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A Class of Interval-Newton-Operators

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Abstract — Zusammenfassung

A Class of Interval-Newton-Operators. A class of interval-Newton-operators N_a will be discussed. One of them, $\hat{N}$, is optimal in the manner that $\hat{N}(X) \subseteq N_a(X)$. With the help of such an interval operator we can give an existence theorem for the solution x^* of the equation $g(x)=0$.

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Eine Klasse von Intervall-Newton-Operatoren. Eine Klasse von Intervalloperatoren N_a wird diskutiert. Einer von ihnen – $\hat{N}$ – ist optimal in dem Sinne, daß $\hat{N}(X) \subseteq N_a(X)$ gilt. Mit Hilfe eines solchen Intervalloperators kann die Existenz einer Lösung x^* einer Gleichung $g(x)=0$ bewiesen werden.

1. Introduction

There are several interval-Newton-operators which are of the form

$$N(X) := x - S(X)\, g(x) \tag{1}$$

where $x \in X$, and $S(X)$ is a sublinear mapping for fixed X; hence $N(X) = x^*$ if $x = x^*$ is a zero of the equation $g(x) = 0$. For example: $S(X)\, g(x) := IGA\big(L(X), g(x)\big)$, where IGA denotes the interval Gauss algorithm and $L(X)$ is a regular Lipschitz matrix of g or an interval extension of the derivative $g'(x)$. In most cases, $x = \tilde{x}$ is the midpoint of X. Instead of the interval Gauss algorithm, we can determine $r := |e - aL|$, where a is a regular real matrix. If the spectral radius of r is less than 1, then $q := r(e-r)^{-1}$ exists and is a nonnegative matrix. Moreover,

$$N(X) := \tilde{x} - [e-q, e+q]\big(a g(\tilde{x})\big) \tag{2}$$

is an interval-Newton-operator which depends on the matrix a. For $a := (\text{mid}\, L)^{-1}$ the method was introduced in [4].

In this paper we will show that the choice of $a = (\text{mid}\, L)^{-1}$ is optimal in that the image interval is contained in all intervals being produced by interval operators (2) with an arbitrary regular matrix a.

In [2] Alefeld has given some existence theorems for the solution of the equation $g(x)=0$. These theorems are based upon the condition $N(X)\subseteq X$ and they are stated for several interval-Newton-operators N. However in all these cases it is assumed that $L(X)$ is a continuous function. We also give an existence theorem for the interval-Newton-operators (2) without the assumption of continuity of $L(X)$.

Remark about notation: We use the same notation as in [7]; but for convenience of the reader, some notations are repeated. Small Latin letters denote real values and capital letters denote sets, intervals and maps. We denote the set of n-dimensional interval vectors and $n\times n$-interval matrices by $\mathbb{I}\,\mathbb{R}^n$ and $\mathbb{I}\,\mathbb{R}^{n\times n}$, respectively, use $\operatorname{mid} X=(\underline{x}+\bar{x})/2$, $\operatorname{mid} A=(\underline{a}+\bar{a})/2$ for the midpoints, $\operatorname{rad} X=(\bar{x}-\underline{x})/2$, $\operatorname{rad} A=(\bar{a}-\underline{a})/2$ for the radius of $X\in\mathbb{I}\,\mathbb{R}^n$ and $A\in\mathbb{I}\,\mathbb{R}^{n\times n}$, respectively.

Moreover, we set $\mathbb{I}\,D:=\{X\in\mathbb{I}\,\mathbb{R}^n\mid X\subseteq D\}$. The unit matrix is written as e. $\sigma(a)$ denotes the spectral radius of $a\in\mathbb{R}^{n\times n}$.

For a discussion of interval arithmetic we refer to Alefeld/Herzberger [1].

2. A Class of Interval-Newton-Operators

Let $g: D\subseteq\mathbb{R}^n\to\mathbb{R}^n$ be a real function which satisfies an interval Lipschitz condition

$$g(x_1)-g(x_2)\in L(x_1-x_2) \text{ for all } x_1,x_2\in X, \tag{3}$$

where $X\in\mathbb{I}\,D$ and L is a regular interval matrix.

Let $S:\mathbb{I}\,\mathbb{R}^n\to\mathbb{I}\,\mathbb{R}^n$ be a sublinear mapping (see Neumaier [8]) with the property

$$l^{-1}z\in Sz \text{ for } l\in L \text{ and } z\in\mathbb{R}^n. \tag{4}$$

Then we call

$$N(X):=\tilde{x}-Sg(\tilde{x}) \text{ with } \tilde{x}=\operatorname{mid}(X) \tag{5}$$

an *interval-Newton-operator* of g.

Remark: Generally, L and S depend on X, but in the following we consider a fixed interval X; therefore the argument X will be deleted.

(3) and (4) immediately yield

$$g(x^*)=0\wedge x^*\in X\Rightarrow x^*\in N(X), \tag{6}$$

because $g(x^*)-g(\tilde{x})=l(x^*-\tilde{x})$ implies

$$x^*=\tilde{x}-l^{-1}g(\tilde{x})\in\tilde{x}-Sg(\tilde{x}).$$

Let $a\in\mathbb{R}^{n\times n}$ be a regular matrix. Then we define

$$r_a:=|e-aL|, \tag{7}$$

and assume that

$$\sigma(r_a)<1; \tag{8}$$

hence the matrix

$$q_a:=r_a(e-r_a)^{-1} \tag{9}$$

exists and is a nonnegative matrix.

The sublinear mapping

$$S_a z := [e - q_a, e + q_a](az) \tag{10}$$

fulfills the condition (4) for each $a \in \mathbb{R}^{n \times n}$ for which the assumption (8) is true. Indeed $b := e - al$ and $l \in L$ imply $b \in e - aL$ and $|b| \leqq |e - aL| = r_a$. On the other hand $b = e - al$ implies that $l^{-1} z = (e-b)^{-1}(az)$, and since

$$(e-b)^{-1} \in [e - q_a, e + q_a] \quad \text{(see (4.17) in [6])},$$

we have $l^{-1} z \in [e - q_a, e + q_a](az)$.

Hence

$$N_a(X) := \tilde{x} - S_a g(\tilde{x}), \tag{11}$$

where S_a is defined by (10), defines a class of interval-Newton-operators.

3. The Optimal Interval-Newton-Operator of the Class (11)

The question is how to choose the matrix a. Before we give an answer we formulate a

Lemma: *If* $r_a := |e - aL|$, $\hat{r} := |e - \hat{a}L|$ *with* $\hat{a} := (\operatorname{mid} L)^{-1}$, $\sigma(\hat{r}) < 1$ *and* $\sigma(r_a) < 1$ *then the following statements are true*:

1. $\sigma(\hat{r}) \leqq \sigma(r_a)$, (12)
2. $|(a - \hat{a}) z| \leqq (r_a - \hat{r})(e - \hat{r})^{-1} |\hat{a} z|$, (13)
3. $|\hat{a} z| \leqq (e - \hat{r})(e - r_a)^{-1} |az|$ *with* $z \in \mathbb{R}^n$. (14)

Proof:

1. (12) was proved by Neumaier (see Theorem 6 in [8]). ($\hat{a}L$ is an H-matrix since $\sigma(\hat{r}) < 1$.)

2. We use the abbreviation $\tilde{l} = \operatorname{mid} L$ and put

$$b := a\tilde{l} - e. \tag{15}$$

Then the relation

$$r_a = |e - aL| = |a| \operatorname{rad} L + |b| \tag{16}$$

holds (see (31) in [3]), which implies

$$\hat{r} = |\hat{a}| \operatorname{rad} L. \tag{17}$$

From (15) follows

$$b\hat{a} = a - \hat{a}. \tag{18}$$

By inserting the inequality $|a| \geqq |\hat{a}| - |b||\hat{a}|$ into (16) we obtain from (17)

$$|b| \leqq r_a - \hat{r} + |b|\hat{r}$$

and (12)

$$|b| \leqq (r_a - \hat{r})(e - \hat{r})^{-1}. \tag{19}$$

(18) and (19) yield (13).

3. Because of $\sigma(|b|) \leqq \sigma(r_a) < 1$, (18) yields

$$|\hat{a}z| \leqq |(e+b)^{-1}|\,|az| \leqq (e-|b|)^{-1}|az|.$$

By inserting (19) into this inequality we get

$$\begin{aligned}|\hat{a}z| &\leqq \left(e-(r_a-\hat{r})(e-\hat{r})^{-1}\right)^{-1}|az| \\ &= (e-\hat{r})(e-r_a)^{-1}|az|.\end{aligned}$$ ■

We next give an answer to the question: "how to choose a?" by using the

Theorem 1: *If* $\hat{a} := (\operatorname{mid} L)^{-1}$, $\hat{r} := |e - \hat{a}L|$, $\hat{q} := \hat{r}(e-\hat{r})^{-1}$ *and*

$$\hat{N}(X) := \tilde{x} - [e-\hat{q}, e+\hat{q}]\,(\hat{a}g(\tilde{x})) \tag{20}$$

then

$$\hat{N}(X) \subseteq N_a(X) \tag{21}$$

for each a satisfying the condition (8).

Proof: It follows by the definition of radius and midpoint and the formulae (6.4), (6.5) in [7] as well as by the lemma

$$\begin{aligned}|\operatorname{mid} N_a(X) - \operatorname{mid} \hat{N}(X)| &= |(a-\hat{a})\,g(\tilde{x})| \\ &\leqq (r_a-\hat{r})(e-\hat{r})^{-1}|\hat{a}g(\tilde{x})| \\ &\leqq r_a(e-\hat{r})^{-1}(e-\hat{r})(e-r_a)^{-1}|ag(\tilde{x})| - \hat{r}(e-\hat{r})^{-1}|\hat{a}g(\tilde{x})| \\ &= q_a|ag(\tilde{x})| - \hat{q}|\hat{a}g(\tilde{x})| \\ &= \operatorname{rad} N_a(X) - \operatorname{rad} \hat{N}(X).\end{aligned}$$

This relation is equivalent to $\hat{N}(X) \subseteq N_a(X)$ (see (2.12) in [7]). ■

4. Existence Theorem

Theorem 2: *If* $N_a(X) \subseteq X$ *then there exists an* $x^* \in X$ *with* $g(x^*) = 0$.

Proof: By Theorem 1 it is sufficient to prove the existence of x^* in the case that $\hat{N}(X) \subseteq X$ for $N_a(X) \subseteq X$ implies $\hat{N}(X) \subseteq X$.

If $\hat{N}(X) \subseteq X$ then by (2.12) and (6.4) in [7] we obtain

$$|\hat{a}g(\tilde{x})| \leqq \operatorname{rad} X - \operatorname{rad} \hat{N}(X) = \operatorname{rad} X - \hat{q}|\hat{a}g(\tilde{x})|,$$

and it follows from $e + \hat{q} = (e-\hat{r})^{-1}$ that

$$(e-\hat{r})^{-1}|\hat{a}g(\tilde{x})| \leqq \operatorname{rad} X.$$

Now we define $Y \subseteq X$ by $\operatorname{mid} Y := \tilde{x}$ and

$$\operatorname{rad} Y = (e-\hat{r})^{-1}|\hat{a}g(\tilde{x})| \leqq \operatorname{rad} X. \tag{22}$$

Then

$$|\hat{a}g(\tilde{x})| = (e-\hat{r})^{-1}|\hat{a}g(\tilde{x})| - \hat{q}|\hat{a}g(\tilde{x})| = \operatorname{rad} Y - \operatorname{rad} \hat{N}(X),$$

i.e. by (2.12) in [7], $\hat{N}(X) \subseteq Y$.

On the other hand it follows from (20) by (22), (17) and $e-\hat{a}L=[-\hat{r},\hat{r}]$

$$\hat{N}(X)=\tilde{x}-\hat{a}g(\tilde{x})+[-\hat{r},\hat{r}]\operatorname{rad}Y$$
$$=\tilde{x}-\hat{a}g(\tilde{x})+(e-\hat{a}L)(Y-\tilde{x})\subseteq Y.$$

$e-\hat{a}L$ is an interval Lipschitz matrix of the function $f(x):=x-\hat{a}g(x)$, i.e.

$$f(x)-f(\tilde{x})\in(e-\hat{a}L)(x-\tilde{x})\subseteq(e-\hat{a}L)(Y-\tilde{x}) \text{ if } x\in Y.$$

Therefore, $f(x)\in\hat{N}(X)\subseteq Y$ for all $x\in Y$, and by Brouwer's fixpoint theorem there exists a fixpoint $x^*\in Y\subseteq X$ which is a zero of $g(x)$ because $\hat{a}$ is regular.

Remark: Theorem 2 is a consequence of the fact that $\hat{N}(X)$ is an interval extension of f on the intersection $X\cap Y$. This is a result of Section 7 in [5].

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