{x}} \star \hat{\mathsf{y}})$, while the Hausdorff error $\rho_H(\hat{\mathsf{z}}, \hat{\mathsf{x}} \star \hat{\mathsf{y}})$ may increase at the same time. But our task is to decrease the joint error $\rho_H(\langle \hat{\mathsf{u}}, \hat{\mathsf{z}} \rangle, \langle \hat{\mathsf{u}}, \hat{\mathsf{x}} \star \hat{\mathsf{y}} \rangle)$, and this can be reached by decreasing $\rho(\hat{\mathsf{z}}, \hat{\mathsf{x}} \star \hat{\mathsf{y}})$.

As in the interval case, we use only the quantities $\hat{\mathsf{x}}, \hat{\mathsf{y}}$ and the operation "$\star$" to compute $\hat{\mathsf{z}}$. Let us define a binary operation on the set of affine quantities which corresponds to the process.

DEFINITION 5.2. An *affine arithmetical operation* corresponding to a real arithmetical operation "$\star$" is a binary operation "$\widehat{\star}$" such that, applied to any two one-dimensional affine quantities v, w, it produces an affine quantity r satisfying

$$\mathsf{r} = a\mathsf{v} + b\mathsf{w} + c + \delta\varepsilon,$$

where ε is a new interval quantity, $\mathrm{ran}\,\varepsilon = [-1, 1]$,

$$\delta = \max_{\langle \mathsf{v}, \mathsf{w} \rangle} |v \star w - (av + bw + c)|,$$

and the real coefficients a, b, c are chosen to minimize δ.

For such v, w, and r, we will write $\mathsf{r} := \mathsf{v} \mathbin{\widehat{\star}} \mathsf{w}$.

The affine arithmetic [3], [9] has several properties that the interval arithmetic does not. If we neglect the roundoff errors, multiplication by scalar, addition and

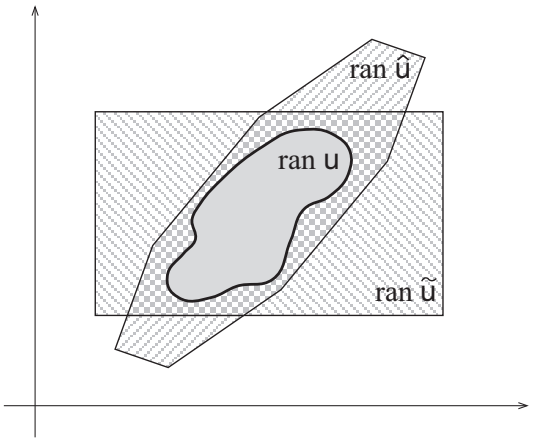

Figure 1. Interval-affine estimate of u in $\mathbb{R}^2$.

subtraction are executed with zero functional error in the affine arithmetic, i.e. exactly. For any one-dimensional affine quantities x, y, the following equalities take place:

$$\begin{aligned}
x \mathbin{\widehat{+}} y &= x + y, \\
x \mathbin{\widehat{-}} y &= x - y, \\
x \mathbin{\widehat{*}} y &= x * y \qquad \big(\mathrm{wid}(\mathrm{ran}\, x) = 0 \text{ or } \mathrm{wid}(\mathrm{ran}\, y) = 0\big), \\
x \mathbin{\widehat{/}} y &= x / y \qquad \big(\mathrm{wid}(\mathrm{ran}\, y) = 0\big).
\end{aligned}$$

In almost all other cases, the affine operations "$\widehat{\star}$" are executed with an error $\delta > 0$. Then, if we consecutively compute $r_1 := x \mathbin{\widehat{\star}} y$, $r_2 := x \mathbin{\widehat{\star}} y$, we will get two different affine quantities r_1 and r_2, often strongly dependent on each other.

5.3. Interval-Affine Arithmetic

The interval and affine estimation methods, based on interval and affine arithmetics, have their own merits and faults. Below, we construct a method that combines the strengths of these methods and, as experiments show, gives a considerable increase in the accuracy of the resulting estimates.

Let u be a quantity. A pair $\{\tilde{u}, \hat{u}\}$, where $\tilde{u}$ and $\hat{u}$ are interval and affine estimates of u, will be called an outer *interval-affine* or *mixed* estimate of the quantity u.

Since $\mathrm{ran}\, u \subseteq \mathrm{ran}\, \tilde{u}$ and $\mathrm{ran}\, u \subseteq \mathrm{ran}\, \hat{u}$, $\mathrm{ran}\, u \subseteq \mathrm{ran}\, \tilde{u} \cap \mathrm{ran}\, \hat{u}$. In the two-dimensional case, the illustrating picture will look like that at Figure 1.

Let $\langle x, y\rangle = \mathrm{pr}_{i,j} u$, $\langle \tilde{x}, \tilde{y}\rangle = \mathrm{pr}_{i,j} \tilde{u}$, $\langle \hat{x}, \hat{y}\rangle = \mathrm{pr}_{i,j} \hat{u}$ for some axes i,j and an operation "$\star$" $\in \{+, -, *, /\}$ is defined over the set $D = \mathrm{ran}\langle \tilde{x}, \tilde{y}\rangle \cap \mathrm{ran}\langle \hat{x}, \hat{y}\rangle$.

We are going to compute an interval-affine estimate for $\langle \mathsf{u}, \mathsf{x} \star \mathsf{y} \rangle$. An interval constituent is easy to compute. This is such a new interval quantity $\tilde{\mathsf{z}}$, that

$$\operatorname{ran} \tilde{\mathsf{z}} = \left[\min_{(\alpha,\beta)\in D} \alpha \star \beta, \max_{(\alpha,\beta)\in D} \alpha \star \beta \right].$$

As an affine estimate, we will take

$$\hat{\mathsf{z}} = a\hat{\mathsf{x}} + b\hat{\mathsf{y}} + c + \delta\varepsilon,$$

where $a, b, c \in \mathbb{R}$, ε is a new interval quantity, $\operatorname{ran} \varepsilon = [-1, 1]$ and

$$\delta = \max_{(\alpha,\beta)\in D} |\alpha \star \beta - (a\alpha + b\beta + c)|.$$

The mixed estimate $\{\langle \tilde{\mathsf{u}}, \tilde{\mathsf{z}} \rangle, \langle \hat{\mathsf{u}}, \hat{\mathsf{z}} \rangle\}$ will be an interval-affine estimate of $\langle \mathsf{u}, \mathsf{x} \star \mathsf{y} \rangle$.

As a result, the problem reduces to searching for reals a, b, c in such a way as to minimize δ. This is, basically, a problem of the *best Tchebychev linear approximation* for the function $f(\alpha, \beta) = \alpha \star \beta$ over the domain D. Since the function $f(\alpha, \beta)$ is already linear for the operations "+" and "−", the problem is to be solved for the operations "$*$" and "/" only.

Taking into account a simple form of D, an approximation close to the best linear approximation of $\alpha \star \beta$ can be found in $O(n)$ computational time, where n is the summary number of elements in the dependency sets of affine $\hat{\mathsf{x}}$ and $\hat{\mathsf{y}}$ (see Section 4). The search problem for the best approximation of $\alpha \star \beta$ over D in close to $O(n)$ computing time is open.

The procedure for the computation of the mixed estimate $\{\tilde{\mathsf{z}}, \hat{\mathsf{z}}\}$ will be called an *interval-affine operation* $\hat{\boxtimes}$ corresponding to a real arithmetical operation "$\star$", and will be written as

$$\{\tilde{\mathsf{z}}, \hat{\mathsf{z}}\} := \{\tilde{\mathsf{x}}, \hat{\mathsf{x}}\} \;\hat{\boxtimes}\; \{\tilde{\mathsf{y}}, \hat{\mathsf{y}}\}.$$

The algorithm for the computation of the operation $\hat{\boxtimes}$ can be represented as follows:

Input

Mixed estimates $\{\tilde{\mathsf{x}}, \hat{\mathsf{x}}\}$ and $\{\tilde{\mathsf{y}}, \hat{\mathsf{y}}\}$, an operation "$\star$".

Output

$\{\tilde{\mathsf{z}}, \hat{\mathsf{z}}\} := \{\tilde{\mathsf{x}}, \hat{\mathsf{x}}\} \;\hat{\boxtimes}\; \{\tilde{\mathsf{y}}, \hat{\mathsf{y}}\}$.

Algorithm

construct the set $D = \operatorname{ran} \langle \tilde{\mathsf{x}}, \tilde{\mathsf{y}} \rangle \cap \operatorname{ran} \langle \hat{\mathsf{x}}, \hat{\mathsf{y}} \rangle$;

compute $\tilde{\mathsf{z}}$, such that $\operatorname{ran} \tilde{\mathsf{z}} = \left[\min_{(\alpha,\beta)\in D} (\alpha \star \beta), \max_{(\alpha,\beta)\in D} (\alpha \star \beta) \right]$;

find a linear approximation $(a\alpha + b\beta + c)$ for $(\alpha \star \beta)$ over D;

compute the approximation error $\delta = \max\limits_{(\alpha,\beta)\in D} |a\alpha + b\beta + c - \alpha \star \beta|$;

construct $\hat{z} = a\hat{x} + b\hat{y} + c + \delta\varepsilon$, where ε is a new interval quantity and $\operatorname{ran}\varepsilon = [-1, 1]$.

6. Interval-Affine Gaussian Algorithm

Let us be given an interval linear system $\boldsymbol{A}x = \boldsymbol{b}$, where $\boldsymbol{A} = (\boldsymbol{a}_{ij}) \in \mathbb{IR}^{n\times n}$, $\boldsymbol{b} = (\boldsymbol{b}_i) \in \mathbb{IR}^n$, and $\Xi_{\text{uni}}(\boldsymbol{A}, \boldsymbol{b})$ is its united solution set.

Using the concept of "interval quantity" introduced in the preceding sections, we can reformulate the problem of outer interval estimation of the set $\Xi_{\text{uni}}(\boldsymbol{A}, \boldsymbol{b})$. We remind that, by "one-dimensional quantity," we call a variable that takes its values from $\mathbb{R}$. Accordingly, an independent variable whose range of values is an interval will be referred to as "one-dimensional interval quantity." In other words, "one-dimensional interval quantity" is an independent variable that is varying in an interval.

We associate, with the intervals $\boldsymbol{a}_{ij}$ and $\boldsymbol{b}_i$, such interval quantities $\tilde{\mathsf{a}}_{ij}$ and $\tilde{\mathsf{b}}_i$ that $\operatorname{ran}\tilde{\mathsf{a}}_{ij} = \boldsymbol{a}_{ij}$, $\operatorname{ran}\tilde{\mathsf{b}}_i = \boldsymbol{b}_i$. Suppose that $\tilde{\mathsf{A}} = (\tilde{\mathsf{a}}_{ij})$ and $\tilde{\mathsf{b}} = (\tilde{\mathsf{b}}_i)$. The matrix $\tilde{\mathsf{A}}$ and the vector $\tilde{\mathsf{b}}$ can be understood as quantities, with the values from $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n\times 1} \stackrel{\text{df}}{=} \mathbb{R}^n$, such that $\operatorname{ran}\tilde{\mathsf{A}} = \boldsymbol{A}$ and $\operatorname{ran}\tilde{\mathsf{b}} = \boldsymbol{b}$.

Let x be a quantity, with its values from $\mathbb{R}^n$, tied with $\tilde{\mathsf{A}}$ and $\tilde{\mathsf{b}}$ by the relation $\tilde{\mathsf{A}}\mathsf{x} = \tilde{\mathsf{b}}$. The range of values of x will have the form

$$\operatorname{ran}\mathsf{x} = \{x \mid (\exists\langle\tilde{A}, \tilde{b}\rangle)\ (\tilde{A}x = \tilde{b})\}. \tag{6.1}$$

One can consider this designation as generalizing (2.1) from Section 2 for the case of an implicit tie between the quantities $\tilde{\mathsf{A}}$, $\tilde{\mathsf{b}}$, and x. Since the interval quantities $\tilde{\mathsf{A}}$ and $\tilde{\mathsf{b}}$ are independent, we can write "$(\exists\tilde{A})(\exists\tilde{b})$" instead of "$(\exists\langle\tilde{A}, \tilde{b}\rangle)$" (see Remark 2.2). Using the standard notation, (6.1) may be rewritten as follows:

$$\operatorname{ran}\mathsf{x} = \{x \in \mathbb{R}^n \mid (\exists\tilde{A} \in \operatorname{ran}\tilde{\mathsf{A}})\ (\exists\tilde{b} \in \operatorname{ran}\tilde{\mathsf{b}})\ (\tilde{A}x = \tilde{b})\},$$

hence,

$$\operatorname{ran}\mathsf{x} = \Xi_{\text{uni}}(\boldsymbol{A}, \boldsymbol{b}).$$

The latter implies, in particular, that, if we have a method for the solution of the *real* system $\tilde{\mathsf{A}}\mathsf{x} = \tilde{\mathsf{b}}$ with respect to x, then rewriting it in interval arithmetic results in a technique for computing interval estimates $\tilde{\mathsf{x}}$ for x. In other words, we get an algorithm for outer interval estimation of the solution set $\Xi_{\text{uni}}(\boldsymbol{A}, \boldsymbol{b})$.

Let us construct a matrix $\hat{\mathsf{A}} = (\hat{\mathsf{a}}_{ij})$ and a vector $\hat{\mathsf{b}} = (\hat{\mathsf{b}}_i)$ consisting of affine quantities, such that $\operatorname{ran}\langle\hat{\mathsf{A}}, \hat{\mathsf{b}}\rangle = \operatorname{ran}\langle\tilde{\mathsf{A}}, \tilde{\mathsf{b}}\rangle$. All the elements of $\tilde{\mathsf{A}}$ and $\tilde{\mathsf{b}}$ are independent from each other, so the elements of $\hat{\mathsf{A}}$ and $\hat{\mathsf{b}}$ are independent from each other too.

We suppose that $\operatorname{ran}\tilde{a}_{ij} = \operatorname{ran}\tilde{a}_{ji}$ for any $1 \le i,j \le n$, i.e. the interval matrix $\boldsymbol{A}$ is symmetric. Then the equalities $\operatorname{ran}\hat{a}_{ij} = \operatorname{ran}\hat{a}_{ji}$ for any i,j are also true for the affine $\hat{A}$. Next, let us assign $\hat{a}_{ji} := \hat{a}_{ij}$ for all $1 \le i < j \le n$. After doing that, we have $\hat{a}_{ji} = \hat{a}_{ij}$ for any $1 \le i, j \le n$, which is an inherent property of affine quantities. For interval quantities, the equality $\tilde{a}_{ji} = \tilde{a}_{ij}$ for different i,j is possible only if $\operatorname{wid}(\tilde{a}_{ij}) = 0$ and $\operatorname{ran}\tilde{a}_{ji} = \operatorname{ran}\tilde{a}_{ij}$, i.e. $\tilde{a}_{ij}$ and $\tilde{a}_{ji}$ are equal constants. After the assignment procedure, we get $\operatorname{ran}\hat{A} \subseteq \operatorname{ran}\tilde{A}$, and if $\operatorname{wid}(\operatorname{ran}\tilde{a}_{ij}) > 0$ for some indexes $i \ne j$, the inclusion is strict.

Let a quantity x' be such that $\hat{A}x' = \hat{b}$. It is easy to show that

$$\operatorname{ran} x' = \Xi_{\mathrm{sym}}(\boldsymbol{A}, \boldsymbol{b}).$$

The system $\hat{A}x' = \hat{b}$ can be solved by the same method as the system $\tilde{A}x = \tilde{b}$. At this point, we have to use either affine arithmetic or more exact interval-affine arithmetic. As a result, we will get an estimate for x', which is trivially converted to an interval estimate for the set $\Xi_{\mathrm{sym}}(\boldsymbol{A}, \boldsymbol{b})$. In solving the system $\hat{A}x' = \hat{b}$, we start from the set $\operatorname{ran}\langle\hat{A}, \hat{b}\rangle$ which is more narrow than the set $\operatorname{ran}\langle\tilde{A}, \tilde{b}\rangle$ in the solution of the system $\tilde{A}x = \tilde{b}$. Therefore, one can expect that an estimate obtained for $\operatorname{ran} x'$ will be more narrow than the estimate for $\operatorname{ran} x$.

We have considered the case of symmetric systems. Using a similar approach, it is possible to take into account, partially or fully, any explicit ties. For example, given the ties on the elements of the matrix $A = (a_{ij}) \in \boldsymbol{A}$ that have the form

$$a_{kl} = f(a_{i_1 j_1}, a_{i_2 j_2}, \ldots, a_{i_s j_s}) \tag{6.2}$$

for some indexes k, l, $\{i_m\}$, and $\{j_m\}$, we can perform the corresponding assignment

$$\hat{a}_{kl} := \hat{f}(\hat{a}_{i_1 j_1}, \hat{a}_{i_2 j_2}, \ldots, \hat{a}_{i_s j_s}), \tag{6.3}$$

where $\hat{f}$ denotes a function obtained from the function f by replacing all the operations to their affine counterparts. For nonlinear f, the quantities $\hat{a}_{kl}$ and $f(\hat{a}_{i_1 j_1}, \hat{a}_{i_2 j_2}, \ldots, \hat{a}_{i_s j_s})$ are not equal, but they are dependent on each other. That gives a possibility to partially take the tie (6.2) into account when constructing an enclosure of the solution set.

As a base method for the solution of linear real systems, we used Gaussian elimination algorithm, having reimplemented it in our interval-affine arithmetic with the calculation of ties. The overall method should be thus called the *interval-affine Gaussian algorithm for constrained systems*. The final pseudocode of the method is presented in Table 1, where mig means mignitude of the quantity.

When we solve, by our interval-affine method, an interval linear system with the only constraint $A = A^{\top}$, it is natural to call the method *symmetric interval-affine Gaussian algorithm*.

Table 1. Interval-affine Gaussian algorithm for constrained systems.

Input

An interval matrix $\boldsymbol{A} = (\boldsymbol{a}_{ij}) \in \mathbb{IR}^{n\times n}$.
An interval vector $\boldsymbol{b} = (\boldsymbol{b}_i) \in \mathbb{IR}^n$.
A set *CS* of constraints, similar to (6.2), written in some form.

Output

Either an interval vector $\boldsymbol{x}$ that is an outer interval estimate for the solution set $\Xi_{\text{tie}}(\boldsymbol{A}, \boldsymbol{b})$ or the message "Method fails."

Algorithm

Let $A = (a_{ij})$ and $b = (b_i)$ be an $n \times n$-matrix and an n-vector composed of interval-affine quantities;
convert $\boldsymbol{A} \to A$ and $\boldsymbol{b} \to b$;
compute the elements of A according to the constraints in *CS* in interval-affine arithmetic in compliance with (6.3);

```
for k := 1 to n do
{   // reducing A to the "upper triangular" form
    m := k;
    for i := k + 1 to n do // find the "leading" element
        if mig(a_ik) > mig(a_mk) then m := i ;
    if mig(a_mk) = 0 then
        stop with the message "Method fails";
    interchange the k-th and m-th rows of the matrix A and the elements b_k and b_m
        of the vector b;
    for j := k + 1 to n do
        a_kj := a_kj ⃞/ a_kk;
    b_k := b_k ⃞/ a_kk;
    for i := k + 1 to n do
    {   // "annuling" the column below the diagonal and modifying the right-hand side
        for j := k to n do
            a_ij := a_ij ⃞− a_ik ⃞* a_kj;
        b_i := b_i ⃞− a_ik ⃞* b_k;
    }
}
let x be a n-vector of interval-affine quantities;
for i := n downto 1 do
{   // backward substitution
    x_i := b_i;
    for j := i + 1 to n do
        x_i := x_i ⃞− a_ij ⃞* x_j;
}
Convert x → x;
```

7. Computational Experiments

Below we present the results of computational experiments with interval linear systems having symmetric matrices, as well as matrices called "semiskewsymmetric." A *semiskewsymmetric* matrix is a matrix $A = (a_{ij})$ such that $a_{ij} = -a_{ij}$ for $i \neq j$.

The main diagonal of the semiskewsymmetric matrix, i.e. the elements a_{ii}, is not necessarily zero.

In all the examples, we solved interval linear systems, subject to various types of constraints, by the interval-affine Gaussian algorithm.

EXAMPLE 7.1. Consider the interval 3×3-system from [4]

$$\begin{pmatrix} [0.7,1.3] & [-0.3,0.3] & [-0.3,0.3] \\ [-0.3,0.3] & [0.7,1.3] & [-0.3,0.3] \\ [-0.3,0.3] & [-0.3,0.3] & [0.7,1.3] \end{pmatrix} x = \begin{pmatrix} [-14,-7] \\ [9,12] \\ [-3,3] \end{pmatrix}.$$

Gaussian algorithm produces the following interval enclosures of Ξ_{uni}:

no constraints	$a_{ij} = a_{ji}$	$a_{ij} = -a_{ji}$ $(i \neq j)$
$\begin{pmatrix} [-101,71] \\ [-62.25,99] \\ [-90,90] \end{pmatrix}$	$\begin{pmatrix} [-101,\underline{64.8}] \\ [-\underline{56.06},99] \\ [-90,90] \end{pmatrix}$	$\begin{pmatrix} [-\underline{46.58},\underline{21.44}] \\ [-\underline{14.98},\underline{42.03}] \\ [-\underline{31.33},\underline{31.33}] \end{pmatrix}$

In the above table, the underlined data show the changes of the estimates that result from taking the constraints into account.

EXAMPLE 7.2. Let us consider a system from [7]

$$\begin{pmatrix} [15,17] & [-3,3.01] & [-3,3.01] & [-3,3.01] \\ [-3,3.01] & [15,17] & [-3,2.99] & [-3,2.99] \\ [-3,2.99] & [-3,2.99] & [15,17] & [-3,3.01] \\ [-3,3.01] & [-3,3.01] & [-3,2.99] & [15,17] \end{pmatrix} x = \begin{pmatrix} [-6,-2] \\ [4,5] \\ [-2,4] \\ [8,10] \end{pmatrix}, \quad (7.1)$$

whose matrix is neither symmetric nor semiskewsymmetric in the original form. Still, we can modify it and consider the two cases: $\boldsymbol{a}_{ji} = \boldsymbol{a}_{ij}$ $(i < j)$ and $\boldsymbol{a}_{ji} = -\boldsymbol{a}_{ij}$ $(i < j)$.

In the first case, the interval-affine method yields

no constraints	$a_{ij} = a_{ji}$
$\begin{pmatrix} [-1.0313,0.4958] \\ [-0.3472,0.9745] \\ [-0.7703,0.9190] \\ [0.1495,1.2524] \end{pmatrix}$	$\begin{pmatrix} [-1.031\underline{2},0.4\underline{363}] \\ [-0.\underline{2897},0.9745] \\ [-0.7\underline{610},0.91\underline{89}] \\ [0.1\underline{735},1.252\underline{3}] \end{pmatrix}$

And in the second case we get

no constraints	$a_{ij} = -a_{ji}$ $(i \neq j)$
$\begin{pmatrix} [-1.0303, 0.4948] \\ [-0.3470, 0.9730] \\ [-0.7708, 0.9170] \\ [0.1495, 1.2510] \end{pmatrix}$	$\begin{pmatrix} [-\underline{0.8536}, 0.\underline{3693}] \\ [-0.\underline{2279}, 0.\underline{7831}] \\ [-0.\underline{6104}, 0.\underline{7370}] \\ [0.1\underline{672}, \underline{0.9932}] \end{pmatrix}$

EXAMPLE 7.3 (The random test with symmetric matrices). For a given order n and a positive integer N, we generated N random interval systems $\boldsymbol{A}x = \boldsymbol{b}$ with symmetric $\boldsymbol{A}$. Each system was consecutively solved by interval, by interval-affine and by symmetric interval-affine Gaussian algorithms.

The random system generation algorithm was taken as follows:

Input

n—order of the system;

$\boldsymbol{c} = [\underline{\boldsymbol{c}}, \overline{\boldsymbol{c}}] \in \mathbb{IR}$, $r_{\max} \in \mathbb{R}$—parameters of the system family (see below).

Output

A random matrix $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and a random vector $\boldsymbol{b} \in \mathbb{IR}^n$.

Algorithm

For all i, j, such that $1 \leq i \leq j \leq n$, compute

$$\boldsymbol{a}_{ji} := \boldsymbol{a}_{ij} := \mathrm{rand}(\underline{\boldsymbol{c}}, \overline{\boldsymbol{c}}) + [\mathrm{rand}(-r_{\max}, 0), \mathrm{rand}(0, r_{\max})].$$

For all $1 \leq i \leq n$, compute

$$\boldsymbol{b}_i := \mathrm{rand}(\underline{\boldsymbol{c}}, \overline{\boldsymbol{c}}) + [\mathrm{rand}(-r_{\max}, 0), \mathrm{rand}(0, r_{\max})].$$

In the above, $\mathrm{rand}(\alpha, \beta)$ is a function for computing an equidistributed pseudorandom number from the interval $[\alpha, \beta]$.

The results of our experiments for several n's are presented in Table 2. The parameter $\boldsymbol{c}$ for all n was taken equal to $[-10^4, 10^4]$. In the table, the meanings of the columns is as follows:

N is the number of generated systems of the order n;

$N_{\mathrm{int}}, N_{\mathrm{iaff}}, N_{\mathrm{siaff}}$ are numbers of systems that have been successfully solved by interval, by interval-affine and by symmetric interval-affine Gaussian algorithms respectively;

K_{ave} is arithmetic average of the ratios of the interval estimates' diameters produced by the interval-affine method to the diameters of the estimates produced by the symmetric method. K_{ave} was computed for the problems successfully solved by both methods.

Table 2 shows that taking constraints into account allows us to increase the number of successfully solved systems and to sharpen the accuracy of the estimates.

Table 2. Random test.

n	$r_{\max}$	N	N_{int}	N_{iaff}	N_{siaff}	K_{ave}
5	500	1000	422	652	668	2.21
10	100	200	0	151	156	9.62
15	35	200	0	157	162	4.79
20	23	200	0	124	144	4.30
25	15	50	0	22	26	2.42

The rating K_{ave} depends on many quantities including n and $r_{\max}$. Therefore, for more qualitative analysis of how the accuracy of the estimates changes one has to perform more experiments, with various algorithms for random system generation.

One can also see that the classical interval Gaussian algorithm was not able to solve any one of our problems of the dimension greater or equal than $n = 10$.

In conclusion, we notice that the price of more qualitative results that the interval-affine Gaussian algorithm computes is its higher complexity, which has the order of $O(n^5)$. There are some ways to speed up the method at the expense of worsening the estimates. We postpone their careful consideration till our next papers.

References

1. Alefeld, G., Kreinovich, V., and Mayer, G.: On Symmetric Solution Sets, in: Herzberger, J. (ed.), *Computing Supplement* **16**, Springer, Wien, New York, 2003, pp. 1–23.
2. Alefeld, G., Kreinovich, V., and Mayer, G.: On the Shape of the Symmetric, Persymmetric, and Skew-Symmetric Solution Set, *SIAM J. Matrix Anal. Appl.* **18** (1997), pp. 693–705.
3. Comba, J. L. D. and Stolfi, J.: Affine Arithmetic and Its Applications to Computer Graphics, in: *Proceedings of VI SIBGRAPI*, 1993, pp. 9–18.
4. Hansen, E.: Bounding the Solution of Interval Linear Equations, *SIAM Journal on Numerical Analysis* **29** (5) (1992), pp. 1493–1503.
5. Kearfott, R. B., Nakao, M. T., Neumaier, A., Rump, S. M., Shary, S. P., and Van Hentensyck, P.: *Standardized Notation in Interval Analysis*, `http://www.mat.univie.ac.at/~neum/software/int`
6. Neumaier, A.: *Interval Methods for Systems of Equations*, Cambridge University Press, Cambridge, 1990.
7. Ning, S. and Kearfott, R. B.: A Comparison of Some Methods for Solving Linear Interval Equations, *SIAM Journal on Numerical Analysis* **34** (4) (1997), pp. 1289–1305.
8. Shary, S. P.: Solving Tied Interval Linear Systems, *Siberian Journal of Computational Mathematics* **7** (4) (2004), pp. 363–376 (in Russian).
9. Stolfi, J. and de Figueiredo, L. H.: Self-Validated Numerical Methods and Applications, in: *Notes of 21st Brazilian Mathematics Colloquium*, 1997.