Topics in Interval Analysis

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9 · Partial Differential Equations

1. Introduction

INTERVAL arithmetic can be used in problems in partial differential equations. But today it seems to be difficult to give a general description of the possibilities. To give some impressions two different examples are selected.

2. Example I

Schröder [6] has studied error estimation for a certain boundary-value problem. The equation has the form

$$-\Delta u + f(x, u) = 0 \quad \text{on } G,$$
$$u = \gamma(x) \quad \text{on } \Gamma. \tag{1}$$

Let ϕ be an approximation for the solution u and define the defect function $d(\phi)$ such that

$$d(\phi) = \Delta y - f(x, y), \quad x \in G;$$
$$\phi = \phi(x, y), \quad x, y \in G. \tag{2}$$

The problem now is to find small bounds for $d(\phi)$ within G in practical cases; here $d(\phi)$ can be a very complicated expression. In the given example we have

$$d(x, y) = d(\phi) = \Delta\phi + \exp\{\phi(x, y) - P(x, y)\}, \tag{3}$$
$$G\colon x \in [-1, +1];\ y \in [-1, +1]$$

with
$$\phi(x, y) = \phi_0(x, y) + \phi_1(x, y) \tag{4}$$

and
$$\phi_0(x, y) = \frac{1}{\pi}\{H(x) + H(y) - H(1)\}, \tag{5}$$

$$H(x) = h(x) + h(-x), \tag{6}$$

$$h(x) = 2(1+x)\ln\{4+(1+x)^2\} + \{4-(1+x)^2\}\arctan\{0{\cdot}5(1+x)\} - \pi, \tag{7}$$

$$\phi_1(x, y) = (1-x^2)(1-y^2)\sum_{i=1}^{6} b_i f_i(x, y), \tag{8}$$

and
$$
\begin{aligned}
f_1 &= 1, & b_1 &= 5{\cdot}6176774\times 10^{-2}, \\
f_2 &= x^2+y^2, & b_2 &= -2{\cdot}0087935\times 10^{-2}, \\
f_3 &= x^4+y^4, & b_3 &= 6{\cdot}2069297\times 10^{-4}, \\
f_4 &= x^2y^2, & b_4 &= 1{\cdot}1764105\times 10^{-2}, \\
f_5 &= x^6+y^6, & b_5 &= -5{\cdot}7364814\times 10^{-4}, \\
f_6 &= x^4y^2+x^2y^4, & b_6 &= -2{\cdot}4416037\times 10^{-3},
\end{aligned} \tag{9}
$$
and
$$P(x,y) = \frac{1}{\pi}(P_1+P_2+P_3+P_4),$$
$$
\begin{aligned}
P_1 &= q(x,y), & P_2 &= q(-y,x), \\
P_3 &= q(-x,-y), & P_4 &= q(y,-x),
\end{aligned} \tag{10}
$$
$$
\begin{aligned}
q(x,y) = {} & (1+x)(1+y)\ln\{(1+x)^2+(1+y)^2\}+ \\
& +\{(1+x)^2-(1+y)^2\}\arctan\{(1+y)/(1+x)\}.
\end{aligned}
$$
It is very easy to compute $d(x,y)$ for a special list of values x_i, y_i. But it seems to be impossible to construct uniform bounds for $d(x,y)$. Interval arithmetic is a successful instrument here. The functions P, ϕ, and $\Delta\phi$ can be described by interval polynomials in two variables in the form
$$
\begin{aligned}
P(x_0+s,y_0+t) &\in \lfloor a_0^* \rfloor + \lfloor a_1^* \rfloor s + \lfloor a_2 \rfloor t = \lfloor Q_a \rfloor, \\
\phi(x_0+s,y_0+t) &\in \lfloor b_0^* \rfloor + \lfloor b_1^* \rfloor s + \lfloor b_2 \rfloor t = \lfloor Q_b \rfloor, \\
\Delta\phi(x_0+s,y_0+t) &\in \lfloor c_0^* \rfloor + \lfloor c_1^* \rfloor s + \lfloor c_2 \rfloor t = \lfloor Q_c \rfloor,
\end{aligned} \tag{11}
$$
for
$$s \in [0,h]; \qquad t \in [0,h]; \qquad s_0,\, t_0 \in G.$$
This is possible if interval polynomial representations of $\ln(z)$, $\arctan(z)$ are known and arithmetic operations with interval polynomials can be performed. It is important that the degree of the resulting interval polynomials can be reduced and bounded by *Vergröberung*. Now from $\lfloor Q_a \rfloor$, $\lfloor Q_b \rfloor$, and $\lfloor Q_c \rfloor$ new interval polynomials can be constructed so that
$$d(x_0+s,y_0+t) \in \lfloor B_0 \rfloor + \lfloor B_1 \rfloor s + \lfloor B_2 \rfloor t \tag{12}$$
and the bounds of $d(x_0+s,y_0+t)$ in the sub-square $[x_0,x_0+h]$, $[y_0,y_0+h]$ are
$$\lfloor B_0+\min(0,\lfloor B_1 h)+\min(0,\lfloor B_2 h) \leqslant d \leqslant B_0\rfloor + B_1\rfloor h + B_2\rfloor h. \tag{13}$$
By dividing G into about 100, 1000, or 10 000 sub-squares and performing this interval-estimation, more or less close bounds for d can be constructed uniformly in G.

According to Schröder [6] more estimations are necessary (see p. 158, equation (4.6) of [6]) for determination of an error-constant α. This problem was also solved by interval arithmetic. The so-constructed value $\alpha^* = 1{\cdot}0471266 \times 10^{-3}$ is only a little larger than the value $\alpha = 1{\cdot}0577 \times 10^{-3}$ which was computed using only a finite set of points x_i, y_i within G. Now it is easy to get correct bounds for u. The interval computations in this problem were performed by Wauschkuhn [7].

From a theoretical point of view it is of interest to use not only interval polynomials for defect estimation. The polynomials can be generalized to certain classes of functions with a range of values within a half-ordered space (see Krückeberg [3]).

3. Example II

In some cases it is possible to construct directly the operator for solving a given partial differential equation. If the problem has the form

$$\frac{\partial u(t,x)}{\partial t} = A(t)\frac{\partial u(t,x)}{\partial x} + B(t)u(t,x) + c(t),$$
$$u(0,x) = \phi(x), \quad u \in R^n \tag{14}$$

and if the special example is

$$\frac{\partial^2 u}{\partial t^2} = t^2\frac{\partial^2 u}{\partial x^2} + |t|\frac{\partial u_0}{\partial x} + t^3 \sin t,$$
$$u(0,x) = x^2 - x + 1, \tag{15}$$
$$\frac{\partial u}{\partial t}(0,x) = 1 - x,$$

the solving operator can be written in general form

$$u(t,x) = \Omega(B,t)\{\Omega(C,t)[\phi] + \Omega(-B,t)c\},$$
$$C(t) = \Omega(-B,t)A(t)\Omega(B,t). \tag{16}$$

By performing (16) for (15) using Formac the following result can be obtained:

$$u(t,x) = x^2 - x + 1 + t(1-x^2) + (2x-1)\frac{|t|t^2}{3!} + \frac{t^4}{6} - \frac{x|t|t^3}{6} -$$
$$-\frac{t^5}{10} + \frac{t^6}{10.9} - \frac{t^7}{9.7.4} + t(18-t^2)\sin t + 6(4-t^2)\cos t - 24 \tag{17}$$

(see V. Scharf [5]). But the coefficients in (17) are computed in Formac with rounding errors. By using interval arithmetic in combination with Formac it is possible to get correct bounds for the coefficients. In this way upper and lower bounds for the solution u can be constructed.

It seems to be a very successful procedure to combine a system like Formac with the ideas of interval analysis. Furthermore, in this way 'inside' intervals can be constructed.

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