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Inverse Problem of the Interval Linear System of Equations

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Abstract

Given a nonsingular central matrix A_c, a central vector b_c and a prescribed interval solution vector x^I, it is required to find the maximum allowable deviation ΔA or Δb so that the solution of the interval linear system $A^I x = b^I$ is contained in x^I. Special cases for ΔA and Δb are considered and bounds on the entries of A^I and b^I are obtained either in a closed form, whenever possible, or via solving a specially designed constrained optimization problem.

Key Words: Linear systems of equations, interval matrices, inequality constraints, constrained optimization.

1. Introduction

The interval linear system of equations has been widely investigated since the pioneer work of Oettli and Prager [7] in 1965. Consider a set of linear equations

$$A^I x = b^I \tag{1}$$

in which A^I is an interval matrix and b^I is an interval vector. Such equations have been of interest for years in both interval and noninterval contexts. As the noninterval case, the interval linear systems of equations are currently of special interest because they arise in many applications. Algorithms for determining the solution set of (1) are found in many references, e.g. [3]–[8].

However, in this paper we are interested in solving the inverse problem of the interval linear system of equations. That is, given the central matrix A_c, the central vector b_c and the interval solution vector x^I, it is required to find the maximum allowable deviations ΔA and Δb from the nominal central values A_c and b_c, respectively, such that the solution of the interval system is contained in the prescribed solution vector x^I.

Starting from the well-known Oettli-Prager Inequality [7], given by

$$|A_c x - b_c| \leq \Delta A|x| + \Delta b \tag{2}$$

each vertex of the convex hull of the solution set of (1), $X = \{x : Ax = b,\ A \in A^I,\ b \in b^I\}$ denoted by ConvX satisfies

$$|A_c x - b_c| = \Delta A|x| + \Delta b \tag{3}$$

which is equivalent to 2^n equations of the form

$$A_c x - b_c = T^\kappa(\Delta A T' x + \Delta b) \tag{4}$$

for $\kappa = 1, 2, \ldots, 2^n$, where T^κ is a diagonal matrix with diagonal elements belonging to $\{-1, 1\}$ and T' is a diagonal matrix with diagonal elements equal to $\mathrm{sgn}(x)$; i.e. $|x| = T'x$.

With the assumption that A_c^{-1} exists, and no change of sign for the solution vector components occurs, multiplying both sides of (4) by A_c^{-1} yields

$$x - x_c = A_c^{-1} T^\kappa(\Delta A T' x + \Delta b) \tag{5}$$

which can be written in the form

$$(I - A_c^{-1} T^\kappa \Delta A T')(x - x_c) = A_c^{-1} T^\kappa(\Delta A T' x + \Delta b) \tag{6}$$

and by assuming that $(I - A_c^{-1} T^\kappa \Delta A T')^{-1}$ exists, yields

$$x - x_c = (I - A_c^{-1} T^\kappa \Delta A T')^{-1} A_c^{-1} T^\kappa(\Delta A T' x + \Delta b) \tag{7}$$

With the requirement that $\underline{x} \leq x \leq \bar{x}$, i.e. $\underline{x} - x_c \leq x - x_c \leq \bar{x} - x_c$, it follows that

$$-\underline{d} \leq (I - A_c^{-1} T^\kappa \Delta A T')^{-1} A_c^{-1} T^\kappa(\Delta A T' x + \Delta b) \leq \bar{d} \tag{8}$$

where $\underline{d} = x_c - \underline{x}$ and $\bar{d} = \bar{x} - x_c$.

The goal is to define the space of maximum allowable deviations ΔA_{ij} and Δb_i, $(i, j = 1, 2, \ldots, n)$ satisfying inequalities (8). The dimension of the space of parameters is, in general, $(n \times n + n)$. It is worth noting that our problem can simply be considered as another approach suggested to tackle the well-known linear tolerance problem [1], [9] and [10]. We believe that our approach can be successfully applied to some practical problems, e.g. in the tolerance design or tuning of a linear circuit with nodal equation $Yv = i$. The tolerance problem is to find the interval entries of the admittance matrix Y so that the output voltage vector v is within the performance acceptability range for some input current vector i. For an illustrative realistic design problem we consider in particular the tunable active filter presented in [1] in which the grounded resistor R_4, shown in [1, Fig.

4], is a single tunable element. The tuning range of R_4 required for the voltage V_2 to stay within the performance acceptability range can be achieved by applying our method for the one nonzero element discussed in Subsection 3.1 of this present work.

In fact, in this paper we consider special cases of the general problem (8). The cases to be considered are divided mainly into two types. Each type is in turn divided into subcases to simplify the problem. In Section 2 the first type is considered where only uncertainty in the right hand side vector, b, is assumed. The second type which considers only uncertainty in the matrix of coefficients, A, is discussed in Section 3. In both types the problem is reduced to solving a finite set of linear inequalities. For the single parameter cases closed form relations are obtained whereas for each of the more than one parameter cases an optimization problem with a suitably selected objective function is defined. The resulting optimization problem is solved in Section 4. Numerical examples are given in Section 5.

2. Uncertainty in the Right Hand Side Vector

Considering $b^I = [b_c - \Delta b, b_c + \Delta b]$, it is required to find Δb so that the solution of the interval system of equations (1) is contained in the given interval solution vector x^I.

Put $\Delta A = 0$ in (8) to obtain the following set of inequalities

$$-\underline{d} \leq A_c^{-1} T^\kappa \Delta b \leq \bar{d} \tag{9}$$

for $\kappa = 1, 2, \ldots, 2^n$, this gives $(2n \times 2^n)$ scalar inequalities. Let $d_i = \min\{\underline{d}_i, \bar{d}_i\}$ for $i = 1, 2, \ldots, n$ and the vector $d = (d_i)$. It can be easily observed that the set of inequalities given by (9) is the set of inequalities of the form

$$|A_c^{-1}| \Delta b \leq d \tag{10}$$

where $|\cdot|$ means the absolute value of the array taken componentwise. To show that (9) lead to (10), we present the following derivation. Inequalities (9) are rewritten in the component form

$$-\underline{d}_i \leq \sum_{j=1}^{n} (A_c^{-1})_{ij} T_{ij}^\kappa \Delta b_j \leq \bar{d}_i \tag{11}$$

for $i = 1, 2, \ldots, n$ and $\kappa = 1, 2, \ldots, 2^n$. For each $i \in \{1, 2, \ldots, n\}$, say $i = i_0$, choose κ_1, where $\kappa_1 \in \{1, 2, \ldots, 2^n\}$, so that $(A_c^{-1})_{i_0 j} T_{jj}^{\kappa_1} \geq 0$ for all $j = 1, 2, \ldots n$. Since we are interested only in nonnegative values of Δb_j, $j = 1, 2, \ldots, n$, and the fact that $\underline{d}_i \geq 0$ and $\bar{d}_i \geq 0$, for all $i = 1, 2, \ldots, n$, inequalities (11) for $i = i_0$ and $\kappa = \kappa_1$ can be written as follows

$$-\underline{d}_{i_0} \leq \sum_{j=1}^{n} \left|(A_c^{-1})_{i_0 j}\right| \Delta b_j \leq \bar{d}_{i_0} \tag{12}$$

Thus the left inequality of (12) becomes redundant. Next, choosing κ_2, where $\kappa_2 \in \{1, 2, \ldots, 2^n\}$, such that $(A_c^{-1})_{i_0 j} T_{jj}^{\kappa_2} \leq 0$ for all $j = 1, 2, \ldots, n$, inequalities (11) for $i = i_0$ and $\kappa = \kappa_2$ can be written as

$$-\underline{d}_{i_0} \leq \sum_{j=1}^{n} \left|(A_c^{-1})_{i_0 j}\right| \Delta b_j \leq \bar{d}_{i_0} \tag{13}$$

where the left inequality of (13) is redundant. From (12) and (13) we deduce that for any i with suitable choices of κ, where $\kappa \in \{1, 2, \ldots, 2^n\}$, inequalities (11) produce the single vector inequality given by

$$\sum_{j=1}^{n} \left|(A_c^{-1})_{i_0 j}\right| \Delta b_j \geq d_i \tag{14}$$

where $d_i = \min\{\underline{d}_i, \bar{d}_i\}$.

Thus inequalities (14) define the parameter space of the uncertainty vector Δb. Thus our problem can be reformulated as an optimization problem with linear constraints and an objective function chosen, linear or nonlinear, according to practical considerations. Alternatively a certain region approximation can be used to approximate the region defined by the above inequalities.

In the special case of constant relative variations in $\Delta b = \varepsilon |b_c|$, for some positive scalar ε, inequalities (14) is written in the form,

$$\sum_{j=1}^{n} \left|(A_c^{-1})_{ij}\right| \varepsilon |b_{c_j}| \leq d_i \tag{15}$$

The solution, which now depends on evaluating ε, can be easily obtained since

$$\varepsilon = \min_{i=1,2,\ldots n} \left\{ \frac{d_i}{\sum_{j=1}^{n} \left|(A_c^{-1})_{ij}\right| |b_{c_j}|} \right\} \tag{16}$$

provided that $\sum_{j=1}^{n} \left|(A_c^{-1})_{ij}\right| |b_{c_j}| \neq 0$ for at least one i.

3. Uncertainty in the Matrix of Coefficients

For the interval matrix A^I, where $A^I = [A_c - \Delta A, A_c + \Delta A]$, it is required to find ΔA such that the interval solution vector of the interval system of equations (1) is contained in the given interval solution vector x^I. Putting $\Delta b = 0$ in (4), we obtain

$$A_c x - b_c = T^\kappa \Delta A T' x$$

or,

$$(A_c - T^\kappa \Delta A T')x = b_c$$

With the assumption that $(A_c - T^\kappa \Delta A T')^{-1}$ exists, then

$$x = (A_c - T^\kappa \Delta A T')^{-1} b_c$$

Moreover, with the requirement that $\underline{x} \leq x \leq \bar{x}$, it follows that ΔA must satisfy the set of inequalities

$$\underline{x} \leq (A_c - T^\kappa \Delta A T')^{-1} b_c \leq \bar{x} \tag{17}$$

for $\kappa = 1, 2, \ldots, 2^n$. Solving (17) for any general ΔA is quite a difficult problem. To simplify the problem and treat a reasonable problem; with respect to practical considerations; we impose the assumption that A undergoes a variation that can be represented by a rank-one matrix ΔA.

To obtain the inverse of the matrix in (17), apply the Sherman–Morrison formula for the inverse of a matrix with a rank-one variation matrix. In fact, a matrix of the form $(B - \sigma u v^T)^{-1}$, with $\sigma \in R$ and $u, v \in R^n$, has the inverse

$$(B - \sigma u v^T)^{-1} = B^{-1} + (\sigma^{-1} - v^T B^{-1} u)^{-1} B^{-1} u v^T B^{-1}$$

provided that $(\sigma^{-1} - v^T B^{-1} u) \neq 0$. Now, let $T^\kappa \Delta A T' = \sigma u^\kappa v^{\kappa^T}$ for some $\sigma \in R$ and $u^\kappa, v^\kappa \in R^n$. Thus,

$$(A_c - T^\kappa \Delta A T')^{-1} = A_c^{-1} + (\sigma^{-1} - v^{k^T} A_c^{-1} u^\kappa)^{-1} A_c^{-1} u^\kappa v^{k^T} A_c^{-1} \tag{18}$$

Using (18), inequalities (17) are written as

$$-\underline{d} \leq (\sigma^{-1} - v^{\kappa^T} A_c^{-1} u^\kappa)^{-1} A_c^{-1} u^\kappa v^{\kappa^T} x_c \leq \bar{d} \tag{19}$$

where $\underline{d} = x_c - \underline{x}$ and $\bar{d} = \bar{x} - x_c$. But

$$(\sigma^{-1} - v^{\kappa^T} A_c^{-1} u^\kappa)^{-1} A_c^{-1} u^\kappa v^{\kappa^T} x_c = \left(\frac{v^{\kappa^T} x_c}{(\sigma^{-1} - v^{\kappa^T} A_c^{-1} u^\kappa)} \right) A_c^{-1} u^\kappa$$

Noting that $\left(\frac{v^{\kappa^T} x_c}{(\sigma^{-1} - v^{\kappa^T} A_c^{-1} u^\kappa)} \right)$ is a scalar, let $A_c^{-1} u^\kappa = \alpha^\kappa$. Substituting in (19) , we get

$$-\underline{d}_i \leq \frac{\sum_{j=1}^{n} v_j^\kappa x_{c_j}}{\left(\sigma^{-1} - \sum_{j=1}^{n} v_j^\kappa \alpha_j^\kappa \right)} \alpha_i^\kappa \leq \bar{d}_i \tag{20}$$

Without loss of generality, assume that $\left(\sigma^{-1} - \sum_{j=1}^{n} v_j^\kappa \alpha_j^\kappa \right) > 0$

Thus, multiplying both sides of (20) by $\left(\sigma^{-1} - \sum_{j=1}^{n} v_j^\kappa \alpha_j^\kappa \right)$ would not alter the sense of the inequalities yielding

$$-\left(\sigma^{-1} - \sum_{j=1}^{n} v_j^\kappa \alpha_j^\kappa \right) \underline{d}_i \leq \sum_{j=1}^{n} v_j^\kappa x_{c_j} \alpha_i^\kappa \leq \left(\sigma^{-1} - \sum_{j=1}^{n} v_j^\kappa \alpha_j^\kappa \right) \bar{d}_i$$

or,

$$\begin{aligned} \sum_{j=1}^{n} v_j^\kappa (\alpha_j^\kappa \bar{d}_i + x_{c_j} \alpha_i^\kappa) &\leq \sigma^{-1} \bar{d}_i \\ \sum_{j=1}^{n} v_j^\kappa (\alpha_j^\kappa \underline{d}_i - x_{c_j} \alpha_i^\kappa) &\leq \sigma^{-1} \underline{d}_i \end{aligned} \tag{21}$$

for $i = 1, 2, \ldots, n$ and $\kappa = 1, 2, \ldots, 2^n$. Thus the set of inequalities (21) consists of $(2n \times 2^n)$ scalar inequalities. Using inequalities (21), we proceed to solve for ΔA in the following cases:

(i) One nonzero element.

ΔA has one nonzero element in position, say (l, m). Denote this element by δ. In this case let $\sigma = \delta$, $u^\kappa = T^\kappa e_l$, and $v^\kappa = T' e_m$ where e_i denotes a unit vector with only the i^{th} component equals 1 and all other components equal 0.

(ii) One nonzero row.

ΔA has one nonzero row, say, the l^{th} row which either

(a) has equal valued components. Let $\sigma = \delta$, where δ is a scalar, $u^{\kappa} = T^{\kappa} e_l$, and $v^{\kappa} = T' e$, where $e \in R^n$ denotes the vector of the form $(1, 1, \ldots, 1)^T$, i.e. all its elements equal one, or

(b) it is a general vector in R^n. In this case $\sigma = 1$, $u^{\kappa} = T^{\kappa} e_l$, and $v^{\kappa} = T'\delta$, where δ is a vector in R^n.

(iii) One nonzero column.

ΔA has one nonzero column, say, the m^{th} column, which either

(a) has equal valued components. Let $\sigma = d$, where δ is a scalar, $u^{\kappa} = T^{\kappa} e$ and $v^{\kappa} = T' e_m$, or,

(b) it is a general vector in R^n. In this case $\sigma = 1$, $u^{\kappa} = T^{\kappa}\delta$ and $v^{\kappa} = T' e_m$, where σ is a vector in R^n.

3.1. One Parameter Cases

First we solve the single parameter cases for which $\sigma = \delta$, where the scalar $\delta > 0$. In inequalities (21), let $\sum_{j=1}^{n} v_j^{\kappa} (\alpha_j^{\kappa} \bar{d}_i + x_{c_j} \alpha_i^{\kappa})$ be denoted by B_i^{κ} be denoted by B_i^{κ} and $\sum_{j=1}^{n} v_j^{\kappa} (\alpha_j \underline{d}_i - x_{c_j} \alpha_i^{\kappa})$ be denoted by C_i^{κ}, thus

$$B_i^{\kappa} \leq \frac{1}{\delta}\bar{d}_i \qquad \text{and} \qquad C_i^{\kappa} \leq \frac{1}{\delta}\underline{d}_i \tag{22a,b}$$

for $i = 1, 2, \ldots, n$. If for any $i \in \{1, 2, \ldots, n\}$ there exists a $\kappa \in \{1, 2, \ldots, 2^n\}$ such that $B_i^{\kappa} > 0$ then (22a) is written as

$$\delta \leq \frac{\bar{d}_i}{B_i^{\kappa}} \tag{23a}$$

Also, for $C_i^{\kappa} > 0$, from (22b)

$$\delta \leq \frac{\underline{d}_i}{C_i^{\kappa}} \tag{23b}$$

In fact, B_i^{κ} and/or $C_i^{\kappa} < 0$ for any $i \in \{1, 2, \ldots, n\}$, will give redundant constraints since this provides negative lower bounds on δ and we are concerned with the nonnegative values of δ only.

Substituting for the different values of u^{κ} and v^{κ} yield:

(i) One nonzero element

For one nonzero element in the (l, m) position, with $\sigma = \delta$, $u^{\kappa} = T^{\kappa} e_l$ and

$v^\kappa = T' e_m$ yields

$$\delta = \min_{i=1,\dots,n} \left\{ \frac{\underline{d}_i}{|(A_c^{-1})_{ml}\underline{d}_i - x_{c_m}(A_c^{-1})_{il}|}, \frac{\bar{d}_i}{|(A_c^{-1})_{ml}\bar{d}_i + x_{c_m}(A_c^{-1})_{il}|} \right\} \quad (24)$$

(ii-a) One nonzero row with equal components

Equal l^{th} row deviation, $\sigma = \delta$, where δ is a scalar, $u^\kappa = T^\kappa e_l$ and $v^\kappa = T' e$ yields

$$\delta = \quad (25)$$

$$\min_{i=1,\dots,n} \left\{ \frac{\underline{d}_i}{\left| \sum_{j=1}^{n} T'_{jj} \left((A_c^{-1})_{jl}\underline{d}_i - (A_c^{-1})_{il} x_{c_j} \right) \right|}, \frac{\bar{d}_i}{\left| \sum_{j=1}^{n} T'_{jj} \left((A_c^{-1})_{jl}\bar{d}_i + (A_c^{-1})_{il} x_{c_j} \right) \right|} \right\}$$

(iii-a) One nonzero column with equal components

Equal m^{th} column deviation, $\sigma = \delta$, where δ is a scalar, $u^\kappa = T^\kappa e$ and $v^\kappa = T' e_m$, yields

$$\delta = \min_{i=1,\dots,n} \left\{ \frac{\underline{d}_i}{\sum_{j=1}^{n} |(A_c^{-1})_{mj}\underline{d}_i - (A_c^{-1})_{ij} x_{c_m}|}, \frac{\bar{d}_i}{\sum_{j=1}^{n} |(A_c^{-1})_{mj}\bar{d}_i + (A_c^{-1})_{ij} x_{c_m}|} \right\} \quad (26)$$

3.2 n-Parameter Cases

For solving the n-parameter cases, let the scalar $\sigma = 1$ in (21), and hence

$$\begin{aligned} \sum_{j=1}^{n} v_j^\kappa (\alpha_j^\kappa \bar{d}_i + x_j^c \alpha_i^\kappa) &\leq \bar{d}_i \\ \sum_{j=1}^{n} v_j^\kappa (\alpha_j^\kappa \underline{d}_i - x_j^c \alpha_i^\kappa) &\leq \underline{d}_i \end{aligned} \quad (27)$$

Substituting for the different values of u^κ and v^κ yields *(ii-b) One nonzero row with equal components*

For unequal l^{th} row deviations, let $u^{\kappa} = T^{\kappa} e_l$, and $v_k = T'\delta$, where δ is a vector in R^n with nonnegative entries. Since

$$\alpha_i^{\kappa} = \sum_{j=1}^{n} (A_c^{-1})_{ij} u_j^{\kappa} = (A_c^{-1})_{il} T_{ll}^{\kappa}$$

substituting in (27), we obtain

$$\begin{aligned} T_{ll}^{\kappa} \sum_{j=1}^{n} T'_{jj}\delta_j \left(x_{c_j} (A_c^{-1})_{il} + (A_c^{-1})_{jl} \bar{d}_i \right) \leq \bar{d}_i \\ T_{ll}^{\kappa} \sum_{j=1}^{n} T'_{jj}\delta_j \left(-x_{c_j} (A_c^{-1})_{il} + (A_c^{-1})_{jl} \underline{d}_i \right) \leq \underline{d}_i \end{aligned} \tag{28}$$

where $i = 1, 2, \ldots n$ and $\kappa = 1, 2, \ldots, 2^n$, i.e. $(2n \times 2^n)$ scalar inequalities define the parameter space. But as $T_{ll}^{\kappa} \in \{-1, 1\}$, i.e. it acquires only two different values for all κ, this results in reducing the number to $(2n \times 2)$ scalar inequalities. In fact, the number of effective inequalities turns out to be only $2n$ due to redundancy. Define

$$\begin{aligned} B_{ij} &= (A_c^{-1})_{jl} \bar{d}_i + (A_c^{-1})_{il} x_{cj} \\ C_{ij} &= (A_c^{-1})_{jl} \underline{d}_i - (A_c^{-1})_{il} x_{cj} \end{aligned} \tag{29}$$

Thus using (29), (28) is rewritten in matrix form as

$$\begin{aligned} BT'\delta \leq \bar{d}, \quad -BT'\delta \leq \bar{d} \\ CT'\delta \leq \underline{d}, \quad -CT'\delta \leq \underline{d} \end{aligned} \tag{30}$$

$\delta \geq 0$ .These can be written in the compact form

$$\mathrm{P}\delta \leq \mathrm{r}$$

for some suitably chosen matrix of coefficients P and right hand side vector r. Thus the problem of finding the maximum allowable deviation of the l^{th} row components of the matrix of coefficients is formulated as an optimization problem with a feasible region defined by (30).

(iii-b) One nonzero column with unequal components

For the nonzero m^{th} column deviations with n parameters, let $u^{\kappa} = T^{\kappa}\delta$, $v^{\kappa} = T'e_m$ and

$$\alpha_i^{\kappa} = \sum_{j=1}^{n} (A_c^{-1})_{ij} u_j^{\kappa} = \sum_{j=1}^{n} (A_c^{-1})_{ij} T_{jj}^{\kappa} \delta_j$$

Substituting in (27) yield

$$T'_{mm}\left(\sum_{j=1}^{n} A_c^{-1}{}_{mj} T_{jj}^{\kappa}\delta_j \bar{d}_i + x_{c_m}\sum_{j=1}^{n} A_c^{-1}{}_{ij} T_{jj}^{\kappa}\delta_j\right) \leq \bar{d}_i$$
$$T'_{mm}\left(\sum_{j=1}^{n} A_c^{-1}{}_{mj} T_{jj}^{\kappa}\delta_j \underline{d}_i - x_{c_m}\sum_{j=1}^{n} A_c^{-1}{}_{ij} T_{jj}^{\kappa}\delta_j\right) \leq \underline{d}_i$$

or,

$$\begin{aligned} T'_{mm}\sum_{j=1}^{n} T_{jj}^{\kappa}\left(A_c^{-1}{}_{mj}\bar{d}_i + A_c^{-1}{}_{ij}x_{c_m}\right)\delta_j &\leq \bar{d}_i \\ T'_{mm}\sum_{j=1}^{n} T_{jj}^{\kappa}\left(A_c^{-1}{}_{mj}\underline{d}_i - A_c^{-1}{}_{ij}x_{c_m}\right)\delta_j &\leq \underline{d}_i \end{aligned} \tag{31}$$

Define

$$\begin{aligned} B'_{ij} &= A_c^{-1}{}_{mj}\bar{d}_i + A_c^{-1}{}_{ij}x_{c_m} \\ C'_{ij} &= A_c^{-1}{}_{mj}\underline{d}_i - A_c^{-1}{}_{ij}x_{c_m} \end{aligned} \tag{32}$$

Using (32), inequalities (31), written in matrix form, become

$$|B'T^{\kappa}\delta| \leq \bar{d} \quad \text{and} \quad |C'T^{\kappa}\delta| \leq \underline{d} \tag{33}$$

for $\kappa = 1, 2, \ldots, 2^n$, defining $(2n \times 2^n)$ scalar inequalities. But since $\delta_i \geq 0$, $\bar{d}_i \geq 0$, and $\underline{d}_i \geq 0$ for all $i = 1, 2, \ldots n$, only the set of inequalities found in the positive orthant is considered. The choices of $\kappa \in \{1, 2, \ldots, 2^n\}$ are such that (33) is reduced to $2n$ inequalities of the form

$$|B'|\delta \leq \bar{d} \quad \text{and} \quad |C'|\delta \leq \underline{d} \tag{34}$$

where $|\cdot|$ defines the absolute value of the array taken component wise. Again (34) can be rewritten in the compact form Pδ $\leq$r for special choices of P and r.

It is worth noting that all the n-parameter problems as shown by (10), (30) and (34), can be reformulated as optimization problems subject to the inequality constraints

$$\mathrm{P}\delta \leq \mathrm{r}, \tag{35}$$

for suitably chosen P and r.

4. Solution of the Associated Optimization Problem

To deal with the set of inequalities (35), Pδ $\leq$ r, we choose a suitable linear or quadratic objective function and solve the resulting optimization problem.

(a) Linear objective function.

In this case the resulting optimization problem is a linear programming problem of the form

$$\begin{aligned} &\text{Maximize} \quad \sum_{i=1}^{n} \delta_i && \text{(P1)} \\ &\text{subject to} \quad \mathrm{P}\delta \leq \mathrm{r}, \qquad \delta \geq 0. \end{aligned}$$

The LP problem (P1) is solved using the Simplex method.

(ii) Quadratic objective function.

Here the resulting optimization problem is a quadratic programming problem of the form

$$\begin{aligned} &\text{Maximize} \quad \sum_{i=1}^{n} 2\omega_i \delta_i - \sum_{i=1}^{n} \delta_i^2 && \text{(P2)} \\ &\text{subject to} \quad \mathrm{P}\delta \leq \mathrm{r}, \qquad \delta \geq 0 \end{aligned}$$

where ω_i, $i = 1, 2, \ldots, n$, are the components of a weight vector. The quadratic programming problem (P2) is solved via a complementary pivoting technique. Note that the quadratic objective function is a concave function. This guarantees that a computed local maximum is the global maximum. The contour levels of the objective function are balls whose center is ω. Two types of optimal solutions are possible depending on the weight vector ω: In *type1* the optimum lies in the interior of the constraint region while in *type2* the optimum is on the boundary of the constraint region. Solution of quadratic programs through solving linear complementary problems is found in [2] together with a guarantee of convergence for the used pivoting algorithm. In fact, the Kuhn–Tucker conditions of the quadratic programming problem (P2) reduce to the linear complementary problem

$$w - Mz = q, \quad wTz = 0, \quad w, z > 0$$

where

$$M = \begin{pmatrix} 0 & -P \\ P^T & 2I \end{pmatrix}, \quad q = \begin{pmatrix} r \\ -2\omega \end{pmatrix}, \quad w = \begin{pmatrix} y \\ v \end{pmatrix}, \quad z = \begin{pmatrix} u \\ \delta \end{pmatrix}.$$

with y denoting the vector of slack variables whereas u and v are the Lagrangian multiplier vectors associated with the constraints $\mathrm{P}\delta \leq r$ and $\delta \geq 0$, respectively. Thus the complementary pivoting algorithm discussed in [2, sec.11.1] can be used to find a Kuhn–Tucker point of (P2) with a guarantee of convergence in a finite number of iterations [2, Theorem 11.2.4].

Remark. It is worth noting that if the quadratic objective function was chosen as the convex function $f(\delta) = \delta^T\delta$, then it leads to the quadratic programming problem

$$\begin{aligned} &\text{Maximize} \quad \sum_{i=1}^{n} \delta_i^2 \\ &\text{s.t.} \quad \mathrm{P}\delta \leq \mathrm{r}, \quad \delta \geq 0 \end{aligned} \tag{P3}$$

Following the notations of [2,sec. 11.1] to solve (P3), it is required to minimize $c^T\delta + \frac{1}{2}(\delta^T H\delta)$ where $H = -2I$ and $c = 0$. Thus H is a negative definite matrix, and there is no guarantee of convergence.

5. Numerical Example

Given

$$A_c = \begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix}, \quad b_c = \begin{pmatrix} 0 \\ 0 \\ 1000 \\ 1000 \end{pmatrix}$$

then

$$x_c = A_c^{-1}b_c = (\,125 \quad 125 \quad 375 \quad 375\,)^T, \quad T = I$$

Given the prescribed interval solution vector

$$x^I = ([110, 140],\ [110, 140],\ [360, 390],\ [360, 390])^T,$$

it is required to analyze the given system of equations $A^I x = b^I$ w.r.t. the maximum allowable deviations of the various system parameters as discussed in Sections 2 and 3.

5.1. Right-Hand-Side Uncertainties

For arbitrary deviation vector, the set of constraints defining Δb is given by (10), thus

$$P = |A_c^{-1}| \tag{i}$$

where

$$A_c^{-1} \approx \begin{pmatrix} 0.291667 & 0.083333 & 0.083333 & 0.0416667 \\ 0.083333 & 0.291667 & 0.0416667 & 0.083333 \\ 0.083333 & 0.0416667 & 0.291667 & 0.083333 \\ 0.0416667 & 0.083333 & 0.083333 & 0.291667 \end{pmatrix}$$

$$r_i = \min\{\underline{d}_i, \bar{d}_i\} = 15, \quad i = 1, 2, 3, 4. \tag{ii}$$

The maximum allowable deviation vector in the right hand side with P and r given by (i) and (ii), respectively, for:

1. LP problem (P1), is

$$\Delta b = (29.999968, 30.00001, 30.00001, 30)^T$$

2. QP problem (P2), with weight vector $\omega = (50, 50, 50, 50)^T$ is

$$\Delta b = (29.999973, 29.999998, 29.999998, 30.000001)^T.$$

x_R^I for the computed Δb as obtained by Rohn's algorithm [8] is

$$([110, 140], [110, 140], [360, 390], [360, 390])^T.$$

Solving (P2) with $\omega_i = 100$, and all other ω_j's equal 0 where $i, j \in \{1, 2, 3, 4\}$ and $i \neq j$, we find that $\Delta b_i = 51.42852$ and all other Δb_j's equal 0.

In the special case in which $\Delta b = \varepsilon|b_c|$, for some positive scalar ε, using (16) gives $\Delta b = (0, 0, 40, 40)^T$ where x_R^I for the computed Δb is,

$$x_R^I = ([120, 130], [120, 130], [360, 390], [360, 390])^T.$$

Table 1. One nonzero deviation in the elements of the matrix of coefficients

(l, m)	ΔA_{lm}	x_R^I
(1, 1)	0.367347	$([112.903, 140], [121.44, 129.286], [371.544, 379.286], [373.272, 377.143])^T$
(1, 2)	0.39779	$([110.963, 140], [120.989, 129.286], [370.989, 379.286], [372.995, 377.143])^T$
(1, 3)	0.135593	$([110.335, 140], [120.81, 129.286], [370.81, 379.285], [372.905, 377.143])^T$
(1, 4)	0.136364	$([110.169, 140], [120.763, 129.286], [370.763, 379.286], [372.881, 377.143])^T$
(2, 1)	0.39779	$([120.989, 129.286], [110.963, 140], [372.995, 377.143], [370.989, 379.286])^T$
(2, 2)	0.367347	$([121.544, 129.286], [112.903, 140], [373.272, 377.143], [371.544, 379.286])^T$
(2, 3)	0.136364	$([120.763, 129.286], [110.169, 140], [372.881, 377.143], [370.763, 379.286])^T$
(2, 4)	0.135593	$([120.81, 129.286], [110.335, 140], [372.905, 377.143], [370.81, 379.285])^T$
(3, 1)	0.39779	$([120.989, 129.286], [122.995, 127.143], [360.963, 390], [370.989, 379.286])^T$
(3, 2)	0.404494	$([120.856, 129.286], [122.928, 127.143], [360.497, 390], [370.86, 379.286])^T$
(3, 3)	0.131868	$([121.032, 129.286], [123.016, 127.143], [361.111, 390], [371.032, 379.286])^T$
(3, 4)	0.135593	$([120.81, 129.286], [122.905, 127.143], [360.335, 390], [370.81, 379.286])^T$
(4, 1)	0.404494	$([122.988, 127.143], [120.856, 129.286], [370.86, 379.286], [360.497, 390])^T$
(4, 2)	0.39779	$([122.995, 127.143], [120.989, 129.286], [370.989, 379.286], [360.963, 390])^T$
(4, 3)	0.135593	$([122.905, 127.143], [120.81, 129.286], [370.81, 379.286], [360.335, 390])^T$
(4, 4)	0.13868	$([123.016, 127.143], [121.032, 129.286], [371.032, 379.286], [361.111, 390])^T$

5.2. Uncertainty in the Matrix of Coefficients

Solving for the different cases considered:

(i) One nonzero element in position (l, m).

ΔA_{lm} is found using (24). The maximum allowable deviation for each coefficient A_{ij}, i and $j = 1, 2, 3, 4$, is found in Table 1 with the corresponding computed interval solution vector x_R^I.

(ii) One nonzero row

For equal row components, the maximum allowable deviation for each row $\Delta A_{l.} = \delta(1, 1, 1, 1)^T$, $l = 1, 2, 3, 4$, where δ is a scalar, computed according to (24), is given in Table 2 with the corresponding computed interval solution vector x_R^I.

Table 2. Equal row deviations

l	d	x_R^I
1	0.050139	$(110.734, 140], [120.924, 129.286], [370.924, 379.286], [372.962, 377.143])^T$
2	0.050139	$([120.924, 129.286], [110.734, 140], [372.962, 377.143], [370.924, 379.286])^T$
3	0.050139	$([120.924, 129.286], [122.962, 127.143], [360.734, 390], [370.924, 379.286])^T$
4	0.050139	$[122.962, 127.143], [120.924, 129.286], [370.924, 379.286], [360.734, 390])^T$

For unequal components, the set of constraints defining the vector $\delta = \Delta A_{l.}$, for each $l = 1, 2, 3, 4$, is given by (30), thus

$$P = \begin{pmatrix} BT' \\ -BT' \\ CT' \\ -CT' \end{pmatrix} \quad r = \begin{pmatrix} \bar{d} \\ \bar{d} \\ \underline{d} \\ \underline{d} \end{pmatrix} \qquad (i), (ii)$$

where $T' = I$. Note that the set of constraints is reduced from 4n to 2n inequality due to redundancy; since the constraints of the form $-BT'\delta \leq \bar{d}$ and $-CT'\delta \leq \underline{d}$ are redundant constraints as $T' = I$, $B \geq 0$, and $C \geq 0$ component wise. For example, for row number one with P and r given by (i) and (ii), respectively, we obtain:

1. LP problem (P1), gives

$$\Delta A_{1.} = (0, 0.397779, 0, 0)^T,$$

2. QP problem (P2), with $\omega = (0.1, 0.1, 0.1, 0.1)^T$, gives

$$\Delta A_{1.} = (0.0777792, 0.079492, 0.039835, 0.040175)^T$$

where x_R^I for the computed $\Delta A_{1.}$ is found to be,

$$x_R^I = ([111, 140], [121, 129.286], [371, 379.286], [373, 377.143])^T$$

Results for unequal row components are given in Table 3.

Table 3. Unequal row deviations. with $\omega = (0.1, 0.1, 0.1, 0.1)^T$

l	$\Delta A_{l.}$
1	$(0.077792, 0.079492, 0.03983, 0.040175)^T$
2	$(0.079492, 0.077792, 0.40175, 0.03983)^T$
3	$(0.079893, 0.080226, 0.039345, 0.041011)^T$
4	$(0.080226, 0.079893, 0.0401011, 0.03934)^T$

(iii) One nonzero column

For equal column components, the maximum allowable deviation for each column $\Delta A_{.m} = \delta(1, 1, 1, 1)^T$, $m = 1, 2, 3, 4$, where δ is a scalar computed according to (26), is found in Table 4 with the corresponding computed interval solution vector x_R^I.

Table 4. Equal column deviations

m	d	x_R^I.
1	0.214286	$([112.903, 140], [112.903, 140], [362.903, 390], [362.903, 390])^T$
2	0.214286	$([112.903, 140], [112.903, 140], [362.903, 390], [362.903, 390])^T$
3	0.076923	$([111.111, 140], [111.111, 140], [361.111, 390], [361.111, 390])^T$
4	0.076923	$([111.111, 140], [111.111, 140], [361.111, 390], [361.111, 390])^T$

For unequal column components, the set of constraints defining the parameters $\Delta A_{.m}$, for each $m = 1, 2, 3, 4$, is given by (34) or (35), with

$$P = \begin{pmatrix} |B'| \\ |C'| \end{pmatrix}, \quad r = \begin{pmatrix} \bar{d} \\ \underline{d} \end{pmatrix} \qquad (i), (ii)$$

where B' and C' are defined in (32). Note that the set of constraints consists of $2n$ inequalities. For example, for column number one, with P and r given by (i) and (ii), respectively, we obtain:

1. LP problem (P1) gives

$$\Delta A_{.1} = (0.214286, 0.214286, 0.214286, 0.214286)^T,$$

2. QP problem (P2), with $\omega = (0.5, 0.5, 0.5, 0.5)^T$ gives

$$\Delta A_{.1} = (0.214286, 0.214286, 0.214286, 0.214286)^T$$

with x_R^I for the computed $\Delta A_{.1}$ given by

$$x_R^I = ([112.9, 140], [112.9, 140], [362.9, 390], [362.9, 390.000031])^T$$

Results for unequal column components are found in Table 5.

Table 5. Unequal column deviations. $\omega = (0.5, 0.5, 0.5, 0.5)^T$

m	$\Delta A_{.m}$
1	$(0.214286, 0.214286, 0.214286, 0.214286)^T$
2	$(0.214286, 0.214286, 0.214286, 0.214286)^T$
3	$(0.076923, 0.076923, 0.076923, 0.076923)^T$
4	$(0.076923, 0.076923, 0.076923, 0.076923)^T$

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