NORTH-HOLLAND

Calculation of Exact Bounds for the Solution Set of Linear Interval Systems

C. Jansson
Technische Universität Hamburg-Harburg
Arbeitsbereich Informatik III
Eissendorfer Strasse 38
D-21071 Hamburg, Germany

Dedicated to Professor Helmut Brakhage.

Submitted by Richard S. Varga

ABSTRACT

This paper presents some topological and graph theoretical properties of the solution set of linear algebraic systems with interval coefficients. Based on these properties, we describe a method which, in a finite number of steps, either calculates exact bounds for each component of the solution set, or finds a singular matrix within the interval coefficients. The calculation of exact bounds of the solution set is known to be NP-hard. Our method needs p calls of a polynomial-time algorithm, where p is the number of nonempty intersections of the solution set with the orthants. Frequently, due to physical or economical requirements, many variables do not change the sign. In those cases p is small, and our method works efficiently. © *Elsevier Science Inc., 1997*

1. INTRODUCTION

A real linear interval system is defined as a family of real linear systems where the coefficients of the system matrix and the righthand side vary between given lower and upper bounds. The corresponding solution set is defined as the set of all solutions of this family and is generally nonconvex. This causes difficulties in computing bounds for the solution set.

LINEAR ALGEBRA AND ITS APPLICATIONS 251:321–340 (1997)

0024-3795/97/$17.00
SSDI 0024-3795(96)00681-8

Many methods are known for calculating lower and upper bounds for each component of the solution set; the interested reader is referred especially to the textbooks of Alefeld and Herzberger [1], Moore [9], and Neumaier [10]. Most of these methods require that the interval system matrix be strongly regular (see for example Neumaier [10, Chapter 4]), and usually at least an approximate inverse has to be calculated. During the last decade, in our opinion, great strides have been made in the following directions:

First, the calculation of bounds for the solution set of sparse linear and nonlinear systems. Here, we mention the methods of Alefeld and Platzöder [2] and Schwandt [25] for special sparse systems which are related for example to M-matrices, and the methods of Rump [23, 24], which work for general sparse systems.

Secondly, the progress made mainly by Rohn. On the one side, Rohn gave many complexity results related to interval problems, and proved their NP-hardness (see [15, 18, 19, 20–22]). On the other side, his papers (see especially [16, 17], and Neumaier [10, Chapter 6]) provide a deep insight into the algebraic properties of linear interval systems, and present algorithms for computing exact bounds and for finding a singular matrix in a given interval matrix.

Rohn's approach [17] for computing exact bounds is mainly based on solving special boundary problems of the linear interval system, such that the convex hull of the solutions of these boundary problems is equal to the convex hull of the solution set. He assumed in his approach that the interval system matrix $[A]$ is regular; i.e., all $A \in [A]$ are regular. Shary [26] proposed a branch-and-bound scheme for calculating bounds of the solution set.

Oettli [12] proved that the intersection of the solution set of a real linear interval system with each orthant is a convex polyhedron. He proposed to use linear programming for computing exact lower and upper bounds of the solution set in *each* orthant. Since there are 2^n orthants, this method can be used only for small dimension n.

In this paper, we present a new algorithm which is related to Oettli's work [12]. This algorithm is based on some topological and graph-theoretical properties of the solution set, and can be viewed as a graph search method applied to an implicitly defined graph.

In a recent paper Rohn [19] proved that, under the conjecture P $\neq$ NP, there exists no polynomial-time algorithm which for each linear interval system:

(a) calculates bounds for the solution set provided it is bounded;

(b) gives an error message provided the solution set is unbounded.

Our algorithm satisfies (a) and (b) for each linear interval system. Therefore, our algorithm cannot calculate bounds for each linear interval system in polynomial time. But the algorithm is strongly related to the structure of the solution set, and additionally has the nice properties that for each linear interval system:

(c) *exact* bounds for each component of the solution set are calculated if and only if the solution set is bounded;

(d) exact bounds can be calculated by p calls of a polynomial-time algorithm, where p is the number of orthants intersecting the corresponding solution set.

To the author, it is not known if there is any other algorithm which satisfies these conditions. It is not assumed that $[A]$ is regular or strongly regular; the algorithm shows *a posteriori* the regularity or singularity of $[A]$. Frequently in practice, due to physical or economic properties, many variables do not change the sign and the solution set intersects only few orthants. In such situations our algorithm may be applied.

We use the following notation. The coefficients of real $m \times n$ matrices A are denoted by A_{ij}, its columns by $A_{.j}$, and its rows by $A_{i.}$. $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{m \times n}$ denote the sets of all real numbers, real n-vectors, and real $m \times n$ matrices, respectively. $I\mathbb{R}$, $I\mathbb{R}^n$, $I\mathbb{R}^{m \times n}$ denote the sets of real compact intervals $[a] = [\underline{a}, \bar{a}]$, real interval vectors $[b] = [\underline{b}, \bar{b}]$, and real $m \times n$ interval matrices $[A] = [\underline{A}, \bar{A}]$, respectively. We also shall use the center-radius notation, that is,

$$[A] = [A^c - \Delta, A^c + \Delta],$$

with the center matrix $A^c := \frac{1}{2}(\underline{A} + \bar{A})$, and the radius matrix $\Delta := \frac{1}{2}(\bar{A} - \underline{A})$. Analogously, interval vectors are described in the form

$$[b] = [b^c - \delta, b^c + \delta].$$

$\rho(A) := \max\{|\lambda| \mid \lambda \text{ an eigenvalue of } A\}$ denotes the spectral radius of A. An interval matrix $[A]$ is called regular if each $A \in [A]$ is nonsingular. Otherwise $[A]$ is called singular. For an arbitrary set $\Sigma \subseteq \mathbb{R}^n$ the components are denoted by $(\Sigma)_k := \{x_k \mid x \in \Sigma\}$, and the interval hull is denoted by $\diamond(\Sigma) := \cap\{[v] \in I\mathbb{R}^n \mid \Sigma \subseteq [v]\}$.

The paper is organized in the following way. In Section 2 some basic topological properties for linear interval systems are discussed. In particular, it is shown that if a nonempty, connected component of the solution set is bounded, then the solution set is equal to this component, and the corresponding interval system matrix is regular. In Section 3 a finite representation graph of the solution set is introduced. This graph describes the intersections of the solution set with the orthants. In Section 4 our method is presented, and some examples are given. Section 5 contains some conclusions.

2. TOPOLOGICAL PROPERTIES

A system of real linear interval equations is defined as a family of linear equations

$$Ax = b \qquad \text{with} \quad A \in [A], \quad b \in [b], \tag{2.1}$$

where $[A] \in I\mathbb{R}^{n\times n}$, $[b] \in I\mathbb{R}^{n}$. Formally, we use for such a system the notation

$$[A]x = [b]. \tag{2.2}$$

The corresponding solution set is defined by

$$\Sigma := \Sigma([A],[b]) := \{x \in \mathbb{R}^n \mid Ax = b,\ A \in [A], b \in [b]\}. \tag{2.3}$$

Here, we do not suppose $[A]$ to be regular. In this section, we develop some topological properties of the solution set of a system of linear interval equations. These results are basic for our method described in Section 4.

The solution set Σ, in general, is not convex and has a complicated shape. See Figure 1, which is taken from Neumaier [10].

Moreover, Σ need not be connected or bounded. This is demonstrated by the linear interval equation $[-1,1]x = 1$, which has the solution set $\Sigma = (-\infty, -1] \cup [1, \infty)$. The following theorem shows that at least Σ is a closed set.

THEOREM 2.1. *The solution set Σ is closed.*

Proof. In the case $\Sigma = \emptyset$ nothing is to be proved. Therefore, let $\tilde{x} \in \mathbb{R}^n$ be an accumulation point of Σ. Then there exist sequences (A^k), (b^k), (x^k) with $A^k \in [A]$, $b^k \in [b]$, $A^k x^k = b^k$, and $x^k \to \tilde{x}$.

Because the sequences (A^k), (b^k) are contained in compact sets, there exists convergent subsequences (A^{k_j}), (b^{k_j}) with $A^{k_j} \to \tilde{A} \in [A]$, $b^{k_j} \to \tilde{b} \in$

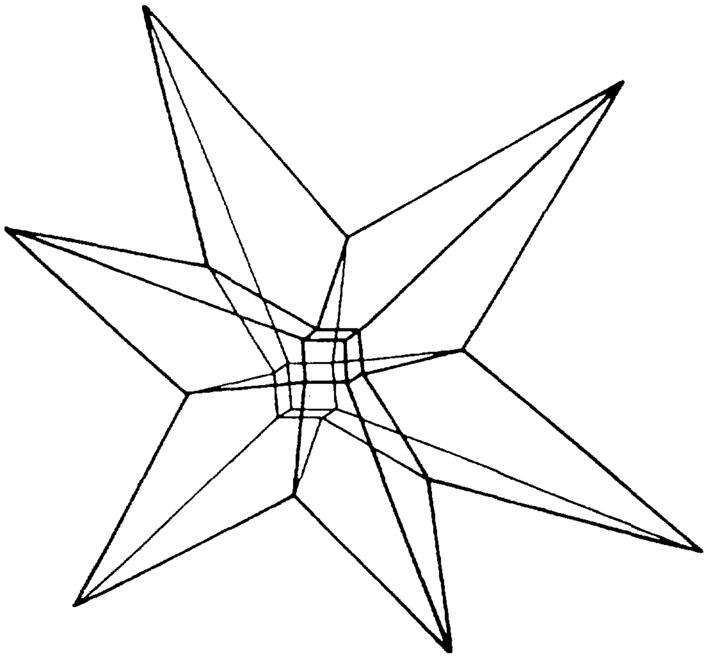

FIG. 1. A projection of a three-dimensional solution set.

$[b]$, $A^{k_j}x^{k_j} = b^{k_j}$, and $x^{k_j} \to \tilde{x}$. The equation

$$\tilde{b} = \lim_{j\to\infty} b^{k_j} = \lim_{j\to\infty} A^{k_j}x^{k_j}$$

$$= \lim_{j\to\infty} A^{k_j} \cdot \lim_{j\to\infty} x^{k_j} = \tilde{A}\tilde{x}$$

yields $\tilde{x} \in \Sigma$. ■

This result was mentioned in [4] but not proved. We mention that this result is also a consequence of Theorem 3.1, which is due to Oettli and Prager [13]. For systems of linear interval equations with regular system matrix $[A]$, more properties hold.

THEOREM 2.2. *Let $[A] \in I\mathbb{R}^{n\times n}$ be regular. Then, for each right-hand side $[b] \in I\mathbb{R}^n$, the corresponding solution set Σ is connected and compact.*

Proof. The function

$$h: [A] \times [b] \to \mathbb{R}^n \qquad \textit{with} \quad h(A, b) := A^{-1}b$$

is well defined, and continuous on a compact, connected set, because $[A]$ is regular. Σ is the range of h. Hence, Σ is connected and compact. ∎

The next theorem shows under mild assumptions that each topologically connected component of Σ is unbounded if and only if Σ is unbounded. This theorem is fundamental for our graph search method, and gives a topological alternative statement for linear interval systems. In fact, our method calculates a connected component of Σ.

THEOREM 2.3. *Suppose that Σ is nonempty. Then exactly one of the following two statements is true*:

(i) *Σ is bounded*;
(ii) *each topologically connected component of Σ is unbounded.*

Proof. If Σ is bounded, then (ii) is not valid. Now, let Σ be unbounded, and assume that there exists a nonempty bounded, topologically connected component $\hat{\Sigma}$ of Σ. Then there exist $A^0 \in [A]$, $b \in [b]$, and $x^0 \in \hat{\Sigma}$ with $A^0x^0 = b$. The matrix A^0 is regular; otherwise the solution set of $A^0x = b$ would be unbounded, yielding the unboundedness of $\hat{\Sigma}$.

Because Σ is not bounded, by Theorem 2.2 it follows that $[A]$ is not regular. Therefore, let $A^1 \in [A]$ be singular.

Because $[A]$ is convex, the set of matrices

$$A(\lambda) := \lambda A^1 + (1 - \lambda) A^0, \; \lambda \in [0, 1]$$

is contained in $[A]$, and $A(0) = A^0$, $A(1) = A^1$. Let

$$\tilde{\lambda} := \inf\{\lambda \in [0, 1] \mid A(\lambda) \text{ is singular}\}.$$

Because $A^0 = A(0)$ is regular, all sufficiently small perturbations of A^0 are regular. Hence, $\tilde{\lambda} > 0$, and $A(\tilde{\lambda})$ must be singular.

Let $x(\lambda)$ be the unique solution of

$$A(\lambda)x(\lambda) = b, \qquad \lambda \in [0, \tilde{\lambda}).$$

Then $x(\lambda) = A^{-1}(\lambda) \cdot b$ depends continuously on λ for all $\lambda \in [0, \tilde{\lambda})$, and all $x(\lambda)$ are connected to x^0. Therefore, $x(\lambda)$ is contained in the connected component $\hat{\Sigma}$ of Σ for all $\lambda \in [0, \tilde{\lambda})$.

Let (λ_j) be a sequence with $\lambda_j \to \tilde{\lambda}$, $\lambda_j < \tilde{\lambda}$. Because $\hat{\Sigma}$ is bounded and $x(\lambda_j) \in \hat{\Sigma}$, it follows that there exists a subsequence $(x(\lambda_j))$ that converges to $\tilde{x} \in \mathbb{R}^n$. Because of Theorem 2.1, Σ is closed, and therefore each connected component of Σ is closed. Hence, $\tilde{x} \in \hat{\Sigma}$. Let $(x(\lambda_j))$ be the convergent subsequence. Then

$$A(\tilde{\lambda})\tilde{x} = \lim_{\lambda_j \to \tilde{\lambda}} A(\lambda_j) \cdot \lim_{\lambda_j \to \tilde{\lambda}} x(\lambda_j)$$

$$= \lim_{\lambda_j \to \tilde{\lambda}} A(\lambda_j) \cdot x(\lambda_j) = b$$

Hence, the equation $A(\tilde{\lambda})x = b$ has a solution $\tilde{x} \in \hat{\Sigma}$.

Because $A(\tilde{\lambda})$ is singular, the solution set of $A(\tilde{\lambda})x = b$ is unbounded and connected with $\tilde{x} \in \hat{\Sigma}$. This contradicts our assumption that $\hat{\Sigma}$ is bounded. ■

By setting $\tilde{A} := A(\tilde{\lambda})$, from the proof of the above theorem we have immediately:

COROLLARY 2.4. *Let $b \in \mathbb{R}^n$. If $[A]$ is not regular but contains a regular matrix $A^0 \in [A]$, then there exists a singular matrix $\tilde{A} \in [A]$ such that the solution set of $\tilde{A}x = b$ is unbounded.*

The following theorem summarizes some properties of the solution set Σ, which are important for our method.

THEOREM 2.5. *Let Σ be nonempty. If a connected component $\hat{\Sigma}$ of Σ is bounded, then the following conditions hold:*

(i) *Σ is compact;*
(ii) *$[A]$ is regular;*
(iii) *Σ is connected, and $\Sigma = \hat{\Sigma}$.*

Proof. (i): By Theorem 2.3 it follows that Σ is bounded, and Theorem 2.1 yields (i).

(ii): Since Σ is nonempty, there exists $A^0 \in [A]$, $b \in [b]$ such that $A^0x = b$ contains a solution. Because of (i), A^0 is regular.

Assume that $[A]$ is not regular. Then using Corollary 2.4, there exists a singular matrix $\tilde{A} \in [A]$ such that $\tilde{A}x = b$ has an unbounded solution set. This contradicts (i). Hence $[A]$ is regular.

(iii): follows by (ii) and Theorem 2.2. ■

Rohn [19] proved that $[A]$ is regular if and only if the solution set is bounded, provided that $[A]$ contains at least one regular matrix. Therefore, the proof of Theorem 2.5 can be shortened by using his theorem.

3. THE REPRESENTATION GRAPH

So far, we have considered only topological properties of the solution set Σ. The following theorem is due to Oettli and Prager [13], and allows the description of Σ from the algebraic point of view.

THEOREM 3.1. *The solution set Σ can be described in the form*

$$\Sigma = \{x \in \mathbb{R}^n \,|\, |A^c x - b^c| \leqslant \Delta |x| + \delta\}. \tag{3.1}$$

In general, Σ is not convex (for examples see [3, 11]). However, it has been observed that the intersection of Σ with each orthant is a convex polyhedron (see for example Beeck [4], Rohn [16, 17]).

To see this, let $\{-1, 1\}^n$ denote the set of all sign vectors with components equal to -1 or 1. For $s \in \{-1, 1\}^n$ the diagonal matrix with diagonal vector s is denoted by S. For $x \in \Sigma$ let $s = s(x)$ be the sign vector of x, that is,

$$s_i = \begin{cases} 1 & \text{if} \quad x_i \geqslant 0, \\ -1 & \text{otherwise}. \end{cases}$$

Then, using Theorem 3.1, it follows that the intersection of Σ with the orthant

$$\mathbb{R}^n(s) := \{x \in \mathbb{R}^n \,|\, Sx \geqslant 0\} \tag{3.2}$$

is given by

$$\begin{aligned} (A^c - \Delta S)x &\leqslant b^c + \delta, \\ (A^c + \Delta S)x &\geqslant b^c - \delta, \\ Sx &\geqslant 0. \end{aligned} \tag{3.3}$$

Therefore

$$\Sigma(s) := \Sigma \cap \mathbb{R}^n(s)$$

is a convex polyhedron described by the system of inequalities (3.3), and Σ is the union of at most 2^n convex polyhedrons.

Our method, described in Section 4, can be viewed as a graph search method (cf. for example [14]) applied to the following implicitly defined graph: According to the solution set Σ, we define a graph $G = (V, E)$ with the set of nodes

$$V := \left\{s \in \{-1, 1\}^n \middle| \Sigma(s) \neq \varnothing\right\}, \tag{3.4}$$

and the set edges

$$E := \left\{\{s, t\} \middle| \begin{array}{l} s, t \in V,\ s \text{ and } t \\ \text{differ in exactly one component}, \\ \text{and } \Sigma(s) \cap \Sigma(t) \neq \varnothing \end{array}\right\}. \tag{3.5}$$

We call G the representation graph of the solution set Σ.

Two nodes s, t are called adjacent if $\{s, t\} \in E$. For $s \in V$, the set $N(s)$ denotes the set of all nodes $t \in V$ which are adjacent to s. The representation graph G is given implicitly by the solution set Σ, and in general G is not connected.

We first establish a basic relationship between the solution set Σ and its representation graph.

THEOREM 3.2. (a) *Each nonempty topologically connected component* $\hat{\Sigma}$ *of* Σ *can be represented in the form*

$$\hat{\Sigma} = \bigcup\{\Sigma(s) | s \in U\} \tag{3.6}$$

where U *is the node set of a connected component of the representation graph* G.

(*b*) *If* Σ *is nonempty and bounded, then* $G = (V, E)$ *is a connected graph and*

$$\Sigma = \bigcup\{\Sigma(s) | s \in V\}. \tag{3.7}$$

Proof. (a): From (3.3) it follows that for each orthant $\mathbb{R}^n(s)$

$$\hat{\Sigma} \cap \mathbb{R}^n(s) = \Sigma \cap \mathbb{R}^n(s) = \Sigma(s)$$

is a convex polyhedron. Since $\hat{\Sigma}$ is connected, two arbitrary points x^0, $x^1 \in \hat{\Sigma}$ can be connected by a (nonlinear) curve $x(\lambda)$ such that $\lambda \in [0, 1]$, $x^0 = x(0)$, $x^1 = x(1)$. Since $\hat{\Sigma} \cap \mathbb{R}^n(s)$ is a convex polyhedron, in each orthant $x(\lambda)$ can be chosen as a straight line. Moreover, w.l.o.g. we can assume that if $x(\lambda)$ passes from $\Sigma(s)$ to $\Sigma(t)$, then the corresponding component of $x(\lambda)$, falling to zero, does not stay zero for some open interval. That is, if $x_{i}^{*}(\lambda^*) = 0$ for some $i^* \in \{1, \dots, n\}$, $\lambda^* \in [0, 1]$, then $x_{i^*}(\lambda^0) \cdot x_{i^*}(\lambda^1) < 0$ provided λ^0, λ^1 are sufficiently close to λ^* and $\lambda^0 < \lambda^* < \lambda^1$.

We call $\lambda^* \in [0, 1]$ a critical point if at least one component $x_{i^*}(\lambda^*)$ is equal to zero and changes sign in λ^*; that is, there exist λ^0, $\lambda^1 \in [0, 1]$ which are sufficiently close to λ^* such that $\lambda^0 < \lambda^* < \lambda^1$ and $x_{i^*}(\lambda^0) \cdot x_{i^*}(\lambda^1) < 0$. Obviously, if the curve $x(\lambda)$ leaves an orthant, then the corresponding parameter λ must be a critical point.

Below, we show that at critical points λ^*, the sign vectors $s(x(\lambda^0))$ and $s(x(\lambda^1))$ are connected by a path in G. Since the sign vector remains constant at points which are not critical, it follows then that to the curve $x(\lambda)$ there corresponds a path $s^0, \dots, s^l$ in G such that $s^0 = s(x^0)$ and $s^l = s(x^1)$. Hence, each topologically connected component $\hat{\Sigma}$ of Σ corresponds to a connected component of G. This proves Theorem 3.2(a).

First, we discuss the simple case where exactly one component $x_i(\lambda^*)$ changes sign. In this case $s := s(x(\lambda^0))$ and $t := s(x(\lambda^1))$ differ in exactly one component, and $x(\lambda^*) \in \Sigma(s) \cap \Sigma(t)$. From (3.5) it follows that s, t is a path in G.

Now, we discuss the more complicated degenerate case where at least two components of $x(\lambda^*)$ change sign (see Figure 2). Because in (3.5) two adjacent nodes differ in exactly one component, it follows that $s(x(\lambda^0))$, $s(x(\lambda^1))$ are not adjacent. Hence, at a first glance, a path in G between these two nodes is not obvious.

Let U^* denote the set of all sign vectors s^* that satisfy

$$s_i^* = \begin{cases} 1 & \text{if} \quad x_i(\lambda^*) > 0, \\ -1 & \text{if} \quad x_i(\lambda^*) < 0, \\ 1 \text{ or } -1 & \text{if} \quad x_i(\lambda^*) = 0. \end{cases}$$

Then $x(\lambda^*) \in \Sigma(s^*)$ for all $s^* \in U^*$, and because $\Sigma(s^*)$ is nonempty, (3.4) yields $U^* \subseteq V$. Because U^* contains *all* vectors with components 1 or -1 if

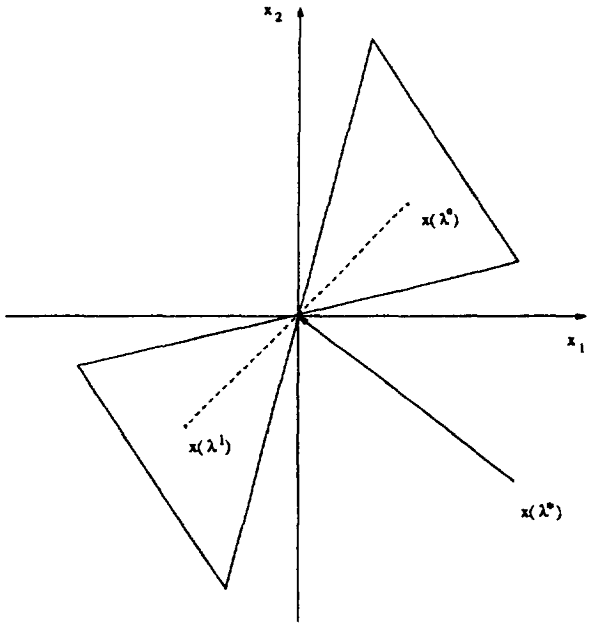

FIG. 2. Two components of $x(\lambda)$ change sign at λ^*.

$x_i(\lambda^*) = 0$, it follows that U^* is connected node set in G. Since $s(x(\lambda^0))$, $s(x(\lambda^1)) \in U^*$, it follows that these two nodes are connected by a path in G.

(b): Let $s, t \in V$. Then by (3.4) $\Sigma(s)$ and $\Sigma(t)$ are nonempty. Let $x^0 \in \Sigma(s)$, $x^1 \in \Sigma(t)$. Because Σ is nonempty and bounded, Theorem 2.5 shows that Σ is connected. Hence, there exists a curve $x(\lambda)$, $\lambda \in [0,1]$, such that $x(0) = x^0$, $x(1) = x^1$. Above, we have shown that there corresponds to this curve a path which connects s, t. Therefore G is connected. ■

The following example demonstrates that, for singular $[A]$, G need not be connected. For the linear interval system

$$x_1 - x_2 = 0,$$

$$[-1,1] \cdot x_2 = 1,$$

the solution set Σ can be described by varying the parameter $\alpha \in [-1,1]$:

If $\alpha \in (0,1]$, then $x_1(\alpha) = x_2(\alpha) = 1/\alpha \geqslant 1$.
If $\alpha = 0$, then the system has no solution.
If $\alpha \in [-1,0)$, then $x_1(\alpha) = x_2(\alpha) = 1/\alpha \leqslant -1$.

Therefore, the corresponding representation G has the node set $V = \{(1,1),(-1,-1)\}$, and from (3.5) it follows that G is not connected.

The following theorem shows how to calculate for a given node $s \in V$ the set of adjacent nodes $N(s)$.

THEOREM 3.3. *Let $s \in V$. Then the following two conditions are equivalent*:

(i) $t \in N(s)$;

(ii) *there exists a* $k \in \{1, \ldots, n\}$ *such that* $t_k = -s_k$, $t_i = s_i$ *for* $i = 1, \ldots, n$, $i \neq k$, *and* $\underline{f}^* := \min_{x \in \Sigma(s)} s_k x_k = 0$.

Proof. By the definition (3.5), it follows that $t \in N(s)$ if and only if t and s differ exactly in one coordinate k, and because

$$\Sigma(s) \subseteq \{x \in \mathbb{R}^n | s_k x_k \geqslant 0\},$$

$$\Sigma(t) \subseteq \{x \in \mathbb{R}^n | -s_k x_k \geqslant 0\},$$

it follows that $x \in \Sigma(s) \cap \Sigma(t)$ if and only if $\min_{x \in \Sigma(s)} s_k x_k = 0$. In the latter case, x is an optimal solution. ■

Now we examine how to calculate exact bounds for $\Sigma(s)$.

THEOREM 3.4. *Let $s \in V$. Then*:

(i) *The quantity* $s_k \cdot \min_{x \in \Sigma(s)} s_k x_k$ *is an exact lower* [*upper*] *bound of the kth component* $(\Sigma(s))_k$ *provided* $s_k = +1$ [$s_k = -1$].

(ii) *The quantity* $s_k \cdot \max s_k x_k$ *is an exact upper* [*lower*] *bound of the kth component* $(\Sigma(s))_k$ *provided* $s_k = +1$ [$s_k = -1$].

(iii) $\Sigma(s)$ *is unbounded if and only if there exists a* $k \in \{1, \ldots, n\}$ *such that* $\max_{x \in \Sigma(s)} s_k x_k$ *is unbounded.*

Proof. Noticing that $\Sigma(s) \subseteq \{x \in \mathbb{R}^n \,|\, s_i x_i \geqslant 0$ for $i = 1, \ldots, n\}$, some simple computations yield (i), (ii), (iii). ■

The last two theorems show that the set of adjacent nodes as well as exact lower and upper bounds of $\Sigma(s)$ can be calculated by using linear programming techniques.

4. THE METHOD

From our previous discussion it follows that we can find a connected component $\hat{\Sigma}$ of Σ by using a graph search method in the following way:

1. We compute a starting node $s \in V$ by solving a linear system $Ax = b$, where $A \in [A]$, $b \in [b]$, and define $s = s(x)$ to be the sign vector of this solution x. Usually, we solve the midpoint system $A^c x = b^c$.
2. Subsequently, all nodes $N(s)$ are calculated by using Theorem 3.3, 3.4 then all nodes of $N(s')$ with $s' \in N(s)$ are calculated, and so on.

Obviously, the graph search method terminates by calculating a set of nodes of V which are connected to the starting node s in a finite number of steps. Moreover, solving the minimization and maximization problems by using Theorem 3.4 yields exact lower and upper bounds $[x(s)] = [\underline{x}(s), \bar{x}(s)]$ of $\Sigma(s)$. These observations yield the following algorithm:

Algorithm GRAPH SEARCH
Input: $[A]$, $[b]$;

Output: list U, and a set of boxes $\{[x(s)] \mid s \in U\}$

solve the midpoint system $A^c x = b^c$;
if A^c is singular **then** STOP: $[A]$ is singular;
set $s := s(x)$; (s is the starting node)
list $L := \{s\}$; (L is a working list)
list $U := \emptyset$; (U shall contain all nodes of V connected to s)
while $L \neq \emptyset$ **do**
  remove an element $s \in L$;
  set $U := U \cup \{s\}$; (update U)
  initialize $\Sigma(s)$ by using (3.3);
  for $k = 1, \dots, n$ **do**
    solve the lp problems $\max\{s_k x_k \mid x \in \Sigma(s)\}$;
    if $\max\{s_k x_k \mid x \in \Sigma(s)\}$ is unbounded
      then GRAPH SEARCH is STOPped: $[A]$ is singular;
    else store the optimal values on the corresponding
    components of $[x(s)]$; (cf. Theorem 3.4)
  end
  for $k = 1, \dots, n$ **do**
    solve the lp problems $\min\{s_k x_k \mid x \in \Sigma(s)\}$;
    if $\min\{s_k x_k \mid x \in \Sigma(s)\} = 0$
      then store according to Theorem 3.3 the
        adjacent node t in list $N(s)$;
    store the optimal values on the corresponding component of $[x(s)]$; (cf.

Theorem 3.4);
end
$L := L \cup \{N(s) - U\}$;
end

THEOREM 4.1. *Algorithm* GRAPH SEARCH *satisfies the following statements*:

(i) *The algorithm terminates after a finite number of steps.*

(ii) *If the algorithm is stopped because the linear programming problem* $\max\{s_k x_k \mid x \in \Sigma(s)\}$ *is unbounded, then* $[A]$ *is singular. Otherwise, the set* U, *calculated by* GRAPH SEARCH, *is equal to the set of nodes* V *of the representation graph* G, G *is connected, and* $[A]$ *is regular. In the latter case,*

$$\diamond(\Sigma(s)) = [x(s)] \qquad \text{for all} \quad s \in V, \tag{4.1}$$

That is, the smallest interval vector containing the intersection of Σ *with the corresponding orthant is calculated.*

(iii) *For each component* $k = 1, \ldots, n$ *the equations*

$$\min\{x_k \mid x \in \Sigma\} = \min\{(\underline{x}(s))_k \mid s \in V\}, \tag{4.2}$$

$$\max\{x_k \mid x \in \Sigma\} = \max\{(\bar{x}(s))_k \mid s \in V\} \tag{4.3}$$

are fulfilled.

(iv) *The algorithm either calculates in polynomial time exact bounds for the solution set* Σ *or proves that* $[A]$ *is singular, provided the number of intersections of* Σ *with the orthants is polynomial bounded.*

Proof. (i): In each step of the while loop a finite number of linear programming problems are solved. There are only a finite number of sign vectors s, which are stored in U by using $U := U \cup \{s\}$ in each step of the while loop. Because $L := L \cup \{N(s) - U\}$, nodes s that are stored in U are not repeated. Therefore, the algorithm terminates in a finite number of steps.

(ii): If the algorithm is stopped because A^c is singular or a linear programming problem $\max\{s_k x_k \mid x \in \Sigma(s)\}$ is unbounded, then Theorem 3.4(iii) and Theorem 2.5 yield the singularity of $[A]$.

Now, we suppose that GRAPH SEARCH terminates in the while loop with $L = \emptyset$. It follows that the graph search method [which computes in each step of the while loop the adjacent nodes $N(s)$] stores in U all nodes of our representation graph G which are connected to the starting node s. Therefore, using Theorem 3.2, the connected component $\hat{\Sigma}$ corresponds to the connected component U calculated by GRAPH SEARCH. Since $\hat{\Sigma}$ is bounded from Theorem 2.5 and Theorem 3.2, it follows that $\Sigma = \hat{\Sigma}$, G is connected, and $[A]$ is regular.

Equation (4.1) follows immediately from Theorem 3.4.

(iii): is an immediate consequence of (ii).

(iv): Since linear programming problems can be solved in polynomial time, and the number of nonempty intersections of Σ with the orthants is equal to the number of nodes in V, this statement follows immediately. ■

Frequently, many of the variables, due to physical or economical requirements, do not change the sign, that is, the solution set intersects only a few orthants. In those cases our algorithm yields exact bounds w.r.t. each orthant in polynomial time, or proves that the interval matrix $[A]$ is singular. In the latter case, using the unbounded solution of the corresponding linear programming problem, a singular matrix $A \in [A]$ can be constructed by using the corresponding simplex tableaus. We will not go into detail here. Theorem 4.1 shows that our method is useful especially for those problems where the number of nonempty intersections of Σ with the orthants is not too large. The performability of our method does not depend on the radii Δ or δ.

At a first glance, the computational costs seemed to be very large, since in each orthant, containing some points of the solution set, $2n$ linear programming problems have to be solved. But, if the optimal solution of $\max\{s_k x_k \mid x \in \Sigma(s)\}$, $\min\{s_k x_k \mid x \in \Sigma(s)\}$ is used as a starting point for $\max\{s_{k+1} x_{k+1} \mid x \in \Sigma(s)\}$, $\min\{s_{k+1} x_{k+1} \mid x \in \Sigma(s)\}$, respectively, then, because of the special structure of $\Sigma(s)$ [cf. (3.3)], in many cases (this was observed in our experiments) only about $O(n^2)$ operations are necessary. Hence the $2n$ linear programming problems can be solved in $O(n^3)$ operations, and the total costs are about $O(|V| \cdot n^3)$. Nevertheless, it should be mentioned that there are very simple examples such that exponentially many orthants must be visited. This is the case if the right-hand side $[b]$ contains the zero vector; then $0 \in \Sigma$ and $\Sigma(s)$ is nonempty for all $s \in \{-1, 1\}^n$.

So far our algorithm calculates exact bounds only if we assume that the execution is done in exact arithmetic or multiple-precision arithmetic. If floating-point arithmetic is in use, then, due to roundings, the bounds are in general not correct, at least in the last digits. But if we apply verification methods for solving the corresponding real linear programming problems (see for example [5, 7]), then verified bounds can be calculated. The additional costs are small and, in almost all cases, do not change the complexity of the algorithm. If Kulisch's arithmetic [8] is in use, then defect corrections can be calculated exactly, and in almost all cases the bounds can be calculated with last-significant-bit accuracy.

Our algorithm is written in MATLAB and uses the IEEE double precision floating-point arithmetic. In the following, we display five decimal digits in our examples.

To demonstrate how our algorithm works, we consider the following system of linear interval equations which is due to Rohn [17]:

$$[1, 1000]x_1 + [1, 1000]x_2 = [1, 2],$$

$$[-1000, -1] + [1, 1000]x_2 = [3, 4].$$

Then the solution of the midpoint system is equal to $x = (-0.001998, 0.004995)^T$, yielding $s = s(x) = (-1, 1)^T$, $L := \{s\}$, and $U := \emptyset$.

The first execution of the while loop gives

$$L = \emptyset, \qquad U = \{(-1, 1)^T\},$$

$$\left[x\left((-1, 1)^T\right)\right] = \begin{pmatrix} [-3.99500, 0] \\ [0.001002, 3.99800] \end{pmatrix},$$

$$N\left((-1, 1)^T\right) = \{(1, 1)^T\}, \quad \text{and} \quad L = \{(1, 1)^T\}.$$

The second execution of the while loop gives

$$L = \emptyset, \qquad U = \{(-1, 1)^T, (1, 1)^T\},$$

$$x\left((1, 1)^T\right) = \begin{pmatrix} [0, 1.99500] \\ [0.00300, 2.00000] \end{pmatrix}.$$

We have $N((1, 1)^T) = \{(-1, 1)^T\}$, and since $N((1, 1)^T) - U = \emptyset$, it follows that $L = \emptyset$.

Therefore our algorithm terminates with proving the regularity of $[A]$ and calculating the exact bounds

$$\Sigma \subseteq \left[x\left((-1, 1)^T\right)\right] \cup \left[x\left((1, 1)^T\right)\right] \subseteq \begin{pmatrix} [-3.99500, 1.99500] \\ [0.001002, 3.99800] \end{pmatrix}.$$

The spectral radius $\rho(|(A^c)^{-1}| \cdot \Delta) = 1.9960$, the smallest singular value of A^c is equal to 707.81, and the largest singular value of Δ is equal to 999.

Thus, neither the strong regularity criterion $\rho(|(A^c)^{-1}| \cdot \Delta) < 1$ nor the regularity criterion of Rump, $\sigma_{\min}(A^c) > \sigma_{max}(\Delta)$ (cf. [23, 24]), is fulfilled.

Now, we generalize the above example for higher dimensions in the following way. The center A^c and the radius matrix Δ have the same structure:

$$\begin{pmatrix}
\times & & & & & & \times & \times & \times \\
 & \times & & & & & & \times & \times \\
 & & \times & & & & & & \times \\
 & & & \cdot & & & & & \\
 & & & & \cdot & & & & \\
 & & & & & \cdot & & & \\
\times & & & & & & \times & & \\
\times & \times & & & & & & \times & \\
\times & \times & \times & & & & & & \times
\end{pmatrix} \nwarrow k\text{th diagonal}$$

$\nwarrow k$th diagonal

Therefore, A^c and Δ have nonzero only entries on the diagonal, and above and below the corresponding kth diagonals.

In the following, we always choose dimension $n = 20$, $k = 18$, $A^c_{ii} = 50$ for $i = 1, \ldots, n$, and $A^c_{ij} = 100$, $A^c_{ij} = -100$ for coefficients above and below the corresponding kth diagonals, respectively. Moreover, the right-hand side $b^c = A^c \cdot e$, where e is the vector with all components equal to 1, and $\delta := 0$.

In our first test case, $\Delta_{ii} = 15$ for $i = 1, \ldots, n$, and $\Delta_{ij} = 15$ above and below the corresponding kth diagonals. Hence, the corresponding interval coefficients are equal to [35, 65] on the diagonal, and are equal to [85, 115] for the other nonzero coefficients.

The spectral radius $\rho(|(A^c)^{-1}| \cdot \Delta) = 0.62356$, the smallest singular value of A^c is equal to 50, and the maximal singular value of Δ is equal to 39.270. Thus both regularity criteria are satisfied.

Our algorithm terminates with $L = \emptyset$, and proves the regularity of $[A]$. The node set V of the representation graph consists of one sign vector, which is equal to e. The exact bounds of the first and last component of the solution set are equal to

$$[0.59560, 1{,}6538], [0.52923, 1.5506],$$

respectively.

In the second test case, $\Delta_{ii} = 40$ for $i = 1, \ldots, n$, and $\Delta_{ij} = 100$ above and below the corresponding kth diagonals. Therefore, the interval coefficients are equal to [10, 90] on the diagonal, and are equal to [0, 200] for the other nonzero coefficients.

The spectral radius $\rho(|A^c)^{-1}| \cdot \Delta) = 3.0225$. The smallest singular value of A^c and the largest singular value of Δ are 50, 201.80, respectively. Thus, neither regularity criterion is fulfilled.

Our algorithm terminates with $L = \emptyset$, proving the regularity of $[A]$. The node set V of the representation graph consists of seven sign vectors, that is, seven orthants are visited. The exact bounds of the first and last component of the solution set are equal to

$$[-0.56750 \times 10^4, 0.42500 \times 10^3], \quad [-0.15000 \times 10^{-2}, 0.27850 \times 10^4]$$

respectively.

In our last test case, $\Delta_{ii} = 40$ for $i = 1, \dots, n$, and $\Delta_{ij} = 110$ above and below the kth diagonals. The spectral radius $\rho(|(A^c)^{-1}| \cdot \Delta) = 3.2484$, the smallest singular value of A^c is equal to 50, and the maximal singular value of Δ is equal to 217.98. Neither regularity criterion is fulfilled.

Our method stops in the first orthant with $U = \{e\}$ because a maximization problem is unbounded. Therefore, it is proved that $[A]$ is singular.

5. CONCLUSIONS

An algorithm for calculating exact bounds for each component of the solution set of a linear interval system has been described. The problem of calculating exact bounds is known to be NP-hard. Additionally, the algorithm has the properties that (i) an error message is given provided that the solution set is unbounded, and (ii) the calculation is done in p calls of a polynomial time algorithm, where p is the number of orthants intersecting the solution set.

In case (i) it was shown that $[A]$ is singular, and moreover, each connected component is unbounded. Therefore, a calculation of finite bounds is not possible. Property (ii) seems to be especially interesting in practice. Our approach also demonstrates from a theoretical point of view that the NP-hardness of the problem stems from the fact that the solution set may intersect exponentially many orthants. In particular, for the class of real linear interval equations where p is small, for example $p \leqslant n$, our method calculates exact bounds for the solution set in polynomial time.

A preliminary version of this paper appeared in [6].

I wish to thank Professor S. M. Rump for many valuable discussions.

REFERENCES

1. G. Alefeld and J. Herzberger, Introduction to Interval Computations, Academic, New York, 1983.
2. G. Alefeld and L. Platzöder, A quadratically convergent Krawczyk-like algorithm, SIAM, J. Numer. Anal. 20(1):210–219 (1983).
3. W. Barth and E. Nuding, Optimale Lösung von Intervallgleichungssystemen, Computing 12:117–125 (1974).
4. H. Beeck, Charakterisierung der Lösungsmenge von Intervallgleichungssystemen, Z. Angew. Math. Mech. 53:T181–T182 (1973).
5. C. Jansson, A Self-validating method for solving linear programming problems, Computing Suppl. 6:33–46 (1988).
6. C. Jansson, Some Properties of Linear Interval Systems and Applications, Technical Report 94.3 Forschungsschwerpunkt Informations- und Kommunikationstechnik, TU Hamburg-Harburg, 1994.
7. R. Krawczyk, Fehlerabschätzung bei linearer Optimierung, Interval Math. 29 (1975).
8. U. Kulisch, Miranker, W. L., Computer Arithmetic in Theory and Practice, Academic, New York, 1981.
9. R. E. Moore, Methods and Applications of Interval Analysis, SIAM, Philadelphia, 1979.
10. A. Neumaier, Interval Methods for Systems of Equations, Encyclopedia Math. Appl., Cambridge U.P., 1990.
11. K. Nickel, Die Überschätzung des Wertebereichs einer Funktion in der Intervallrechnung mit Anwendungen auf lineare Gleichungstysteme, Computing 18:15–36 (1977).
12. W. Oettli, On the solution set of a linear system with inaccurate coefficients, SIAM J. Numer. Anal. 2:115–118 (1965).
13. W. Oettli and W. Prager, Compatibility of approximate solution of linear equations with given error bounds for coefficients and right-hand sides, Numer. Math. 6:405–409 (1964).
14. C. H. Papadimitriou and K. Steiglitz, Combinatorial Optimization: Algorithms and Complexity, Prentice-Hall, Englewood Cliffs, N.J., 1982.
15. S. Poljak and J. Rohn, Checking robust nonsingularity is NP-hard, Math. Control Signals Systems 6:1–9 (1993).
16. J. Rohn, Solving interval linear systems, Freiburger Intervallberichte 84(7):1–14 (1984).
17. J. Rohn, Systems of linear interval equations, Linear Algebra Appl. 126:39–78 (1989).
18. J. Rohn, Checking robust stability of symmetric interval matrices is NP-hard, Comment. Math. Univ. Carol. in 35:795–797 (1994).
19. J. Rohn, Enclosing solutions of linear interval equations is NP-hard, Computing 53:365–368 (1994).
20. J. Rohn, Checking bounds on solutions of linear interval equations is NP-hard, Linear Algebra Appl., to appear.

21. J. Rohn, NP-hardness results for linear algebraic problems with interval data, in Topics in Validated Computation (J. Herzberger, Ed.), North-Holland, 1994.
22. J. Rohn and V. Kreinovich, Computing exact componentwise bounds on solutions of linear systems with interval data is NP-hard, SIAM J. Matrix Anal. Appl. 16:415–420 (1995).
23. S. M. Rump, Validated solution of large linear systems, in Computing Supplementum 9, Validation Numerics (R. Albrecht, G. Alefeld, and H. J. Stetter, Eds.), Springer-Verlag, 1993, pp. 191–212.
24. S. M. Rump, Verification methods for dense and sparse systems of equations, in Topics in Validated Computations—Studies in Computational Mathematics (J. Herzberger, Ed.), Elsevier, Amsterdam, 1994, pp. 63–136.
25. H. Schwandt, An interval arithmetic approach for the construction of an almost globally convergent method for the solution of the nonlinear Poisson equation on the unit square, SIAM J. Sci. Statist. Comput. 5(2):427–452 (1984).
26. S. P. Shary, Optimal solutions of interval linear algebraic systems, Interval Comput. 2:7–30 (1991).

Received 2 March 1995; revised 26 July 1995