Reliable Computing **7:** 409–424, 2001.

On Maximal Inner Estimation of the Solution Sets of Linear Systems with Interval Parameters

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(Received: 2 May 1998; accepted: 17 November 2000)

Abstract. The purpose of this paper is to inquire the connection between maximal inner interval estimates of the solution sets to interval linear system and solutions of the dualization equation in Kaucher interval arithmetic. The results of our work are as follows: 1) a criterion of inner interval estimate of the solution set, 2) a criterion for a solution of dualization equation to be a maximal inner interval estimate of the solution set, 3) a criterion for multiplication by an interval matrix to be upper strictly isotone.

1. Notation

We use Latin letters for real objects: small for numbers and vectors $(a, b, c, \dots)$ and capital for matrices $(A, B, C, \dots)$.

By *interval*, we call an object of the form $[\underline{x}, \overline{x}]$ with $\underline{x}, \overline{x} \in \mathbb{R}$ (not necessarily $\underline{x} \leq \overline{x}$). If $\underline{x} \leq \overline{x}$, then $[\underline{x}, \overline{x}]$ is said to be a *proper interval*. At the same time, we think of a proper interval $[\underline{x}, \overline{x}]$ as the set of all real numbers between the points $\underline{x}$ and $\overline{x}$, i.e., $\{x \in \mathbb{R} \mid \underline{x} \leq x \leq \overline{x}\}$.

A vector (matrix) with interval components is called an *interval vector (matrix)*. An interval vector (matrix) is said to be *proper* if all its components are proper intervals. Similar to the one-dimensional case, we think of a proper interval vector $[\underline{x}, \overline{x}] := ([\underline{x}_1, \overline{x}_1], \dots, [\underline{x}_n, \overline{x}_n])^\top$ as the set of all real vectors bounded by the vectors $\underline{x}$ and $\overline{x}$, i.e., $\{x \in \mathbb{R}^n \mid \underline{x}_j \leq x_j \leq \overline{x}_j,\ j = 1, \dots, n\}$.

We use boldface Latin letters for interval objects: small for intervals and interval vectors $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots)$ and capital for interval matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots)$. The bold symbol $\mathbf{0}$ designates the interval $[0, 0]$.

2. Introduction

In interval analysis, it is traditionally assumed that the data uncertainty expressed in the form of intervals of possible values has the same nature for all the parameters under consideration. Put it differently, we suppose that all these intervally uncertain parameters do not "conflict" with each other, so that the resulting uncertainty caused by all the input factors is summed over the separate parameters. However, it is not difficult to point out another practical situation: different parameters act on the system in such a way that their actions suppress each other, while the interval data uncertainties mutually compensate. The latter is especially typical e.g. in control systems or in the decision making situations described by antagonistic games, when the intervals of possible values of the parameters represent actions on the system that are different by their nature: outer perturbations as opposed to our controls.

A mathematical formalization of this kind of problem statements within the interval framework can be given by using the so-called *generalized solution sets* and in particular *AE-solution sets* [9], [12]–[14]. In the present paper, we are going to consider the simplest interval linear systems whose parameters are subject to conflicting actions. The first results in this field are also due to S. Shary [9], [11] and we substantially rely upon these works. Let us turn to formal definitions.

Specifically, we deal with the following Main Problem:

Given:

1) a linear system of equations $Ax = b$ as a symbolic description of an operating Object (in this system, x is an n-vector of unknown variables, A is an $m \times n$-matrix of parameters, b is an m-vector of parameters);
2) a decomposition of all the parameters of the system into two nonintersecting subsets. We shall call the parameters of the first subset by *A-parameters* and those of the second subset by *E-parameters* [a];
3) proper intervals of values for the parameters, i.e., a proper interval matrix $\mathbf{A} = (\mathbf{a}_{ij})$ and a proper interval vector $\mathbf{b} = (\mathbf{b}_i)$.

Wanted:

Proper intervals of values for the unknown variables, such that for all values of unknowns and A-parameters from the respective intervals one can find values of E-parameters from the respective intervals for the Object operating. The intervals for values of unknown variables should be as wide as possible.

[a] The terms “A-parameters” and “E-parameters” originate from the names of the logical quantifiers “$\forall$” and “$\exists$”, that is, *ALL* (universal) quantifier and *EXISTS* (existential) quantifier [3]. The distinction between A- and E-parameters would be clear from the `Wanted` of the Main Problem.

The above formulation is borrowed from [9], [12], [13], and its particular case is, for instance, a popular linear tolerance problem [6], [7], [15].

In [11], S. Shary proposed to find solutions of the Main Problem among the solutions of the so-called *dualization equation* (defined in Section 6 later), and the purpose of this paper is to inquire the connection between these solutions and the solution sets to interval linear systems under estimation. The results of our study will be formulated in Section 6, while Sections 3–5 are auxiliary: Section 3 represents the Main Problem in strict mathematical terms, Section 4 reminds some necessary facts of interval mathematics that are not widely known even among specialists, and Section 5 expresses the Main Problem in interval form.

3. Reformulation of the Main Problem with the Use of Logical Formulas and Set Inclusions

We are going to rewrite the Main Problem in convenient mathematical terms. First, we shall transform the *Given* item into a universal form suitable for every decomposition of parameters into A- and E-groups. Let c be a parameter and a proper interval $\mathbf{c}$ represents its possible values. We associate with the parameter c two parameters $c^{\forall}$ and $c^{\exists}$ by

$$c^{\forall} := \begin{cases} c, & \text{if } c \text{ is an A-parameter,} \\ 0, & \text{if } c \text{ is an E-parameter,} \end{cases}$$

$$c^{\exists} := \begin{cases} 0, & \text{if } c \text{ is an A-parameter,} \\ c, & \text{if } c \text{ is an E-parameter.} \end{cases}$$

Analogously, we associate with the interval $\mathbf{c}$ two proper intervals $\mathbf{c}^{\forall}$ and $\mathbf{c}^{\exists}$ by

$$\mathbf{c}^{\forall} := \begin{cases} \mathbf{c}, & \text{if } c \text{ is an A-parameter,} \\ \mathbf{0}, & \text{if } c \text{ is an E-parameter,} \end{cases}$$

$$\mathbf{c}^{\exists} := \begin{cases} \mathbf{0}, & \text{if } c \text{ is an A-parameter,} \\ \mathbf{c}, & \text{if } c \text{ is an E-parameter.} \end{cases}$$

Hence, we arrive at

Given:

1) a symbolic system $(A^{\forall} + A^{\exists})x = b^{\forall} + b^{\exists}$, with $A^{\forall} = (a_{ij}^{\forall})$, $A^{\exists} = (a_{ij}^{\exists})$, $b^{\forall} = (b_i^{\forall})$, $b^{\exists} = (b_i^{\exists})$;
2) proper interval matrices $\mathbf{A}^{\forall} = (\mathbf{a}_{ij}^{\forall})$, $\mathbf{A}^{\exists} = (\mathbf{a}_{ij}^{\exists})$ and proper interval vectors $\mathbf{b}^{\forall} = (\mathbf{b}_i^{\forall})$, $\mathbf{b}^{\exists} = (\mathbf{b}_i^{\exists})$.

To reformulate what is *Wanted* we introduce the following definitions:

DEFINITION 3.1. The *solution set of the linear system* $(A^{\forall} + A^{\exists})x = b^{\forall} + b^{\exists}$ *with interval parameters* $\mathbf{A}^{\forall}$, $\mathbf{b}^{\forall}$, $\mathbf{A}^{\exists}$, $\mathbf{b}^{\exists}$ is the set*

$$\Xi = \{x \in \mathbb{R}^n \mid (\forall A^{\forall} \in \mathbf{A}^{\forall})\,(\forall b^{\forall} \in \mathbf{b}^{\forall})\,(\exists A^{\exists} \in \mathbf{A}^{\exists})\,(\exists b^{\exists} \in \mathbf{b}^{\exists}) \\ \big((A^{\forall} + A^{\exists})x = b^{\forall} + b^{\exists}\big)\}. \quad (3.1)$$

* It is also called *AE-solution set of αβ-type* in [14] or *αβ-solution set* in the previous works [9], [11]–[13].

DEFINITION 3.2. An *inner interval estimate* for the set Ξ is a proper interval vector $\mathbf{x}$ such that

$$\mathbf{x} \subseteq \Xi.$$

DEFINITION 3.3. An interval vector $\mathbf{x}$ is said to be a *maximal* vector with a property P, if

1) P holds for $\mathbf{x}$ and
2) for all the interval vectors strictly including $\mathbf{x}$ the property P does not hold.*

Hence, we arrive at

Wanted:
Maximal inner interval estimate for the set Ξ.

4. Kaucher Interval Arithmetic

4.1. Interval Algebraic System

Kaucher interval arithmetic is an the algebraic system

$$\langle \mathbb{IR}, \subseteq, \leq, \vee, \wedge, \mathrm{dual}, \mathrm{pro}, +, -, \cdot, / \rangle$$

with its components defined as follows:

Basic set. *Intervals* and *proper intervals* have been defined in Section 1, and the basic set $\mathbb{IR}$ consists of all the intervals $[\underline{x}, \overline{x}]$:

$$\mathbb{IR} = \{\mathbf{x} = [\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in \mathbb{R}\}.$$

The numbers $\underline{x}$ and $\overline{x}$ are called *left endpoint* and *right endpoint* of the interval $\mathbf{x} = [\underline{x}, \overline{x}]$ respectively. Two intervals are considered equal if their corresponding endpoints are equal:

$$\mathbf{x} = \mathbf{y} \quad \overset{def}{\Longleftrightarrow} \quad (\underline{x} = \underline{y} \ \text{ and } \ \overline{x} = \overline{y}).$$

The set of all proper intervals is denoted by $\mathrm{I}\mathbb{R}$:

$$\mathrm{I}\mathbb{R} = \{\mathbf{x} = [\underline{x}, \overline{x}] \mid \underline{x} \leq \overline{x}, \ \underline{x}, \overline{x} \in \mathbb{R}\}.$$

An interval $[\underline{x}, \overline{x}]$ is called

improper	if $\underline{x} > \overline{x}$,
backward	if $\underline{x} \geq \overline{x}$,
degenerate	if $\underline{x} = \overline{x}$.

* Strict inclusion of interval vectors will be defined in Section 4. Here it is sufficient to know, that for proper interval vectors it coincides with strict inclusion of sets.

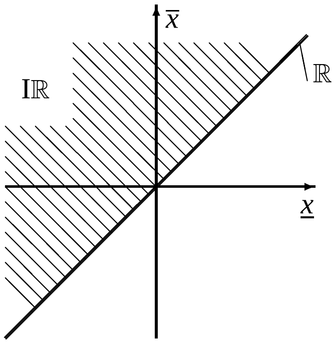

Figure 1. The set $\mathbb{IR}$.

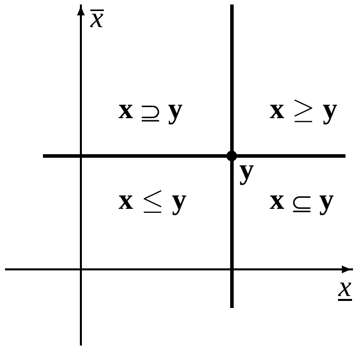

Figure 2. The partial order relations on $\mathbb{IR}$.

The set of degenerate intervals represents real numbers in $\mathbb{IR}$. For brevity, we shall use x instead $[x, x]$ in interval operations.

Lattice structure. "$\subseteq$", "$\leq$" are partial order relations on $\mathbb{IR}$ defined by the order relation "$\leq$" on $\mathbb{R}$ (see Figures 1 and 2):

$$\mathbf{x} \subseteq \mathbf{y} \overset{def}{\Longleftrightarrow} (\underline{x} \geq \underline{y} \text{ and } \overline{x} \leq \overline{y}),$$
$$\mathbf{x} \leq \mathbf{y} \overset{def}{\Longleftrightarrow} (\underline{x} \leq \underline{y} \text{ and } \overline{x} \leq \overline{y}).$$

"$\vee$", "$\wedge$" are lattice operations of inclusion supremum and inclusion infimum respectively. They are defined by supremum and infimum operations on $\mathbb{R}$ as follows:

$$\bigvee_{i\in I} \mathbf{x}_i = \sup_{i\in I}{}_{\subseteq} \mathbf{x}_i = \left[\inf_{i\in I} \underline{x}_i, \sup_{i\in I} \overline{x}_i\right] \quad \text{for any family of intervals } \{\mathbf{x}_i\}_{i\in I} \text{ that is upper bounded by inclusion,}$$

$$\bigwedge_{i\in I} \mathbf{x}_i = \inf_{i\in I}{}_{\subseteq} \mathbf{x}_i = \left[\sup_{i\in I} \underline{x}_i, \inf_{i\in I} \overline{x}_i\right] \quad \text{for any family of intervals } \{\mathbf{x}_i\}_{i\in I} \text{ that is lower bounded by inclusion.}$$

Unary operations.

"dual" is *dualization*:

$$\operatorname{dual}[\underline{x}, \overline{x}] = [\overline{x}, \underline{x}].$$

"pro" is *proper projection*:

$$\operatorname{pro}\mathbf{x} = \begin{cases} \mathbf{x}, & \text{if } \mathbf{x} \text{ is proper,} \\ \operatorname{dual}\mathbf{x}, & \text{if } \mathbf{x} \text{ is improper.} \end{cases}$$

Binary operations. We shall use the symbol "$\bigvee\!\!\bigwedge^{\mathbf{x}}$" (pronounced as "supinf ([su:'pinf]) on x") for *conditional lattice operation over the interval* $\mathbf{x}$:

$$\bigvee\!\!\bigwedge^{\mathbf{x}} = \begin{cases} \bigvee\limits_{x \in \mathbf{x}}, & \text{if } \mathbf{x} \text{ is proper,} \\ \bigwedge\limits_{x \in \operatorname{dual}\mathbf{x}}, & \text{if } \mathbf{x} \text{ is improper.} \end{cases}$$

The arithmetic operations $+$, $-$, $\cdot$, $/$ are defined on $\mathbb{IR}$ through the corresponding real operations and lattice operations $\vee$, $\wedge$ [1], [2] so that

$$\forall * \in \{+, -, \cdot, /\} \qquad \mathbf{x} * \mathbf{y} = \bigvee\!\!\bigwedge^{\mathbf{x}} \bigvee\!\!\bigwedge^{\mathbf{y}} (x * y). \tag{4.1}$$

Interval sum and product are commutative and associative operations on $\mathbb{IR}$ [1], [2].

Also, the following lattice operation distributivity will be useful for us:

$$\mathbf{x} + (\mathbf{y} \vee \mathbf{z}) = (\mathbf{x} + \mathbf{y}) \vee (\mathbf{x} + \mathbf{z}). \tag{4.2}$$

4.2. Interval Vectors and Interval Matrices

The *interval vectors* and *interval matrices* have been introduced in Section 1. We denote by $\mathbb{IR}^n$ the set of all n-dimensional interval vectors and by $\mathbb{IR}^{m\times n}$ the set of all interval $m \times n$-matrices. The operations dual, pro, $\vee$, $\wedge$, $+$, $-$ as well as relations $=$, $\subseteq$, $\leq$ on $\mathbb{IR}^n$ and on $\mathbb{IR}^{m\times n}$ are defined component-wise. For example, dual matrix is the matrix of dual components, inclusion supremum of the interval vectors $\mathbf{x}, \mathbf{y} \in \mathbb{IR}^n$ is the vector $\mathbf{x} \vee \mathbf{y} \in \mathbb{IR}^n$ with $(\mathbf{x} \vee \mathbf{y})_i = \mathbf{x}_i \vee \mathbf{y}_i$, $i = 1, \ldots, n$.

For a vector $\mathbf{x} \in \mathbb{IR}^n$, the result of *multiplication by a matrix* $\mathbf{C} \in \mathbb{IR}^{m\times n}$ is a vector from $\mathbb{IR}^m$ defined as follows:

$$(\mathbf{C} \cdot \mathbf{x})_i = \sum_{j=1}^{n} \mathbf{c}_{ij}\mathbf{x}_j, \qquad i = 1, \ldots, m.$$

4.3. Strict Inclusion Isotonicity

We define the relation of *strict inclusion* ($\subset$) in $\mathbb{IR}^n$ by the rule

$$\mathbf{x} \subset \mathbf{y} \quad \overset{def}{\Longleftrightarrow} \quad (\mathbf{x} \subseteq \mathbf{y} \text{ and } \mathbf{x} \neq \mathbf{y}).$$

For interval vectors, $\mathbf{x} \subset \mathbf{y}$ means also that

$$\bigl(\forall i \in \{1, \dots, n\}\ (\mathbf{x}_i \subseteq \mathbf{y}_i)\bigr) \qquad \text{and} \qquad \bigl(\exists k \in \{1, \dots, n\}\ (\mathbf{x}_k \subset \mathbf{y}_k)\bigr). \tag{4.3}$$

DEFINITION 4.1. A function $F : D(F) \to \mathbb{IR}^m$, $D(F) \subseteq \mathbb{IR}^n$, is called

1) *(inclusion) isotone** if it retains the relation $\subseteq$:

$$\forall \mathbf{x}, \mathbf{y} \in D(F) \qquad \bigl(\mathbf{x} \subseteq \mathbf{y} \;\Rightarrow\; F(\mathbf{x}) \subseteq F(\mathbf{y})\bigr),$$

2) *strictly (inclusion) isotone* if it retains the relation $\subset$:

$$\forall \mathbf{x}, \mathbf{y} \in D(F) \qquad \bigl(\mathbf{x} \subset \mathbf{y} \;\Rightarrow\; F(\mathbf{x}) \subset F(\mathbf{y})\bigr).$$

EXAMPLE 4.1.

1) The arithmetic operations $+$, $-$, $\cdot$, $/$ in $\mathbb{IR}$ are isotone [1], [2]. This fundamental property of Kaucher interval arithmetic implies that multiplication by an interval matrix is isotone.
2) The arithmetic operation $+$ in $\mathbb{IR}^n$ is strictly isotone, i.e.,

$$\forall\, \mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}' \in \mathbb{IR}^n \quad \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \subset \begin{pmatrix} \mathbf{x}' \\ \mathbf{y}' \end{pmatrix} \;\Rightarrow\; \mathbf{x} + \mathbf{y} \subset \mathbf{x}' + \mathbf{y}'.$$

This follows from the endpoint representation of the interval sum

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{IR} \quad \mathbf{x} + \mathbf{y} = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$$

and from the definition of strict inclusion:

for $\mathbf{z}, \mathbf{z}' \in \mathbb{IR}^k$, $\mathbf{z} \subset \mathbf{z}'$ means that the inequalities $\underline{z}_i \geq \underline{z}'_i$, $\overline{z}_i \leq \overline{z}'_i$, $i \in \{1, \dots, k\}$, hold and at least one of them is strict.

3) In general, multiplication by an interval is not strictly isotone. For example, $[-1, 1] \cdot [1, -1] = [-1, 1] \cdot [0, 0]$.

DEFINITION 4.2. We shall call the function $F : D(F) \to \mathbb{IR}^m$, $D(F) \subseteq \mathbb{IR}^n$, *upper strictly isotone in* $\mathbf{x}$ if

$$\forall \mathbf{y} \in D(F) \qquad \bigl(\mathbf{x} \subset \mathbf{y} \;\Rightarrow\; F(\mathbf{x}) \subset F(\mathbf{y})\bigr).$$

DEFINITION 4.3. We shall call the function $F : D(F) \to \mathbb{IR}^m$, $D(F) \subseteq \mathbb{IR}$, upper strictly isotone in $\mathbf{x}$

1) *over left endpoint* if

$$\forall \mathbf{y} \in D(F) \qquad \bigl((\underline{x} > \underline{y} \text{ and } \overline{x} \leq \overline{y}) \;\Rightarrow\; F(\mathbf{x}) \subset F(\mathbf{y})\bigr),$$

* In classical interval arithmetic, only inclusion isotone functions are considered, so they are often called "(inclusion) monotone". In Kaucher arithmetic, a necessity arises to consider both (inclusion) isotone and antitone functions.

2) *over right endpoint* if

$$\forall \mathbf{y} \in D(F) \qquad \big((\underline{x} \geq \underline{y} \text{ and } \overline{x} < \overline{y}) \Rightarrow F(\mathbf{x}) \subset F(\mathbf{y})\big).$$

Since for all intervals $\mathbf{x}$ and $\mathbf{y}$

$$\mathbf{x} \subset \mathbf{y} \iff \big((\underline{x} > \underline{y} \text{ and } \overline{x} \leq \overline{y}) \text{ or } (\underline{x} \geq \underline{y} \text{ and } \overline{x} < \overline{y})\big),$$

a function is upper strictly isotone in $\mathbf{x} \in \mathbb{IR}$ if and only if it is upper strictly isotone in $\mathbf{x}$ over both left and right endpoints.

LEMMA 4.1. *Let $\mathbf{x} \in \mathbb{IR}^n$. Multiplication by an interval matrix $\mathbf{C} \in \mathbb{IR}^{m\times n}$ is upper strictly isotone in $\mathbf{x}$ if and only if for each $k \in \{1, \ldots, n\}$ both the following conditions hold:*

1) *there exists such a component $\mathbf{c}_{lk}$ that multiplication by it is upper strictly isotone in $\mathbf{x}_k$ over left endpoint, i.e.,*

$$\exists l \in \{1, \ldots, m\}\ \forall \mathbf{u} \in \mathbb{IR} \quad \big((\underline{x}_k > \underline{u} \text{ and } \overline{x}_k \leq \overline{u}) \Rightarrow \mathbf{c}_{lk}\mathbf{x}_k \subset \mathbf{c}_{lk}\mathbf{u}\big), \quad (4.4)$$

2) *there exists such a component $\mathbf{c}_{rk}$ that multiplication by it is upper strictly isotone in $\mathbf{x}_k$ over right endpoint, i.e.,*

$$\exists r \in \{1, \ldots, m\}\ \forall \mathbf{v} \in \mathbb{IR} \quad \big((\underline{x}_k \geq \underline{v} \text{ and } \overline{x}_k < \overline{v}) \Rightarrow \mathbf{c}_{rk}\mathbf{x}_k \subset \mathbf{c}_{rk}\mathbf{v}\big). \quad (4.5)$$

Proof. To begin with, let us spell out the assertion of the lemma. In point of fact, we need to substantiate that the condition

$$\forall k \ \big((4.4) \text{ and } (4.5)\big) \tag{4.6}$$

is equaivalent for

$$\forall \mathbf{y} \in \mathbb{IR}^n \quad (\mathbf{x} \subset \mathbf{y} \Rightarrow \mathbf{C}\mathbf{x} \subset \mathbf{C}\mathbf{y}) \tag{4.7}$$

to be true.

$\Longrightarrow$ Let $\mathbf{y} \in \mathbb{IR}^n$ and $\mathbf{x} \subset \mathbf{y}$. It means

$$\forall j \in \{1, \ldots, n\} \quad (\mathbf{x}_j \subseteq \mathbf{y}_j) \tag{4.8}$$

and

$$\exists k \in \{1, \ldots, n\} \quad (\mathbf{x}_k \subset \mathbf{y}_k). \tag{4.9}$$

Obviously, $\mathbf{x}_k \subset \mathbf{y}_k$ means

$$(\underline{x}_k > \underline{y}_k \text{ and } \overline{x}_k \leq \overline{y}_k) \quad \text{or} \quad (\underline{x}_k \geq \underline{y}_k \text{ and } \overline{x}_k < \overline{y}_k).$$

Applying (4.4) if ($\underline{x}_k > \underline{y}_k$ and $\overline{x}_k \leq \overline{y}_k$) or (4.5) if ($\underline{x}_k \geq \underline{y}_k$ and $\overline{x}_k < \overline{y}_k$) gives

$$\exists p \in \{1, \ldots, m\} \quad \mathbf{c}_{pk}\mathbf{x}_k \subset \mathbf{c}_{pk}\mathbf{y}_k. \tag{4.10}$$

Since interval product and sum are isotone, (4.8) implies

$$\forall i \in \{1, \ldots, m\} \quad \big((\mathbf{Cx})_i \subseteq (\mathbf{Cy})_i\big) \tag{4.11}$$

and

$$\sum_{j \neq k} \mathbf{c}_{pj}\mathbf{x}_j \subseteq \sum_{j \neq k} \mathbf{c}_{pj}\mathbf{y}_j. \tag{4.12}$$

It follows from the strict isotonicity of the interval sum, from (4.10) and from (4.12) that

$$(\mathbf{Cx})_p \subset (\mathbf{Cy})_p. \tag{4.13}$$

Combining (4.11) and (4.13) yields

$$\mathbf{Cx} \subset \mathbf{Cy}.$$

$\Longleftarrow$ To prove that the condition (4.6) is necessary for (4.7), we suppose that (4.6) is wrong, i.e.,

$$\exists k \Big(\big((4.4) \text{ does not hold}\big) \text{ or } \big((4.5) \text{ does not hold}\big)\Big).$$

((4.4) does not hold) means

$$\forall l \in \{1, \ldots, m\}\ \exists \mathbf{u}^l \in \mathbb{IR}\ (\underline{x}_k > \underline{u}^l \text{ and } \overline{x}_k \leq \overline{u}^l \text{ and } \mathbf{c}_{lk}\mathbf{x}_k = \mathbf{c}_{lk}\mathbf{u}^l). \tag{4.14}$$

((4.5) does not hold) means

$$\forall r \in \{1, \ldots, m\}\ \exists \mathbf{v}^r \in \mathbb{IR}\ (\underline{x}_k \geq \underline{v}^r \text{ and } \overline{x}_k < \overline{v}^r \text{ and } \mathbf{c}_{rk}\mathbf{x}_k = \mathbf{c}_{rk}\mathbf{v}^r). \tag{4.15}$$

Let us introduce a vector $\mathbf{y} \in \mathbb{IR}^n$ as follows:

$$\mathbf{y}_j = \begin{cases} \mathbf{x}_j, & \text{if } j \neq k, \\ \left[\max\limits_l \underline{u}^l, \overline{x}_k\right], & \text{if } j = k \text{ and (4.14)}, \\ \left[\underline{x}_k, \min\limits_r \overline{v}^r\right], & \text{if } j = k \text{ and (4.4) and (4.15)}. \end{cases}$$

The vector $\mathbf{y}$ is such that $\mathbf{x} \subset \mathbf{y}$ and $\mathbf{Cx} = \mathbf{Cy}$, which means (4.7) is false. We have thus obtained that (4.7) is not valid without (4.6). □

LEMMA 4.2. *Let a function $F : \mathbb{IR} \to \mathbb{IR}^m$ be isotone.*

1) *The function F is upper strictly isotone in* $\mathbf{x}$ *over left endpoint if and only if*

$$\exists \varepsilon > 0\ \ \forall \delta \quad \big(0 < \delta \leq \varepsilon \ \Rightarrow\ F(\mathbf{x}) \neq F([\underline{x} - \delta, \overline{x}])\big). \tag{4.16}$$

2) *The function F is upper strictly isotone in* $\mathbf{x}$ *over right endpoint if and only if*

$$\exists \varepsilon > 0\ \ \forall \delta \quad \big(0 < \delta \leq \varepsilon \ \Rightarrow\ F(\mathbf{x}) \neq F([\underline{x}, \overline{x} + \delta])\big). \tag{4.17}$$

Table 1.

$\cdot$	$\underline{u} > 0$	$\overline{u} < 0$	$\mathbf{0} \subseteq \mathbf{u}$
$\underline{c}, \overline{c} > 0$	$[\underline{c}\,\underline{u}, \overline{c}\,\overline{u}]$	$[\overline{c}\,\underline{u}, \underline{c}\overline{u}]$	$[\overline{c}\,\underline{u}, \overline{c}\,\overline{u}]$
$\underline{c}, \overline{c} < 0$	$[\underline{c}\,\overline{u}, \overline{c}\,\underline{u}]$	$[\overline{c}\,\overline{u}, \underline{c}\,\underline{u}]$	$[\underline{c}\,\overline{u}, \underline{c}\,\underline{u}]$
$\mathbf{0} \subset \mathbf{c}$	$[\underline{c}\,\overline{u}, \overline{c}\,\overline{u}]$	$[\overline{c}\,\underline{u}, \underline{c}\,\underline{u}]$	$[\min\{\overline{c}\,\underline{u}, \underline{c}\,\overline{u}\}, \max\{\underline{c}\,\underline{u}, \overline{c}\,\overline{u}\}]$
$\mathbf{c} \subset \mathbf{0}$	$[\underline{c}\,\underline{u}, \overline{c}\,\underline{u}]$	$[\overline{c}\,\overline{u}, \underline{c}\,\overline{u}]$	$\mathbf{0}$

Proof.

1) Obviously, the condition (4.16) is necessary for the function F to be upper strictly isotone in $\mathbf{x}$ over left endpoint. We shall prove that it is sufficient too.

   Let (4.16) hold. For every interval $\mathbf{y}$ such that $\underline{x} > \underline{y}$ and $\overline{x} \le \overline{y}$, we can introduce an interval $\mathbf{z} = [\underline{x} - \min\{\varepsilon, \underline{x} - \underline{y}\}, \overline{x}]$. The function F is inclusion isotone and $\mathbf{x} \subset \mathbf{z} \subseteq \mathbf{y}$, therefore $F(\mathbf{x}) \subseteq F(\mathbf{z}) \subseteq F(\mathbf{y})$. The condition (4.16) implies $F(\mathbf{x}) \neq F(\mathbf{z})$. Hence, $F(\mathbf{x}) \subset F(\mathbf{y})$.

2) The second part of Lemma 4.2 is proved analogously. □

To formulate our next result, we need the following notation:

$$\lceil \mathbf{x} \rceil = \max\{|\underline{x}|, |\overline{x}|\}, \quad \lfloor \mathbf{x} \rfloor = \min\{|\underline{x}|, |\overline{x}|\} \quad \text{for } \mathbf{x} \in \mathbb{IR}. \tag{4.18}$$

THEOREM 4.1. Criterion for Multiplication by an Interval Matrix to Be Upper Strictly Isotone. *Let* $\mathbf{x} \in \mathbb{IR}^n$. *Multiplication by an interval matrix* $\mathbf{C} \in \mathbb{IR}^{m\times n}$ *is upper strictly isotone in* $\mathbf{x}$ *if and only if, for each* $k \in \{1, \dots, n\}$, *at least one of the following three conditions holds:*

(a) $\exists \mathbf{c} \in \{\mathbf{c}_{1k}, \dots, \mathbf{c}_{mk}\}\ (0 \notin \operatorname{pro} \mathbf{c})$;

(b) $(\mathbf{0} \not\subseteq \mathbf{x}_k)$ *and* $\big(\exists \mathbf{c}', \mathbf{c}'' \in \{\mathbf{c}_{1k}, \dots, \mathbf{c}_{mk}\}\ (\mathbf{0} \subset \mathbf{c}'$ *and* $\mathbf{c}'' \subset \mathbf{0})\big)$;

(c) $(\mathbf{0} \subseteq \mathbf{x}_k)$ *and* $\big(\exists \mathbf{c} \in \{\mathbf{c}_{1k}, \dots, \mathbf{c}_{mk}\}\ (\mathbf{0} \subset \mathbf{c}$ *and* $\lceil \mathbf{c} \rceil \cdot \lfloor \mathbf{x}_k \rfloor \ge \lfloor \mathbf{c} \rfloor \cdot \lceil \mathbf{x}_k \rceil)\big)$.

Proof consists of Preliminaries and Modification of Lemma 4.1.

1) Preliminaries.

Look at the Table 1 of multiplication of a proper interval $\mathbf{u}$ by a nonzero interval $\mathbf{c}$. One can derive this table from

- either Kaucher table for interval product published in [2],
- or Lakeyev formulas for interval product [5]

$$\begin{aligned}\mathbf{x} \cdot \mathbf{y} \;=\; \big[&\max\{\underline{x}^+\underline{y}^+, \overline{x}^-\overline{y}^-\} - \max\{\overline{x}^+\underline{y}^-, \underline{x}^-\overline{y}^+\},\\ &\max\{\overline{x}^+\overline{y}^+, \underline{x}^-\underline{y}^-\} - \max\{\underline{x}^+\overline{y}^-, \overline{x}^-\underline{y}^+\}\big],\\ &\qquad\qquad \text{where } x^+ = \max\{0, x\}, \;\; x^- = \max\{0, -x\},\end{aligned}$$

- or directly from the representation (4.1) of the interval product.

The structure of the table is such that any formula for the endpoints of the interval product is unchanged for *sufficiently small* inflation of the interval $\mathbf{u}$. Given $\mathbf{c}$ and $\mathbf{u}$, Table 1 and Lemma 4.2 implies that

(I) if at least one of the endpoints of $\mathbf{cu}$ is equal to $\tilde{c}\underline{u}$ with $\tilde{c} \in \{\underline{c}, \overline{c}\}$, $\tilde{c} \neq 0$, then the multiplication by $\mathbf{c}$ is upper strictly isotone in $\mathbf{u}$ over left endpoint,

(II) otherwise, the endpoints of $\mathbf{cu}$ do not depend on $\underline{u}$ and the multiplication by $\mathbf{c}$ is not upper strictly isotone in $\mathbf{u}$ over left endpoint.

The similar facts hold for isotonicity over right endpoint.

2) Modification of Lemma 4.1.

Now one can obtain Theorem 4.1 from Lemma 4.1 and Preliminaries. Given $\mathbf{x}_k$, the condition (a) of Theorem 4.1 corresponds to the first and second rows of Table; if the condition (a) of Theorem 4.1 is false, then the condition (b) of Theorem 4.1 corresponds to the first and second columns and the condition (c) matches to the last column of Table. The conditions (a) and (b) of Theorem 4.1 arise in an obvious way. We need only to explain how the condition (c) comes into being.

The condition (c) describes the case

$$(\mathbf{0} \subseteq \mathbf{x}_k) \quad \text{and} \quad \big(\forall \mathbf{c} \in \{\mathbf{c}_{1k}, \ldots, \mathbf{c}_{mk}\}\ (\underline{c} \cdot \overline{c} \not> 0)\big). \tag{4.19}$$

Let us denote $\mathbf{x}_k$ by $\mathbf{u}$ and assume $\mathbf{c}', \mathbf{c}'' \in \{\mathbf{c}_{1k}, \ldots, \mathbf{c}_{mk}\}$. According to Preliminaries, the multiplication by $\mathbf{c}'$ is upper strictly isotone in $\mathbf{u}$ over left endpoint if and only if $\mathbf{0} \subset \mathbf{c}'$ and

$$(\overline{c}'\underline{u} \leq \underline{c}'\overline{u} \ \text{ and } \ \overline{c}' \neq 0) \quad \text{or} \quad (\underline{c}'\underline{u} \geq \overline{c}'\overline{u} \ \text{ and } \ \underline{c}' \neq 0). \tag{4.20}$$

For $\mathbf{0} \subseteq \mathbf{u}$ and $\mathbf{0} \subset \mathbf{c}'$, (4.20) is equivalent to

$$(|\overline{c}'||\underline{u}| \geq |\underline{c}'||\overline{u}| \ \text{ and } \ |\overline{c}'| \neq 0) \quad \text{or} \quad (|\underline{c}'||\underline{u}| \geq |\overline{c}'||\overline{u}| \ \text{ and } \ |\underline{c}'| \neq 0). \tag{4.21}$$

The formula (4.21) is equivalent to

$$\max\{|\underline{c}'|, |\overline{c}'|\} \cdot |\underline{u}| \geq \min\{|\underline{c}'|, |\overline{c}'|\} \cdot |\overline{u}| \quad \text{and} \quad \max\{|\underline{c}'|, |\overline{c}'|\} \neq 0. \tag{4.22}$$

For $\mathbf{0} \subset \mathbf{c}'$, the condition $\max\{|\underline{c}'|, |\overline{c}'|\} \neq 0$ is always true and may be omitted, so (4.22) in notations (4.18) is

$$\lceil \mathbf{c}' \rceil |\underline{u}| \geq \lfloor \mathbf{c}' \rfloor |\overline{u}|. \tag{4.23}$$

We have obtained that in the case (4.19) the first condition of Lemma 4.1 is equivalent to

$$\exists \mathbf{c}' \quad (\mathbf{0} \subset \mathbf{c}' \ \text{ and } \ \lceil \mathbf{c}' \rceil |\underline{u}| \geq \lfloor \mathbf{c}' \rfloor |\overline{u}|). \tag{4.24}$$

Analogously, in the case (4.19) the second condition of Lemma 4.1 is equivalent to

$$\exists \mathbf{c}'' \quad (\mathbf{0} \subset \mathbf{c}'' \ \text{ and } \ \lceil \mathbf{c}'' \rceil |\overline{u}| \geq \lfloor \mathbf{c}'' \rfloor |\underline{u}|). \tag{4.25}$$

Since

$$|\underline{u}| \le |\overline{u}| \;\Rightarrow\; \lceil \mathbf{c}'' \rceil |\overline{u}| \ge \lfloor \mathbf{c}'' \rfloor |\underline{u}|$$

and

$$|\underline{u}| \ge |\overline{u}| \;\Rightarrow\; \lceil \mathbf{c}' \rceil |\underline{u}| \ge \lfloor \mathbf{c}' \rfloor |\overline{u}|,$$

combining the formulas (4.24) and (4.25) yields

$$\exists \mathbf{c} \in \{\mathbf{c}_{1k}, \ldots, \mathbf{c}_{mk}\} \quad \big(\mathbf{0} \subset \mathbf{c} \text{ and } \lceil \mathbf{c} \rceil \cdot \lfloor \mathbf{u} \rfloor \ge \lfloor \mathbf{c} \rfloor \cdot \lceil \mathbf{u} \rceil\big). \qquad \square$$

5. Reformulation of the Main Problem in Interval Terms

Let us return to the Main Problem.

In Kaucher interval arithmetic, the formula (3.1) for the set Ξ can be rewritten [11] in a simpler form

$$\Xi = \{\, x \in \mathbb{R}^n \mid (\operatorname{dual} \mathbf{A}^{\exists} + \mathbf{A}^{\forall}) \cdot x \subseteq \mathbf{b}^{\exists} + \operatorname{dual} \mathbf{b}^{\forall} \}$$

or, briefly,

$$\Xi = \Xi(\mathbf{A}^{\mathfrak{c}}, \mathbf{b}^{\mathfrak{c}}) = \{x \in \mathbb{R}^n \mid \mathbf{A}^{\mathfrak{c}} x \subseteq \mathbf{b}^{\mathfrak{c}}\}, \tag{5.1}$$

where $\mathbf{A}^{\mathfrak{c}} = (\mathbf{a}^{\mathfrak{c}}_{ij})$, $\mathbf{b}^{\mathfrak{c}} = (\mathbf{b}^{\mathfrak{c}}_i)$,

$$\mathbf{a}^{\mathfrak{c}}_{ij} = \begin{cases} \mathbf{a}_{ij}, & \text{if } a_{ij} \text{ is an A-parameter,} \\ \operatorname{dual} \mathbf{a}_{ij}, & \text{if } a_{ij} \text{ is an E-parameter,} \end{cases} \tag{5.2}$$

$$\mathbf{b}^{\mathfrak{c}}_{i} = \begin{cases} \operatorname{dual} \mathbf{b}_{i}, & \text{if } b_{i} \text{ is an A-parameter,} \\ \mathbf{b}_{i}, & \text{if } b_{i} \text{ is an E-parameter.} \end{cases} \tag{5.3}$$

THEOREM 5.1 Criterion of Inner Interval Estimate. *Let* $\mathbf{C} \in \mathbb{IR}^{m\times n}$, $\mathbf{d} \in \mathbb{IR}^{m}$. *A vector* $\mathbf{y} \in \mathbb{IR}^n$ *is an inner estimate for the solution set* $\Xi(\mathbf{C}, \mathbf{d}) = \{x \in \mathbb{R}^n \mid \mathbf{C}x \subseteq \mathbf{d}\}$ *if and only if* $\mathbf{C}\mathbf{y} \subseteq \mathbf{d}$.

Proof. We need to prove

$$\big(\forall y \in \mathbf{y}\ (\mathbf{C}y \subseteq \mathbf{d})\big) \iff \mathbf{C}\mathbf{y} \subseteq \mathbf{d}.$$

$\Longleftarrow$ The multiplication by an interval matrix is isotone, so $\forall y \in \mathbf{y}\ (\mathbf{C}y \subseteq \mathbf{C}\mathbf{y})$. If $\mathbf{C}\mathbf{y} \subseteq \mathbf{d}$, then $\forall y \in \mathbf{y}\ (\mathbf{C}y \subseteq \mathbf{d})$ due to transitivity of inclusion.

$\Longrightarrow$ The least upper bound for an upper bounded family satisfies a common nonstrict upper boundary, therefore

$$\big(\forall y \in \mathbf{y}\ (\mathbf{C}y \subseteq \mathbf{d})\big) \quad \Rightarrow \quad \left(\bigvee_{y \in \mathbf{y}} \mathbf{C}y \subseteq \mathbf{d}\right).$$

At the same time, it is easy to make sure that

$$\bigvee_{y \in \mathbf{y}} \mathbf{C}y = \mathbf{C}\mathbf{y}$$

(see, for example, [11]*). Indeed, using (4.1) and distributivity of the operation "$\vee$" with respect to addition (4.2), we get

$$\left(\bigvee_{y \in \mathbf{y}} (\mathbf{C} \cdot y)\right)_i = \bigvee_{y \in \mathbf{y}} (\mathbf{C} \cdot y)_i = \bigvee_{y \in \mathbf{y}} \sum_{j=1}^{n} \mathbf{c}_{ij} y_j = \bigvee_{y_1 \in \mathbf{y}_1} \bigvee_{y_2 \in \mathbf{y}_2} \cdots \bigvee_{y_n \in \mathbf{y}_n} \sum_{j=1}^{n} \mathbf{c}_{ij} y_j$$

$$= \sum_{j=1}^{n} \bigvee_{y_j \in \mathbf{y}_j} \mathbf{c}_{ij} y_j = \sum_{j=1}^{n} \mathbf{c}_{ij} \mathbf{y}_j = (\mathbf{C} \cdot \mathbf{y})_i.$$

This completes our proof. □

Remark. Theorem 5.1 allows one to reveal whether a proper interval vector is an inner estimate for the set $\Xi(\mathbf{C}, \mathbf{d})$ without actual finding this set. It turns out that, to make sure that the inner estimation is really the case, we need only to carry out certain operations in Kaucher arithmetic! Namely, we should multiply this vector by an interval matrix and check the inclusion of two vectors.

Now we can reformulate the Main Problem in purely interval terms as follows:

Given:

An interval matrix $\mathbf{A}^{c}$ and an interval vector $\mathbf{b}^{c}$ (defined by the input data through (5.2) and (5.3)).

Wanted:

A maximal proper solution of the interval inclusion $\mathbf{A}^{c}\mathbf{x} \subseteq \mathbf{b}^{c}$.

6. Reduced Problem

DEFINITION 6.1. The interval system $\mathbf{A}^{c}\,\mathbf{x} = \mathbf{b}^{c}$ is called *dualization equation* for the Main Problem. An interval vector $\mathbf{x}^{a}$ is an *(algebraic) solution of dualization equation* if $\mathbf{A}^{c} \cdot \mathbf{x}^{a} = \mathbf{b}^{c}$ in Kaucher arithmetic.

S. Shary proposed in [11] to find solutions of the Main Problem among solutions of the dualization equation, that is, to change searching solutions of the inclusion for searching solutions of the equation. The main reasons of such a reduction are:

* Shary's proof of the above fact in [11], being quite correct, is based on the equality (4.2). However, the property $\mathbf{x} \vee (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \vee \mathbf{y}) + (\mathbf{x} \vee \mathbf{z})$ Shary points out in the introductory part of the paper [11] is wrong, which confuses much readers.

1) Computing algebraic solutions to an equation is a computationally easier problem, and a number of efficient numerical methods for this purpose have been elaborated for the years elapsed [8], [10].*
2) A proper solution of dualization equation is often a solution of the Main Problem. For example,
   - L. Kupriyanova proved [4] that if all parameters of the Main Problem are E-parameters and each column of the initial interval matrix $\mathbf{A}$ has at least one component that does not contain zero, then any proper solution of the system $(\mathrm{dual}\,\mathbf{A}) \cdot \mathbf{x} = \mathbf{b}$ is a solution of the Main Problem.
   - S. Shary proved [9], [11] that if a proper solution of the dualization equation is maximal then it is a solution to the Main Problem.

Is there a necessary and sufficient condition for the solution of the dualization equation to be a solution of the Main Problem?

THEOREM 6.1. Criterion for a Solution of Dualization Equation to Be a Maximal Inner Interval Estimate. *If* $\mathbf{x}^a$ *is a proper solution of the dualization equation, then the following conditions are equivalent:*

(a) $\mathbf{x}^a$ *is a solution of the Main Problem;*

(b) $\mathbf{x}^a$ *is a maximal inner interval estimate for the set* Ξ*;*

(c) $\mathbf{x}^a$ *is a maximal solution of the interval inclusion* $\mathbf{A}^{\mathrm{c}} \cdot \mathbf{x} \subseteq \mathbf{b}^{\mathrm{c}}$*;*

(d) $\mathbf{x}^a$ *is a maximal solution of the dualization equation;*

(e) *multiplication by the matrix* $\mathbf{A}^{\mathrm{c}}$ *is upper strictly isotone in* $\mathbf{x}^a$.

Proof.

(a) $\Leftrightarrow$ (b) by reformulation of the Main Problem with the use of logical formulas and set inclusions (Section 3).

(b) $\Leftrightarrow$ (c) by reformulation of the Main Problem in the interval terms (Section 5).

Let us prove that (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e). We are given that $\mathbf{x}^a$ is a solution of the dualization equation and, obviously, a solution of the interval inclusion $\mathbf{A}^{\mathrm{c}} \cdot \mathbf{x} \subseteq \mathbf{b}^{\mathrm{c}}$. Let $\mathbf{y}$ be an interval vector and $\mathbf{x}^a \subset \mathbf{y}$. Since multiplication by an interval matrix is isotone, $\mathbf{A}^{\mathrm{c}}\mathbf{x}^a \subseteq \mathbf{A}^{\mathrm{c}}\mathbf{y}$. Therefore,

$$(\mathbf{A}^{\mathrm{c}}\mathbf{x}^a \not\supseteq \mathbf{A}^{\mathrm{c}}\mathbf{y}) \;\Leftrightarrow\; (\mathbf{A}^{\mathrm{c}}\mathbf{x}^a \neq \mathbf{A}^{\mathrm{c}}\mathbf{y}) \;\Leftrightarrow\; (\mathbf{A}^{\mathrm{c}}\mathbf{x}^a \subset \mathbf{A}^{\mathrm{c}}\mathbf{y}).$$

Substituting $\mathbf{b}^{\mathrm{c}}$ for $\mathbf{A}^{\mathrm{c}}\mathbf{x}^a$ in the first and second formulas of the above equivalence chain, we get

$$(\mathbf{A}^{\mathrm{c}}\mathbf{y} \not\subseteq \mathbf{b}^{\mathrm{c}}) \;\Leftrightarrow\; (\mathbf{A}^{\mathrm{c}}\mathbf{y} \neq \mathbf{b}^{\mathrm{c}}) \;\Leftrightarrow\; (\mathbf{A}^{\mathrm{c}}\mathbf{x}^a \subset \mathbf{A}^{\mathrm{c}}\mathbf{y}).$$

* One can download Shary's algorithms for computing algebraic solutions to interval linear systems from `http://www.ict.nsc.ru/ftp/ict/interval`, files `subdiff.c`, `subdiff.exe`, `re_split.c`, `re_split.exe`. They are "public domain."

Finally, applying Definitions 3.3 and 4.2 (of maximality and upper strict isotonicity) completes the proof. □

In Theorem 6.1, each of the conditions (b)–(e) is both necessary and sufficient for (a). The condition (e) can be easily verified using Theorem 4.1 when $\mathbf{x}^a$ is known. To put it differently, (e) is an easily verified *a posteriori* condition.

Still, sometimes we need to know whether the solution of the dualization equation is a solution of the Main Problem before the dualization equation is solved. Corollary 6.1 gives such a sufficient *a priori* condition.

COROLLARY 6.1. *If the proper interval matrix* $\mathbf{A}$ *of the initial Main Problem has, in each column, at least one component that does not contain zero, then every proper solution of the dualization equation is a solution of the Main Problem.*

Proof. Make use of the following facts:

1) Theorem 6.1 (the equivalence (a)⇔(e)),
2) Theorem 4.1,
3) pro $\mathbf{A}^c = \mathbf{A}$. □

If the matrix A of the initial Main Problem consists only of E-parameters, then the condition claimed in Corollary 6.1 is also necessary, and we arrive at

COROLLARY 6.2. *Let the matrix* A *of the initial Main Problem consist only of E-parameters. A proper solution of the dualization equation is a solution of the Main Problem if and only if the interval matrix* $\mathbf{A}$ *has, in each column, at least one component that does not contain zero.*

Proof. Make use of the following facts:

1) Theorem 6.1 (the equivalence (a)⇔(e)),
2) Theorem 4.1,
3) Since the matrix A of the initial Main Problem consists only of E-parameters, (5.2) implies $\mathbf{A}^c = \text{dual}\,\mathbf{A}$. The initial interval matrix $\mathbf{A}$ is proper, therefore all the components of the matrix $\mathbf{A}^c$ are backward intervals and cannot strictly include zero. We can use this fact applying Theorem 4.1. □

Remark. In particular, Corollary 6.2 is applicable for maximal inner interval estimation of the *united solution set*

$$\{x \in \mathbb{R}^n \mid (\exists A \in \mathbf{A})(\exists b \in \mathbf{b})\,(Ax = b)\},$$

as well as of the *controllable solution set*

$$\{x \in \mathbb{R}^n \mid (\forall b \in \mathbf{b})(\exists A \in \mathbf{A})\,(Ax = b)\}.$$

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