Pseudo-Intervals Arithmetic based Vector Space
&
Probabilist Set Inversion Algorithm

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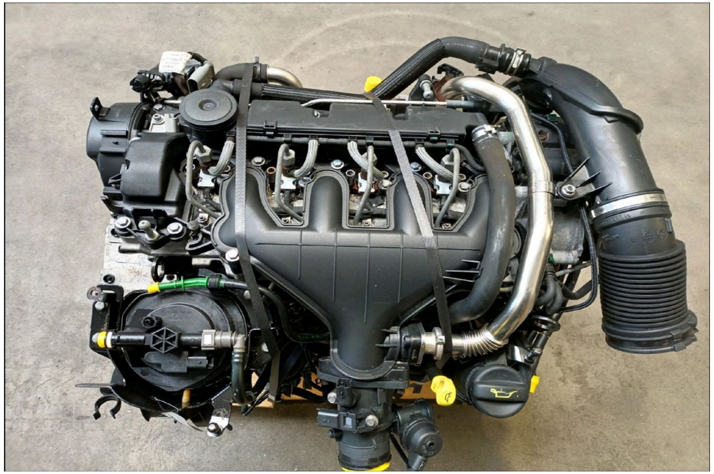
2008 : Peugeot gasoline car engine



Adjustments (~100 parameter intervals)

Exhaust gases densities
and debits
(~20 output intervals)

Set Inversion Problem

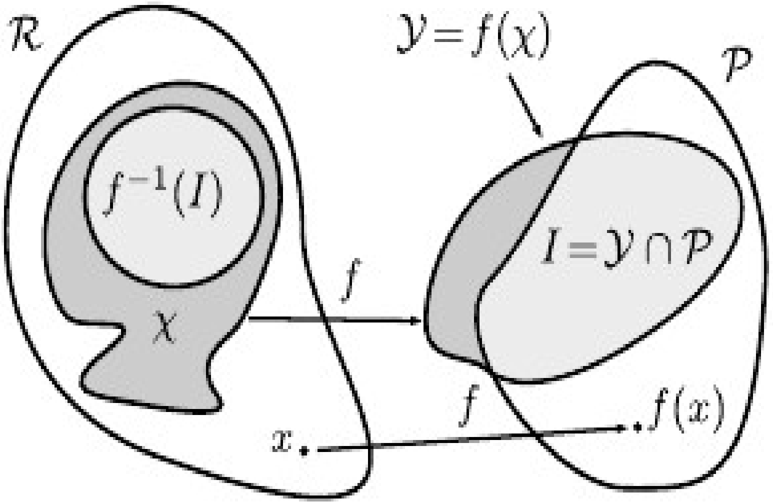


Figure 1: $\mathcal{R} \subset \mathbb{R}^n$ is the set of feasible adjustments, and $\mathcal{P} \subset \mathbb{R}^p$ the set of desired performance of a system. Set inversion computes $S = f^{-1}(\mathcal{P}) \cap \mathcal{R}$.

Looking for an input-output relation

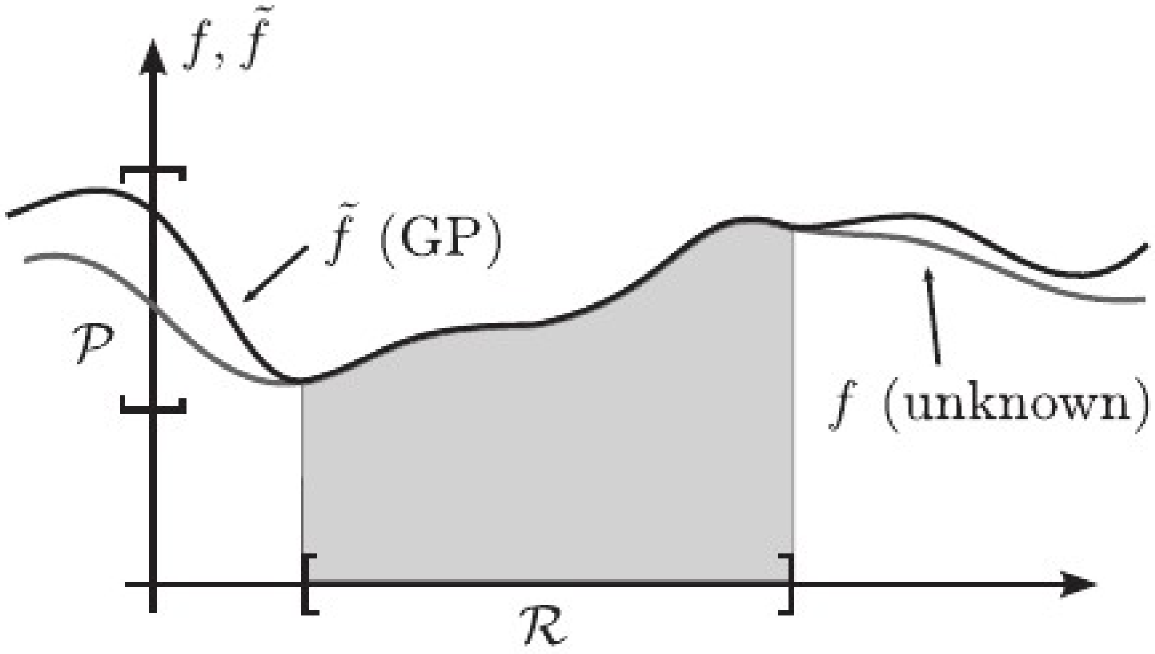


Figure 2. Symbolic Regression *via* Genetic Programming permits to build an approximation of an unknown function defined on $\mathcal{R}$, and valued on $\mathcal{P}$.

Symbolic Regression *via* Genetic Programming combined with Set Inversion

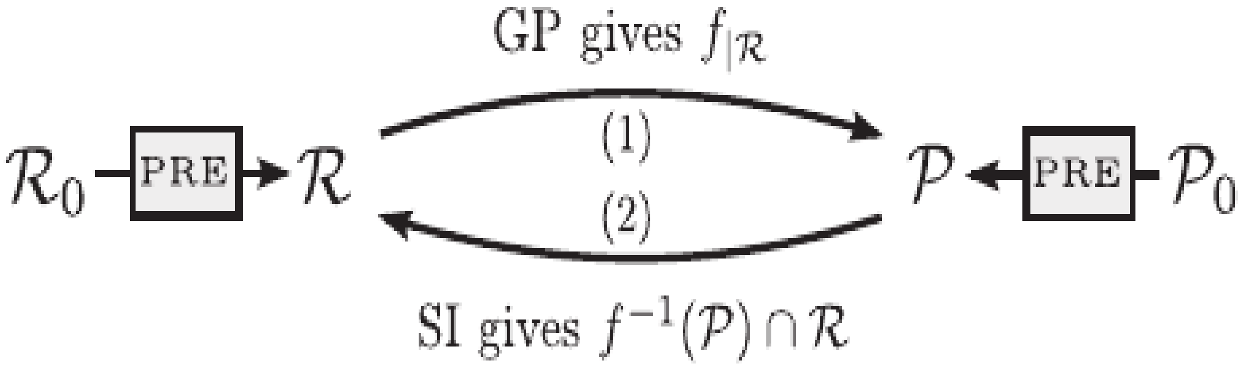


Figure 3. Combining Genetic Programming (GP) and Set Inversion (SI) permits first to find an approximation of the model and in a second step to invert performance set in order to find the efficient and possible adjustments. Experimental input–output data could be filtered and decimated if it is needed in order to remove some frequencies and to avoid overfitting phenomena.

Example of Symbolic Regression

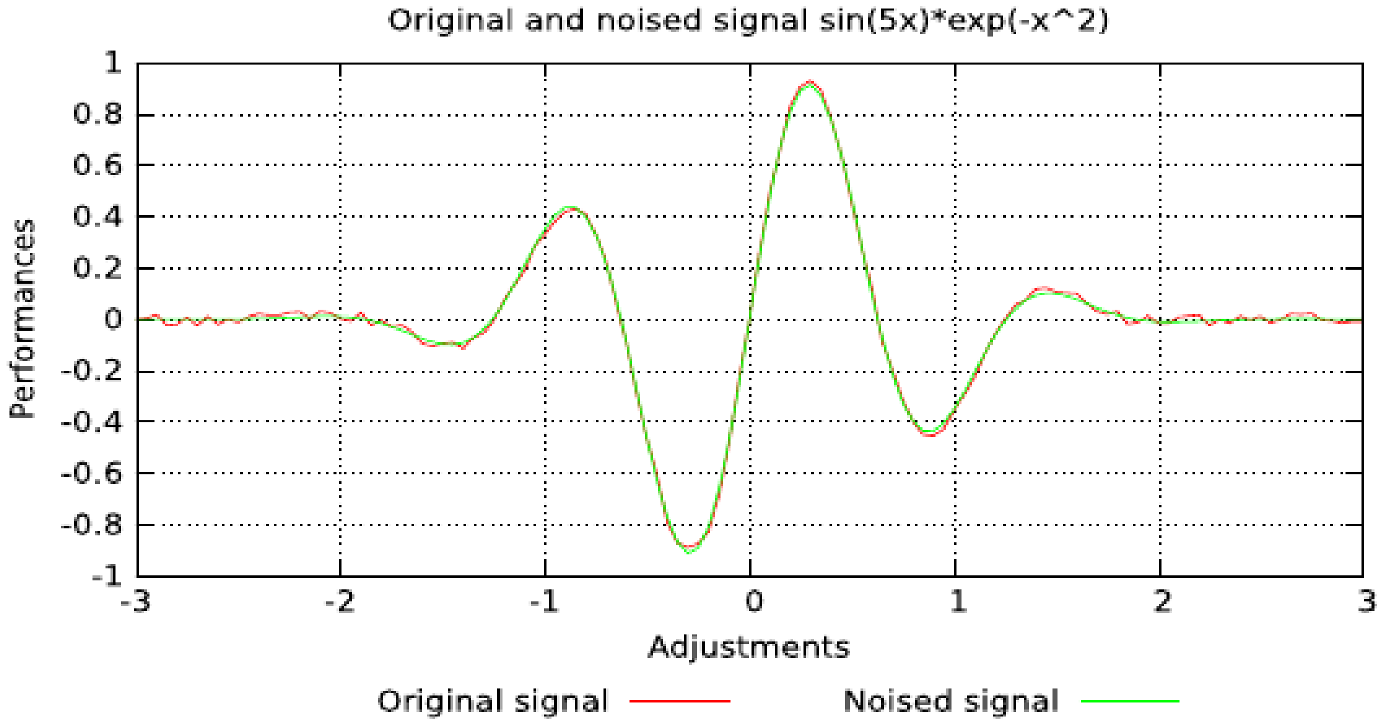


Figure 4. Performance set for the function $f(x) = \sin(5 \cdot x) \cdot e^{-x^2}$. An uniform random noise $\epsilon(x) \in [-0.25, 0.25]$ for $x \in [-3, 3]$ has been added to the initial output data-set f([$-$3,3]).

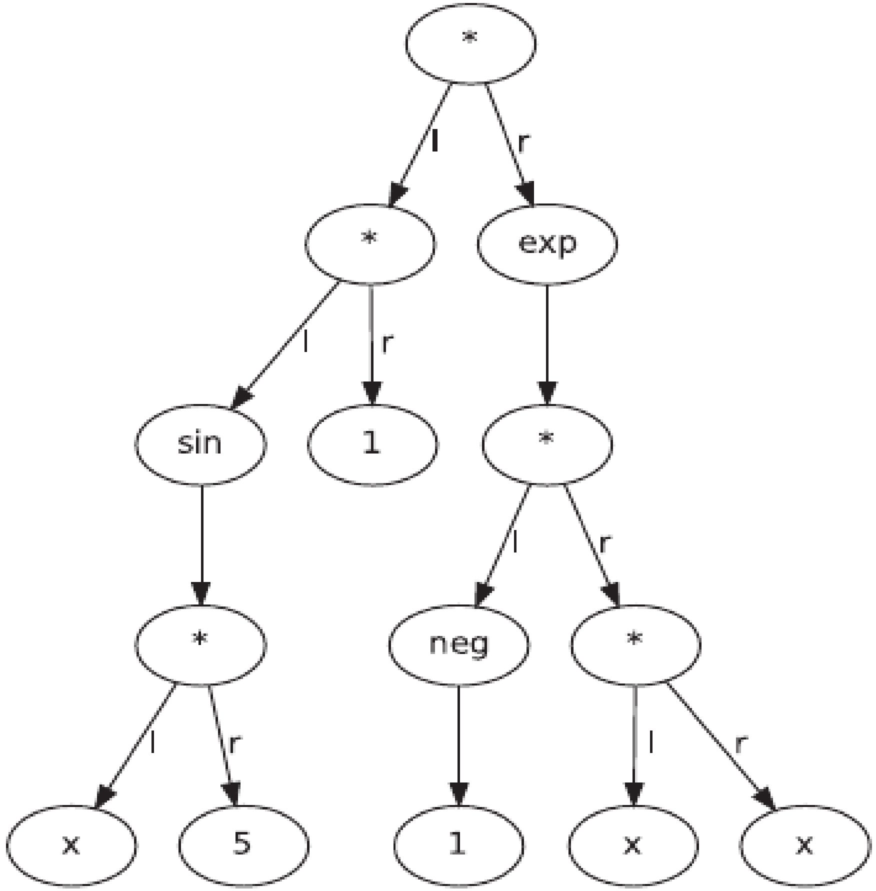


Figure 5. Result for the genetic programming of the $f(x) = \sin(5x) \cdot e^{-x^2}$ noised example. This tree corresponds to the approximation of f, $\tilde{f}(x) = \sin(5x) \cdot e^{-x^2} = f(x)$ on $[-3, 3]$ with function basis set $\{+, -, \cdot, /, \exp, \log, \sin, \cos, \tan, \mathrm{neg}\}$. The letters ‘l’ and ‘r’ correspond respectively to left and right positions of operands in a binary expression, respectively.

Brief Historical Overview

- R. Moore (1966), T. Sunaga (1958) , M. Warmus (1956, 1961) : Generalized Interval Arithmetic (GIA).

- H.-J. Ortolf, E. Kaucher in the 70’s : the intervals form a group with respect to addition and a complete lattice with respect to inclusion.

- In order to adapt it to semantic problems, in the beginning of 80’s, E. Gardenes *et al.* developed an approach called modal interval arithmetic.

- S. Markov and others in the 90’s : Directed Interval Arithmetic, in which Kaucher’s Generalized Intervals can be viewed as classic intervals plus direction. Proper/Improper Intervals : Intervals with sign.

Another approach

- Generalized intervals : pseudo-intervals and anti-pseudo-intervals correspond respectively to the proper and improper ones.

- Canonical construction based on the semi-group completion into a group.

- Associated Real Vector Space. Analogy with Directed Intervals.

- Banach Vector Space Structure : Linear Algebra and Differential Calculus.

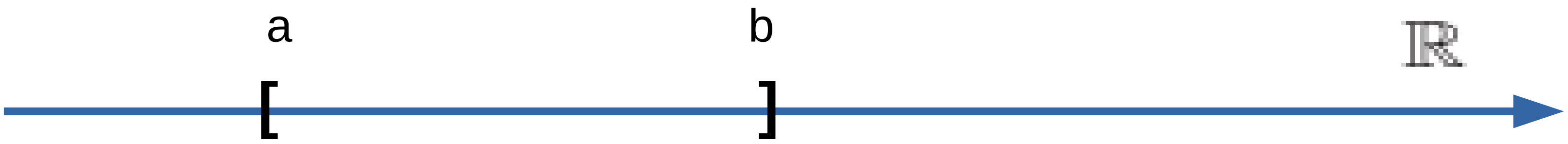
- Extensions to Pseudo-Intervals N-dimensional Free Algebra with N=4,5 or 7.

Minkowski operations

An interval X is defined as a non-empty, closed and connected set of real numbers. One writes real numbers as intervals with same bounds, $\forall a \in \mathbb{R}$, $a \equiv [a, a]$. We denote by $\mathbb{IR} = \mathcal{P}_1$ the set of intervals of $\mathbb{R}$. The arithmetic operations on intervals, called ***Minkowski or classical operations***, are defined such that the result of the corresponding operation on elements belonging to operand intervals belongs to the resulting interval. That is, if $\diamond \in \{+, -, *, /\}$ denotes one of the usual operations, one has, if X and Y are bounded intervals of $\mathbb{R}$,

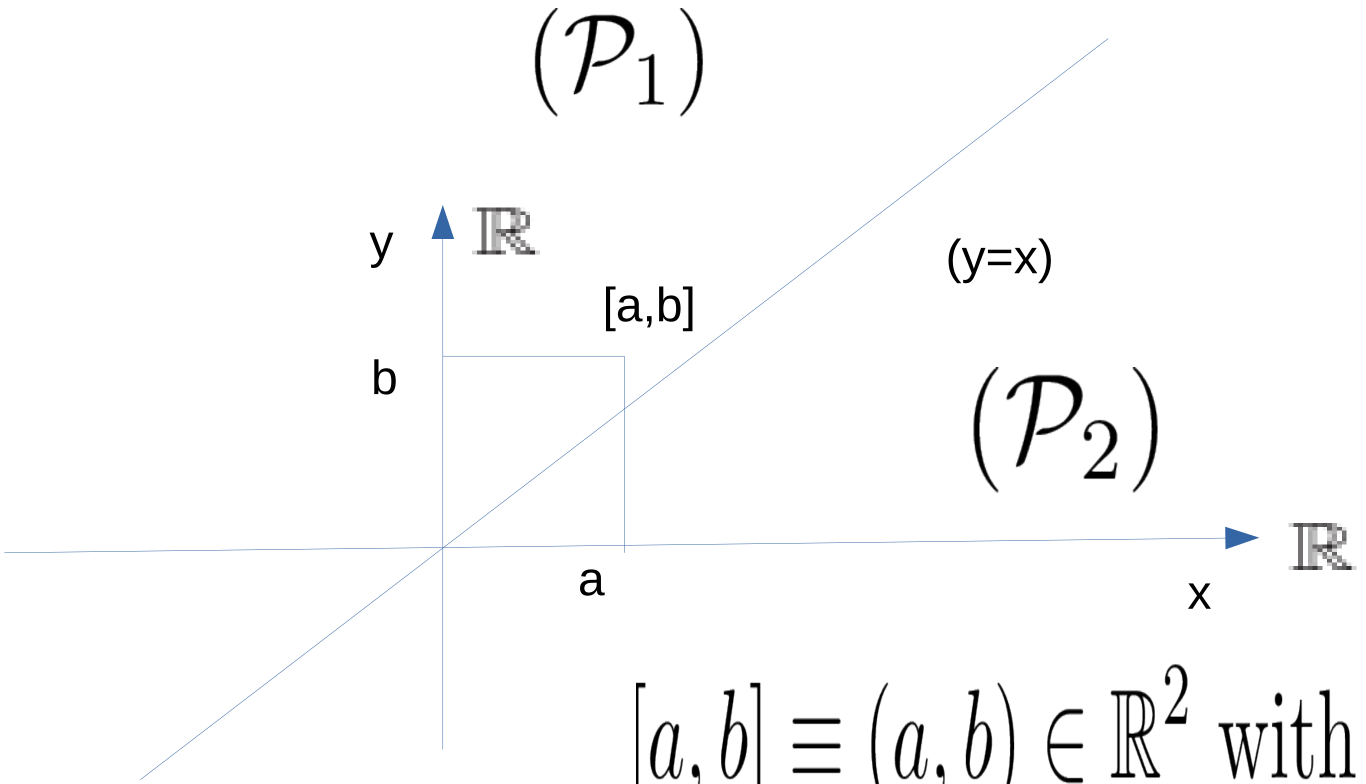
$$X \diamond Y = \{x \diamond y \,/\, x \in X,\ y \in Y\}$$

Usual Representation of Intervals



$$[a,b] = \{x \in \mathbb{R},\ a \leq x \leq b\} \subset \mathbb{R}$$

Another representation of Intervals



$$[a,b] \equiv (a,b) \in \mathbb{R}^2 \text{ with } a \leq b$$

Algebraic completion

As $(\mathbb{IR}, +)$ is a commutative and regular semi-group, the quotient set, denoted by $\overline{\mathbb{IR}}$, associated with the equivalence relations:

$$(A, B) \sim (C, D) \Longleftrightarrow A + D = B + C,$$

for all $A, B, C, D \in \mathbb{IR}$, is provided with a structure of abelian group for the natural addition:

$$\overline{(A, B)} + \overline{(C, D)} = \overline{(A + C, B + D)}$$

where $\overline{(A, B)}$ is the equivalence class of (A, B). We denote by $\setminus\overline{(A, B)}$ the inverse of $\overline{(A, B)}$ for the interval addition.

We have $\setminus\overline{(A,B)} = \overline{(B,A)}$. If $X = [a,a]$, $a \in \mathbb{R}$, then $\overline{(X,0)} = \overline{(0,-X)}$ where $-X = [-a,-a]$, and $\setminus\overline{(X,0)} = \overline{(0,X)}$. In this case, we identify $X = [a,a]$ with a and we denote always by $\mathbb{R}$ the subset of intervals of type $[a,a]$.

Naturally, the group $\overline{\mathbb{IR}}$ is isomorphic to the additive group $\mathbb{R}^2$ by the isomorphism $\overline{([a,b],[c,d])} \rightarrow (a-c, b-d)$. We find the notion of generalized interval and this yields immediately to the following result:

Proposition 1. *Let $X = \overline{(X,Y)} \in \overline{\mathbb{IR}}$, and $l : \mathbb{IR} \longmapsto \mathbb{R}$ which gives the interval length. Thus*

1. *If $l(Y) < l(X)$, there is an unique $A \in \mathbb{IR} \setminus \mathbb{R}$ such that $X = \overline{(A,0)}$,*
2. *If $l(Y) > l(X)$, there is an unique $A \in \mathbb{IR} \setminus \mathbb{R}$ such that $X = \overline{(0,A)} = \setminus\overline{(A,0)}$,*
3. *If $l(Y) = l(X)$, there is an unique $A = \alpha \in \mathbb{R}$ such that $X = \overline{(\alpha,0)} = \overline{(0,-\alpha)}$.*

Intervals with signs

Any element $X = \overline{(A, 0)}$ with $A \in \mathbb{IR} - \mathbb{R}$ is said positive and we write $X > 0$. Any element $X = \overline{(0, A)}$ with $A \in \mathbb{IR} - \mathbb{R}$ is said negative and we write $X < 0$. We write $X \underline{\geq} X'$ if $X \setminus X' \underline{\geq} 0$. For example if X and X' are positive, $X \underline{\geq} X' \Longleftrightarrow l(X) \underline{\geq} l(X')$. The elements $\overline{(\alpha, 0)}$ with $\alpha \in \mathbb{R}^*$ are neither positive nor negative.

Real Vector Space Construction

External multiplication : $\cdot : \mathbb{R} \times \overline{\mathbb{IR}} \longrightarrow \overline{\mathbb{IR}}$

defined, for all $A \in \mathbb{IR}$, by

$$\begin{cases} \alpha \cdot \overline{(A, 0)} = \overline{(\alpha A, 0)}, \\ \alpha \cdot \overline{(0, A)} = \overline{(0, \alpha A)}, \end{cases}$$

for all $\alpha > 0$. If $\alpha < 0$ we put $\beta = -\alpha$. So we put:

$$\begin{cases} \alpha \cdot \overline{(A, 0)} = \overline{(0, \beta A)}, \\ \alpha \cdot \overline{(0, A)} = \overline{(\beta A, 0)}. \end{cases}$$

We denote αX instead of $\alpha \cdot X$. This operation satisfies

1. For any $\alpha \in \mathbb{R}$ and $X \in \overline{\mathbb{IR}}$ we have:

$$\begin{cases} \alpha(\smallsetminus X) = \smallsetminus(\alpha X), \\ (-\alpha)X = \smallsetminus(\alpha X). \end{cases}$$

2. For all $\alpha, \beta \in \mathbb{R}$, and for all $X, X' \in \overline{\mathbb{IR}}$, we have

$$\begin{cases} (\alpha + \beta)X = \alpha X + \beta X, \\ \alpha(X + X') = \alpha X + \alpha X', \\ (\alpha\beta)X = \alpha(\beta X). \end{cases}$$

Pseudo-Intervals Vector Space

Theorem 1. *The triplet* $(\overline{\mathbb{IR}}, +, \cdot)$ *is a real vector space and the vectors* $\mathcal{X}_1 = \overline{([0,1], 0)}$ *and* $\mathcal{X}_2 = \overline{([1,1], 0)}$ *of* $\overline{\mathbb{IR}}$ *determine a basis of* $\overline{\mathbb{IR}}$. *So* $\dim_{\mathbb{R}} \overline{\mathbb{IR}} = 2$.

The linear map

$$\varphi : \overline{\mathbb{IR}} \longrightarrow \mathbb{R}^2$$

defined by

$$\begin{cases} \varphi(\,\overline{([a,b],0)}\,) = (b-a, a), \\ \varphi(\,\overline{(0,[c,d])}\,) = (c-d, -c) \end{cases}$$

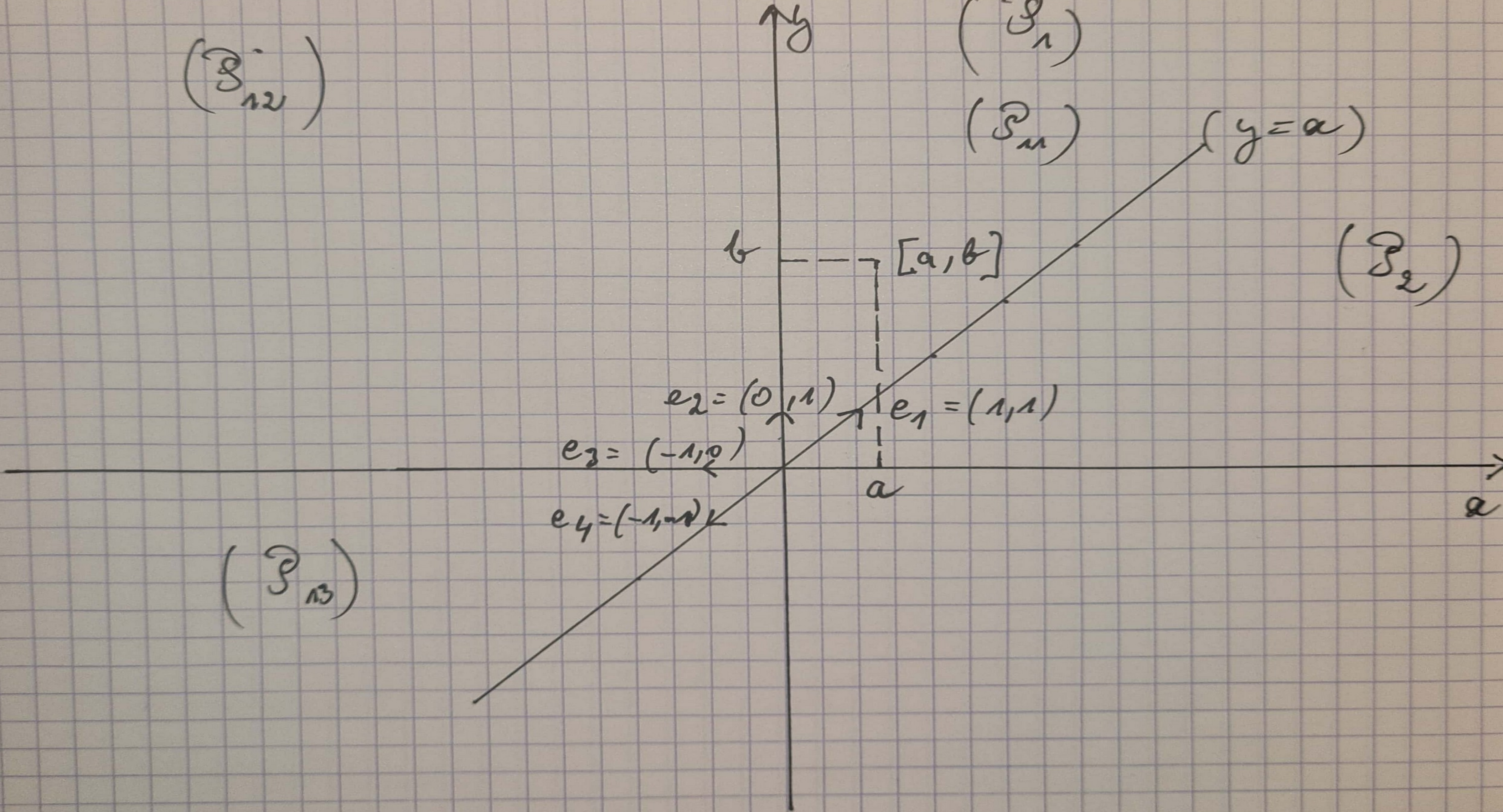
is a linear isomorphism and $\overline{\mathbb{IR}}$ is canonically isomorphic to $\mathbb{R}^2$.

Definition 1. $(\overline{\mathbb{IR}}, +, \cdot)$ *is called the vector space of pseudo-intervals.*

Free Algebra associated to PI-Vector Space

Goals :

- To build a well-defined and distributive interval product.
- To avoid data dependencies in inclusion functions definitions.
- To build unequivocally inclusion functions from native ones.


y
x
(ℨ1)
(ℨ11)
(ℨ12)
(ℨ13)
(ℨ2)
(y=x)
b
a
[a,b]
e1=(1,1)
e2=(0,1)
e3=(-1,0)
e4=(-1,-1)

One observes that the semi-group $\mathbb{IR}$ can be identified to $\mathcal{P}_{1,1} \cup \mathcal{P}_{1,2} \cup \mathcal{P}_{1,3}$. Let us consider as well the following vectors of $\mathbb{R}^2$

$$\begin{cases} e_1 = (1, 1), \\ e_2 = (0, 1), \\ e_3 = (-1, 0), \\ e_4 = (-1, -1). \end{cases}$$

They correspond to the intervals $[1, 1]$, $[0, 1]$, $[-1, 0]$, and $[-1, -1]$. Any point of $\mathcal{P}_{1,1} \cup \mathcal{P}_{1,2} \cup \mathcal{P}_{1,3}$ admits the decomposition

$$(a, b) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4$$

with $\alpha_i \geq 0$. The dependance relations between the vectors e_i are

$$\begin{cases} e_2 = e_3 + e_1 \\ e_4 = -e_1. \end{cases}$$

Pseudo-Intervals Product in A_4

The multiplication table is

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	e_2	e_3	e_3
e_3	e_3	e_3	e_2	e_2
e_4	e_4	e_3	e_2	e_1

This algebra is associative and its elements are called pseudo-intervals.

Pseudo-Intervals Division

Proposition 2. *The multiplicative group $\mathcal{A}_4^*$ of invertible elements of $\mathcal{A}_4$ is the set of elements $x = (x_1, x_2, x_3, x_4)$ such that*

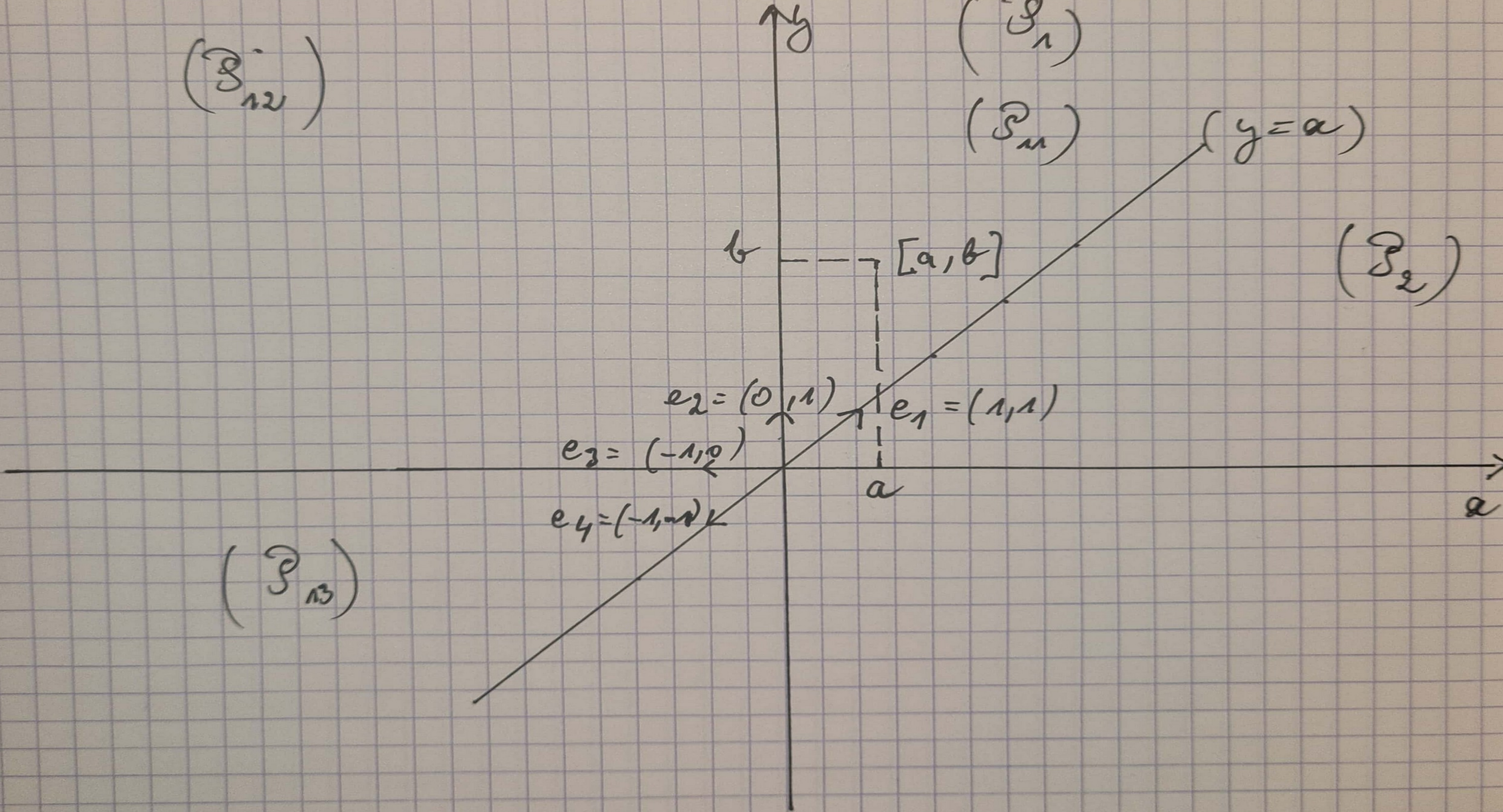
$$\begin{cases} x_4 \neq \pm x_1, \\ x_3 \neq \pm x_2. \end{cases}$$

This means that the invertible intervals do not contain 0. If $x \in \mathcal{A}_4^*$ we have:

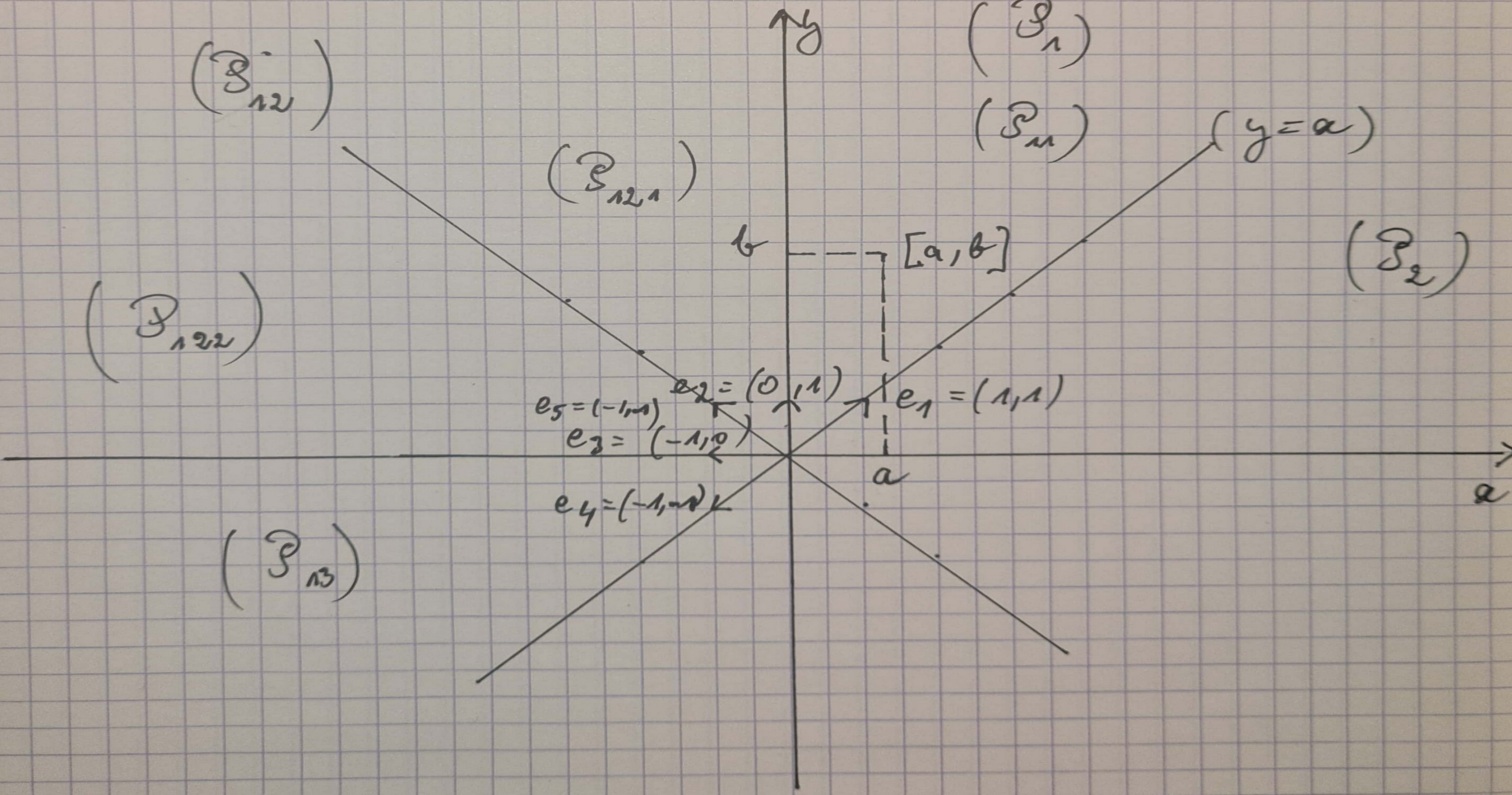
$$x^{-1} = \left(\frac{x_1}{x_1^2 - x_4^2}, \frac{x_2}{x_2^2 - x_3^2}, \frac{x_3}{x_2^2 - x_3^2}, \frac{x_4}{x_1^2 - x_4^2} \right).$$

Proposition 3. *Monotony property: Let* $X_1, X_2 \in \overline{\mathbb{IR}}$*. Then*

$$X_1 \subset X_2 \Longrightarrow X_1 \bullet Z \subset X_2 \bullet Z \textit{ for all } Z \in \overline{\mathbb{IR}}.$$



$(\mathcal{Z}_{12})$
y
$(\mathcal{Z}_1)$
$(\mathcal{Z}_{11})$
$(y=x)$
b
$[a,b]$
$(\mathcal{Z}_2)$
$e_2=(0,1)$
$e_1=(1,1)$
$e_3=(-1,0)$
a
x
$e_4=(-1,-1)$
$(\mathcal{Z}_{13})$



y
x
$(\mathcal{B}_1)$
$(\mathcal{B}_{11})$
$(\mathcal{B}_{12})$
$(\mathcal{B}_{121})$
$(\mathcal{B}_{122})$
$(\mathcal{B}_{13})$
$(\mathcal{B}_2)$
$(y = x)$
b
a
$[a, b]$
$e_1 = (1, 1)$
$e_2 = (0, 1)$
$e_3 = (-1, 0)$
$e_4 = (-1, -1)$
$e_5 = (-1, 1)$

A_5 Free Algebra

We can refine our result of the product to come closer to the result of Minkowski. Consider the one dimensional extension $\mathcal{A}_4 \oplus \mathbb{R}e_5 = \mathcal{A}_5$, where e_5 is a vector corresponding to the interval $[-1, 1]$ of $\mathcal{P}_{1,2}$. The multiplication table of $\mathcal{A}_5$ is

	e_1	e_2	e_3	e_4	e_5
e_1	e_1	e_2	e_3	e_4	e_5
e_2	e_2	e_2	e_3	e_3	e_5
e_3	e_3	e_3	e_2	e_2	e_5
e_4	e_4	e_3	e_2	e_1	e_5
e_5	e_5	e_5	e_5	e_5	e_5

Looking for a Minkowski product

Example. Let $X = [-2, 3]$ and $Y = [-4, 2]$. We have $X \in \mathcal{P}_{1,2,1}$ and $Y \in \mathcal{P}_{1,2,2}$. The product in $\mathcal{A}_4$ gives

$$X \bullet Y = [-16, 14].$$

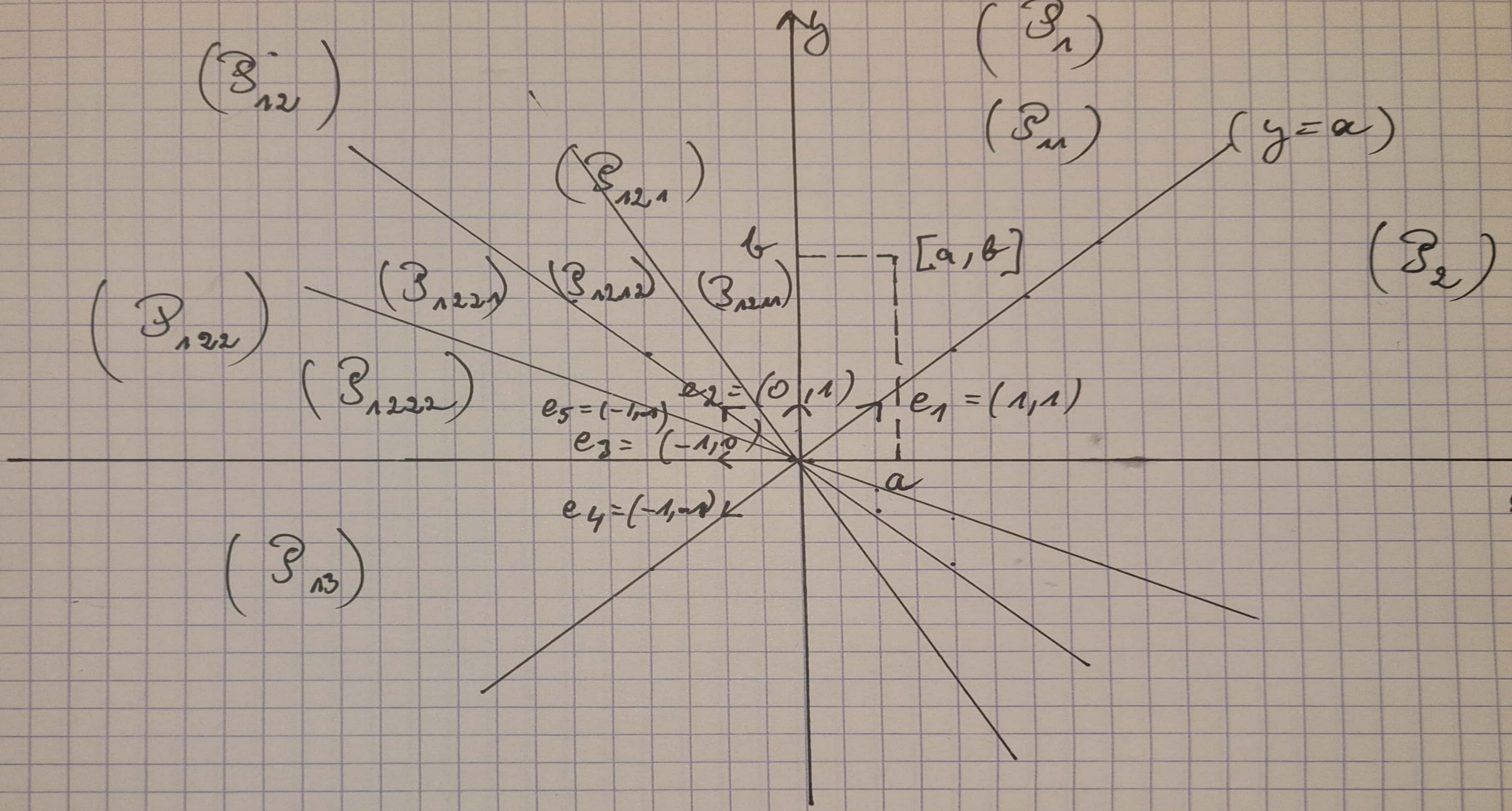
The product in $\mathcal{A}_5$ gives

$$X \bullet Y = [-12, 10].$$

The Minkowski product is

$$[-2, 3].[-4, 2] = [-12, 8].$$

Thus the product in $\mathcal{A}_5$ is better.



y
(ℬ₁)
(ℬ₁₁)
(y = x)
(ℬ₁₂)
(ℬ₁₂₁)
(ℬ₂)
b
[a, b]
(ℬ₁₂₂₁)
(ℬ₁₂₁₂)
(ℬ₁₂₁₁)
(ℬ₁₂₂)
(ℬ₁₂₂₂)
e₂ = (0, 1)
e₁ = (1, 1)
e₅ = (-1, -1)
e₃ = (-1, 0)
a
e₄ = (-1, -1)
(ℬ₁₃)

Conclusion. Considering a partition of $\mathcal{P}_{1,2}$, we can define an extension of $\mathcal{A}_4$ of dimension n, the choice of n depends on the approach wanted of the Minkowski product. For example, let us consider the vector e_6 corresponding to the interval $[-1, \frac{1}{2}]$. Thus the Minkowsky product gives $e_6 \cdot e_6 = e_7$ where e_7 corresponds to $[-\frac{1}{2}, 1]$. This yields to the fact that $\mathcal{A}_6$ is not an associative algebra but it is the case for $\mathcal{A}_7$ whose table of multiplication is

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_2	e_2	e_2	e_3	e_3	e_5	e_6	e_7
e_3	e_3	e_3	e_2	e_2	e_5	e_7	e_6
e_4	e_4	e_3	e_2	e_1	e_5	e_7	e_6
e_5	e_5	e_5	e_5	e_5	e_5	e_5	e_5
e_6	e_6	e_6	e_7	e_7	e_5	e_7	e_6
e_7	e_7	e_7	e_6	e_6	e_5	e_6	e_7

A_7 is the right Free Algebra

Example. Let $X = [-2, 3]$ and $Y = [-4, 2]$. The decomposition on the basis $\{e_1, \cdots, e_7\}$ with positive coefficients writes

$$X = e_5 + 2e_7, \quad Y = 2e_6.$$

$$X \bullet Y = (e_5 + 2e_7)(4e_6) = 4e_5 + 8e_6 = [-12, 8].$$

We obtain now the Minkowski product.

Inclusion functions

$$f_0(x) = x^2 - 2x + 1,\ f_1(x) = (x-1)^2,\ f_2(x) = x(x-2) + 1.$$

$X = [3,4]$ are $[f]_0(X) = [2,11]$, $[f]_1(X) = [4,9]$ and $[f]_2(X) = [6,12]$.

$$\varphi(X) = (3, 4-3, 0, 0) = (3, 1, 0, 0) = 3e_1 + e_2.$$

Since $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$ and with means of product table, one has

$$\begin{aligned}
\varphi([f]_0(X)) &= (3e_1 + e_2)^2 - 2(3e_1 + e_2) + 1 \\
&= 9e_1^2 + 2 \cdot 3e_1 e_2 + e_2^2 - 2 \cdot 3e_1 - 2e_2 + 1 \\
&= 9e_1 + 6e_2 + e_2 - 6e_1 - 2e_2 + e_1 = 4e_1 + 5e_2 \\
&= \varphi([4, 9]),
\end{aligned}$$

$$\begin{aligned}
\varphi([f]_1(X)) &= (3e_1 + e_2 - 1)^2 \\
&= (2e_1 + e_2)^2 \\
&= 4e_1^2 + 4e_1 e_2 + e_2^2 \\
&= 4e_1 + 4e_2 + e_2 \\
&= 4e_1 + 5e_2 \\
&= \varphi([4, 9]),
\end{aligned}$$

$$
\begin{aligned}
\varphi([f]_2(X)) &= (3e_1 + e_2) \cdot (3e_1 + e_2 - 2) + 1 \\
&= 9e_1 + 3e_1e_2 - 6e_1 + 3e1e_2 + e_2^2 - 2e_2 + e_1 \\
&= 4e_1 + 3e_2 + 3e_2 + e_2 - 2e_2 \\
&= 4e_1 + 5e_2 \\
&= \varphi([4, 9]).
\end{aligned}
$$

Thus, $[f]_0(X) = [f]_1(X) = [f]_2(X) = [4, 9]$ and the inclusion function is defined univocally regardless the way to write the original one.

Matrix diagonalization
(Iterated Power Method)

$$M = \begin{pmatrix} [2-\epsilon, 2+\epsilon] & [6-\epsilon, 6+\epsilon] & [5-\epsilon, 5+\epsilon] \\ [6-\epsilon, 6+\epsilon] & [2-\epsilon, 2+\epsilon] & [8-\epsilon, 8+\epsilon] \\ [5-\epsilon, 5+\epsilon] & [8-\epsilon, 8+\epsilon] & [6-\epsilon, 6+\epsilon] \end{pmatrix}.$$

If one uses *scilab* to compute the spectrum of the previous matrix without radius ($\epsilon = 0$), the highest eigenvalue is approximatively 16.345903 and the corresponding eigenvector is (0.4728057, 0.5716783, 0.6705510). In order to show that arithmetics and interval algebra developed above are robust and stable, let's try to compute the highest eigenvalue of an interval matrix. One uses here the iterate power method, which is very simple and constitute the basis of several powerful methods such as deflation and others.

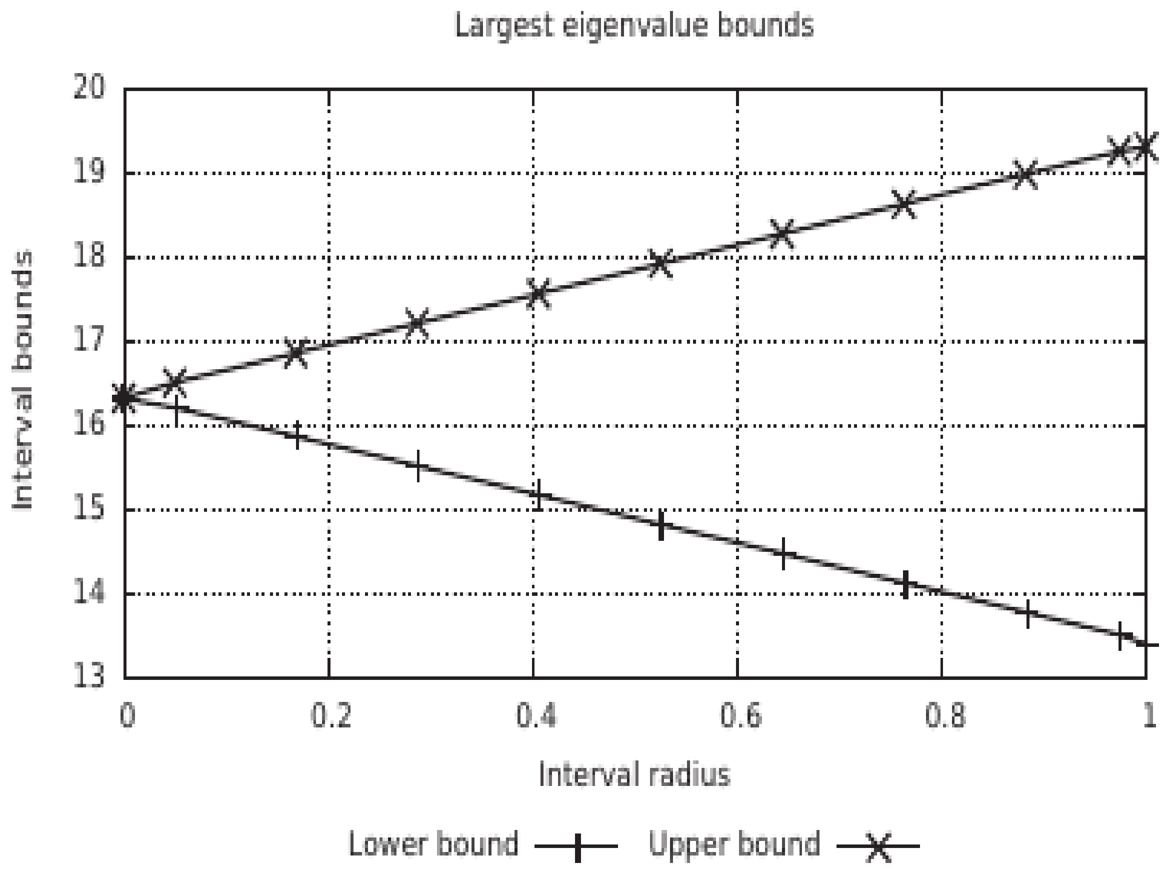


Figure 1: Largest eigenvalue convergence computed with iterate power method to the value computed with *scilab* 16.345903.

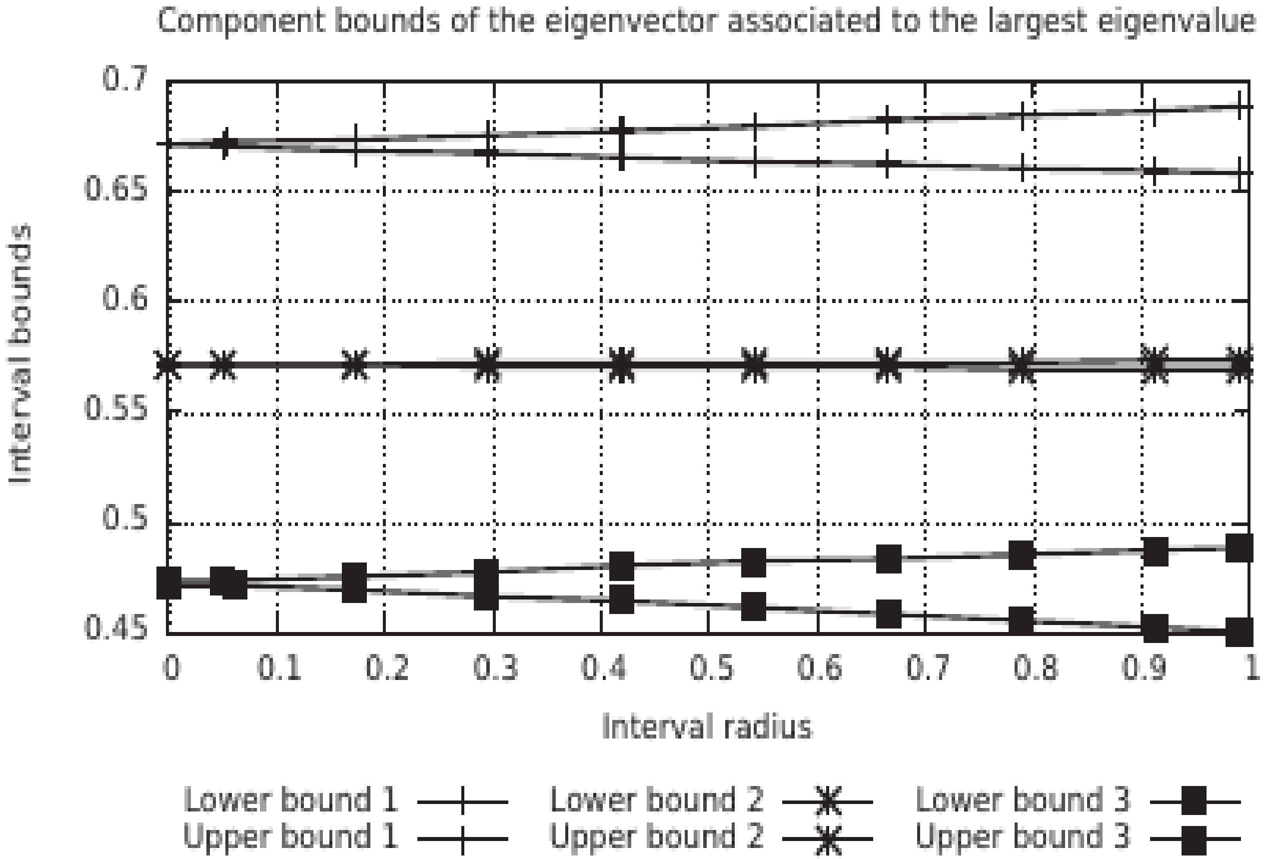


Figure 2: Eigenvector components associated to the largest eigenvalue convergence computed with iterate power method to the eigenvector computed with *scilab* (0.4728057, 0.5716783, 0.6705510).

Matrix Inversion (Schutz-Hotelling Algorithm)

$$X = \begin{pmatrix} [-2-\epsilon, -2+\epsilon] & [-7-\epsilon, 7+\epsilon] & [4-\epsilon, 4+\epsilon] \\ [5-\epsilon, 5+\epsilon] & [-1-\epsilon, -1+\epsilon] & [6-\epsilon, 6+\epsilon] \\ [9-\epsilon, 9+\epsilon] & [-8-\epsilon, -8+\epsilon] & [3-\epsilon, 3+\epsilon] \end{pmatrix} \quad X_0 = \frac{X^t}{\sum_{i,j} A_{ij}^2}, \; X_j = X_{j-1}(2 - A \cdot X_{j-1}), \; \forall n \geq 1$$

Scilab inversion function gives numerically for $\epsilon = 0$

$$X^{-1} = \begin{pmatrix} -0.0924025 & 0.0225873 & 0.0780287 \\ -0.0800821 & 0.0862423 & -0.0657084 \\ 0.0636550 & 0.1622177 & -0.0759754 \end{pmatrix}.$$

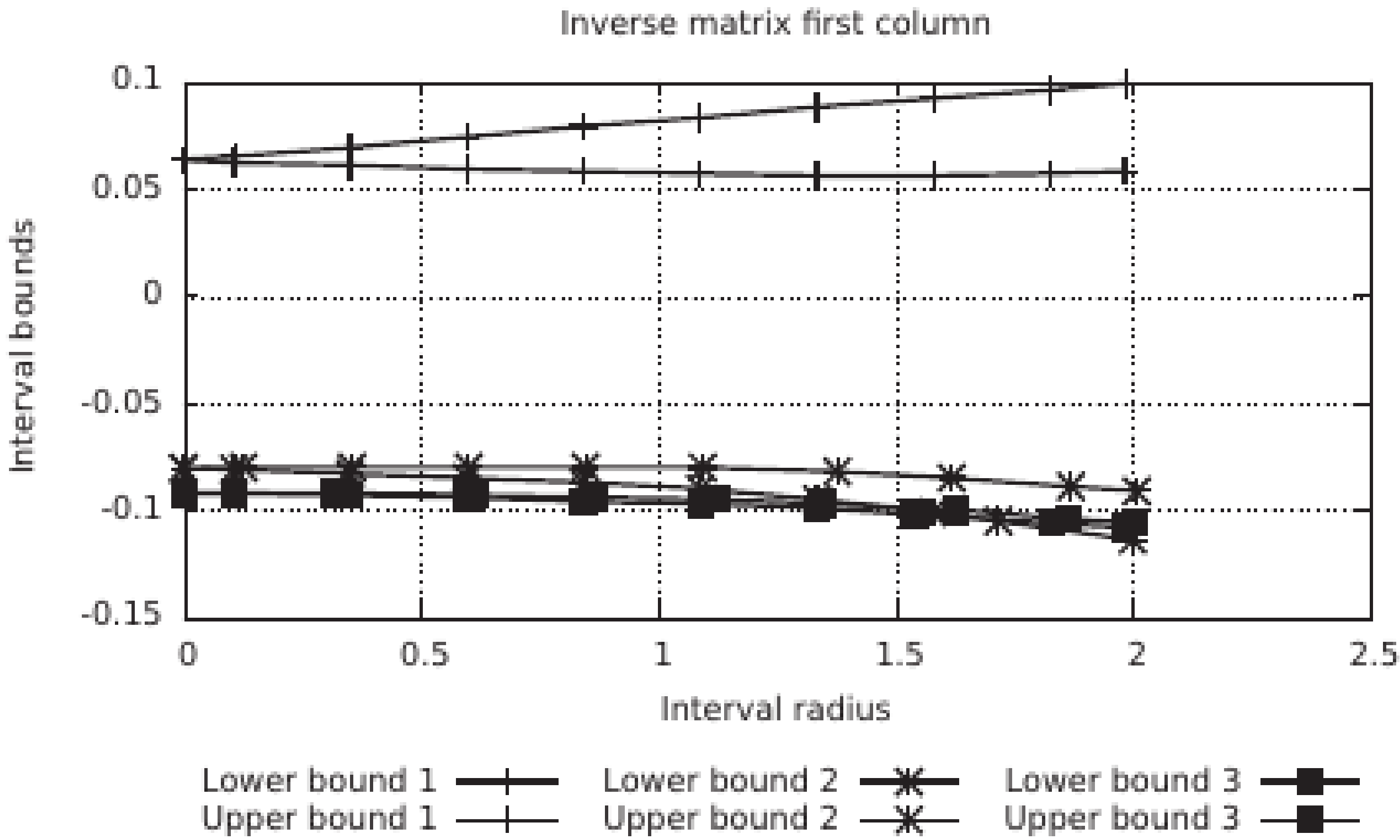


Figure 3: First column elements bounds convergence according interval radius ϵ.

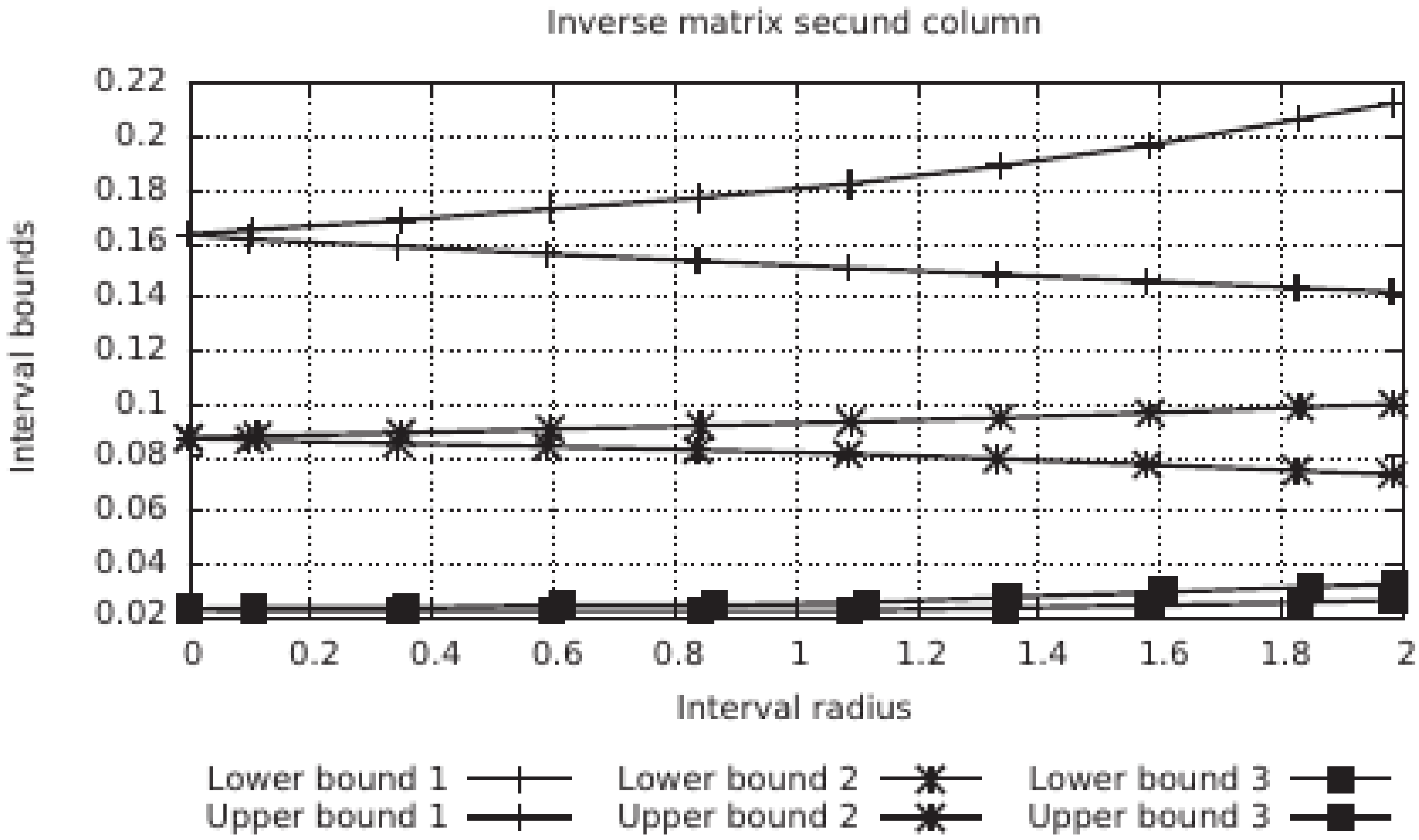


Figure 4: Second column elements bounds convergence according interval radius ϵ.

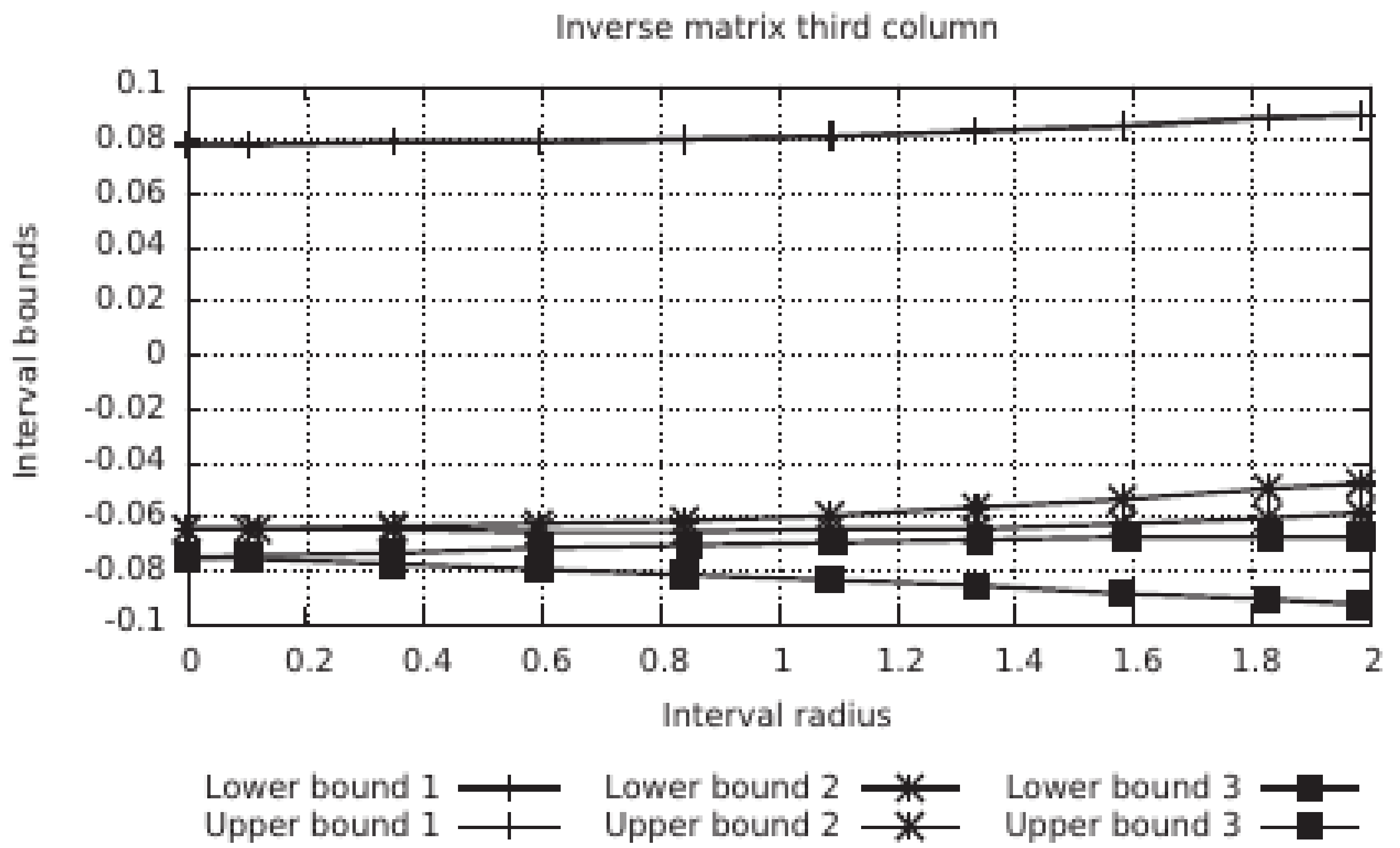


Figure 5: Third column elements bounds convergence according interval radius ϵ.

Topology

Any element $\mathcal{X} \in \overline{\mathbb{IR}}$ is written $\overline{(A,0)}$ or $\overline{(0,A)}$. We define its length $l(\mathcal{X})$ as the length of A and its center as $c(A)$ or $-c(A)$ in the second case.

Theorem 3. *The map* $|| \; || : \overline{\mathbb{IR}} \longrightarrow \mathbb{R}^+$ *given by*

$$||\mathcal{X}|| = l(\mathcal{X}) + |c(\mathcal{X})|$$

for any $\mathcal{X} \in \overline{\mathbb{IR}}$ *is a norm.*

Theorem 4. *The normed vector space* $\overline{\mathbb{IR}}$ *is a Banach space.*

We can consider another equivalent norms on $\overline{\mathbb{IR}}$. For example

$$||\mathcal{X}|| = || \setminus \mathcal{X}|| = \max\{|x|, |y|\}$$

where $\mathcal{X} = \overline{([x, y], 0)}$, but the initial one has a better geometrical interpretation.

Continuity & Differentiability

Definition 3. *A function $f : \overline{\mathbb{R}} \longrightarrow \overline{\mathbb{R}}$ is continuous at X_0 if*

$$\forall \varepsilon > 0, \exists \eta > 0 \ \textit{such that} \ ||X \setminus X_0|| < \eta \ \textit{implies} \ ||f(X) \setminus f(X_0)|| < \varepsilon.$$

Definition 4. *Consider X_0 in $\overline{\mathbb{R}}$ and $f : \overline{\mathbb{R}} \longrightarrow \overline{\mathbb{R}}$ continuous. We say that f is differentiable at X_0 if there exists a linear function $g : \overline{\mathbb{R}} \longrightarrow \overline{\mathbb{R}}$ such that*

$$||f(X) \setminus f(X_0) \setminus g(X \setminus X_0)|| = o(||X \setminus X_0||).$$

Examples.

- $f(\mathcal{X}) = \mathcal{X}$. This function is continuous and differentiable at any point. Its derivative is $f'(\mathcal{X}) = 1$.

- $f(\mathcal{X}) = \mathcal{X}^2$. Consider $\mathcal{X}_0 = \overline{(X_0, 0)} = \overline{([a, b], 0)}$ and $\mathcal{X} \in \mathcal{B}(\mathcal{X}_0, \varepsilon)$. We have

$$\begin{aligned} ||\mathcal{X}^2 \diagdown \mathcal{X}_0^2|| &= ||(\mathcal{X} \diagdown \mathcal{X}_0)(\mathcal{X} + \mathcal{X}_0)|| \\ &\leq ||\mathcal{X} \diagdown \mathcal{X}_0||\,||\mathcal{X} + \mathcal{X}_0||. \end{aligned}$$

  Given $\varepsilon > 0$, let $\eta = \dfrac{\varepsilon}{||\mathcal{X} + \mathcal{X}_0||}$, thus if $||\mathcal{X} \diagdown \mathcal{X}_0|| < \eta$, we have $||\mathcal{X}^2 \diagdown \mathcal{X}_0^2|| < \varepsilon$ and f is continuous and differentiable. It is easy to prove that $f'(\mathcal{X}) = 2\mathcal{X}$ is its derivative.

- Consider $P = a_0 + a_1 X + \cdots + a_n X^n \in \mathbb{R}[\mathbb{X}]$. We define $f : \overline{\mathbb{IR}} \longrightarrow \overline{\mathbb{IR}}$ with $f(\mathcal{X}) = a_0 \mathcal{X}_2 + a_1 \mathcal{X} + \cdots + a_n^n \mathcal{X}^n$ where $\mathcal{X}^n = \mathcal{X} \cdot \mathcal{X}^{n-1}$. From the previous example, all monomials are continuous and differentiable, it implies that f is continuous and differentiable as well.

- Consider the function Q_2 given by $Q_2([x, y]) = [x^2, y^2]$ if $|x| < |y|$ and $Q_2([x, y] = [y^2, x^2]$ in the other case. This function is not differentiable.

Optimization examples

Let's minimize the function $x \longmapsto x \cdot \exp(x)$ with fixed-step gradient method which belongs to the so-called gradient descent method [35]. This example is very simple but it shows that the result is garanted to be found within the final interval. One sets initial guess interval to $[-5, 2]$, fixed-step of descent to 10^{-2}, finite difference step to 10^{-3} and accuracy of gradient to 10^{-3}.

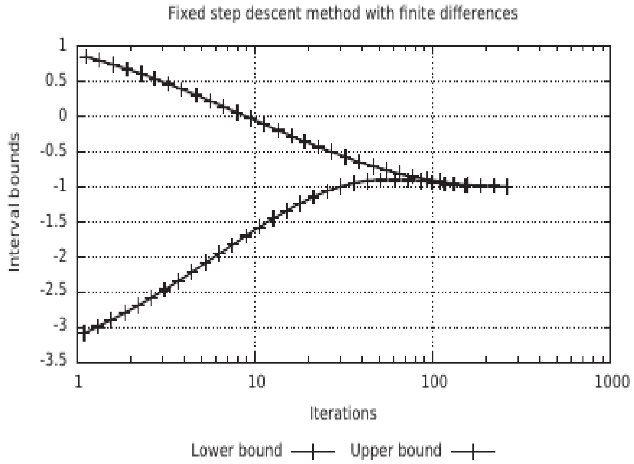


Figure 6: Convergence of the fixed-step gradient algorithm for the function $x \longmapsto x \cdot \exp(x)$ to an interval centered around $[-1, -1] = -1$.

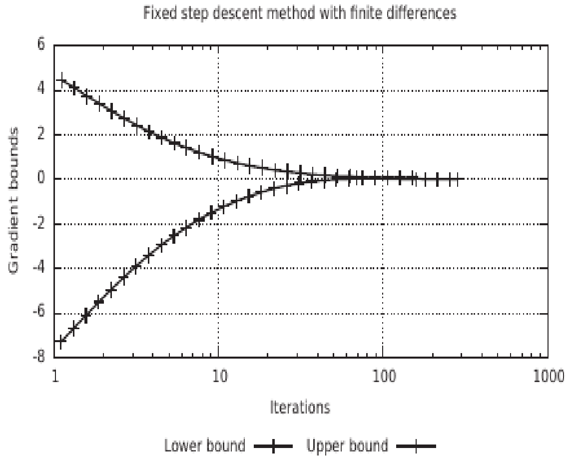


Figure 7: Convergence of the fixed-step gradient algorithm for the derivative of the function $x \longmapsto x \cdot \exp(x)$ to an interval centered around $[0, 0] = 0$.

Let's optimize the same function $x \longmapsto x \cdot \exp(x)$ with a second order method such as the Newton-Raphson one, which is the basis of all second order methods such as Newton or quasi-Newton's ones [35]. One sets initial guess interval to [0, 10], finite difference step to 10^{-3} and accuracy of gradient to 10^{-9}. One can state on Figures (8,9) that it finds the same minimum which is an interval centered around -1.

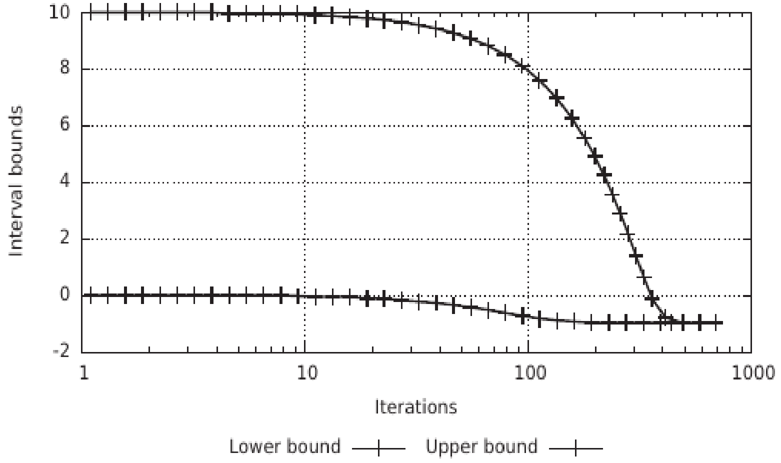


Figure 8: Convergence of the Newton-Raphson algorithm for the function $x \longmapsto x \cdot \exp(x)$ to an interval centered around $[-1, -1] = -1$.

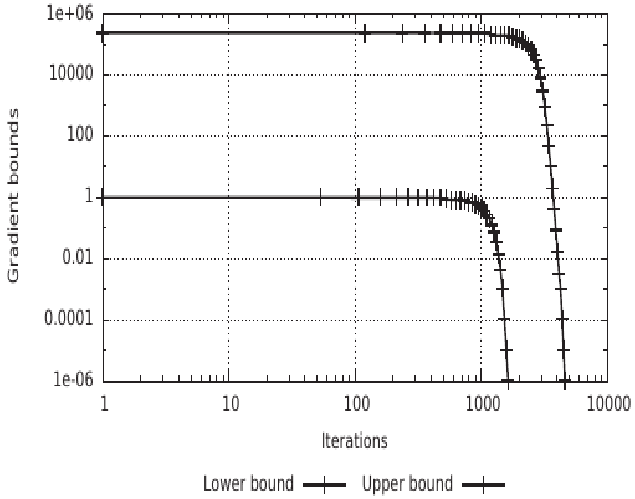


Figure 9: Convergence of the Newton-Raphson algorithm for the derivative of the function $x \longmapsto x \cdot \exp(x)$ to an interval centered around $[0, 0] = 0$.

PROBABILIST SET INVERSION: ψ-algorithm

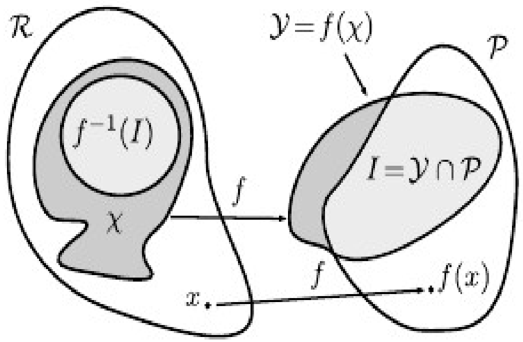


Figure 1: $\mathcal{R} \subset \mathbb{R}^n$ is the set of feasible adjustments, and $\mathcal{P} \subset \mathbb{R}^p$ the set of desired performance of a system. Set inversion computes $S = f^{-1}(\mathcal{P}) \cap \mathcal{R}$.

$$
\begin{aligned}
p(\mathcal{X}) &= p([f](\mathcal{X}) \subset \mathcal{P} \mid f(x) \in [f](\mathcal{X}),\ \forall x \in \mathcal{X}) \\
&= \frac{mes([f](\mathcal{X}) \cap \mathcal{P})}{mes([f](\mathcal{X}))} \\
&= \frac{mes(\mathcal{Y} \cap \mathcal{P})}{mes(\mathcal{Y})} = \frac{mes(\mathcal{I})}{mes(\mathcal{Y})}
\end{aligned}
$$

Set Inversion Examples

$$f_1(x, y) = (x^2 + y^2, x + y), \ \mathcal{R}_1 = [-1, 2]^2, \ \mathcal{P}_1 = [1, 2] \times [1, 4]$$

$$f_2(x, y) = (x^2 - y^2, \frac{y}{1 + x}), \ \mathcal{R}_2 = [0, 6] \times [-10, 10], \ \mathcal{P}_2 = [0, 5] \times [-4, 4].$$

$$f_3(x, y) = (x^2 - y^2 \cdot \exp(x) + x \cdot \exp(y), x \cdot (x + y) - y^2), \ \mathcal{R}_3 = \mathcal{P}_3 = [-5, 5]^2$$

$$f_4(x, y, z) = (x, y, z, x^2 - y^2 + z^2) \text{ with } \mathcal{R}_4 = [-5, 5]^3 \text{ and } \mathcal{P}_4 = [-10, 10]^4.$$

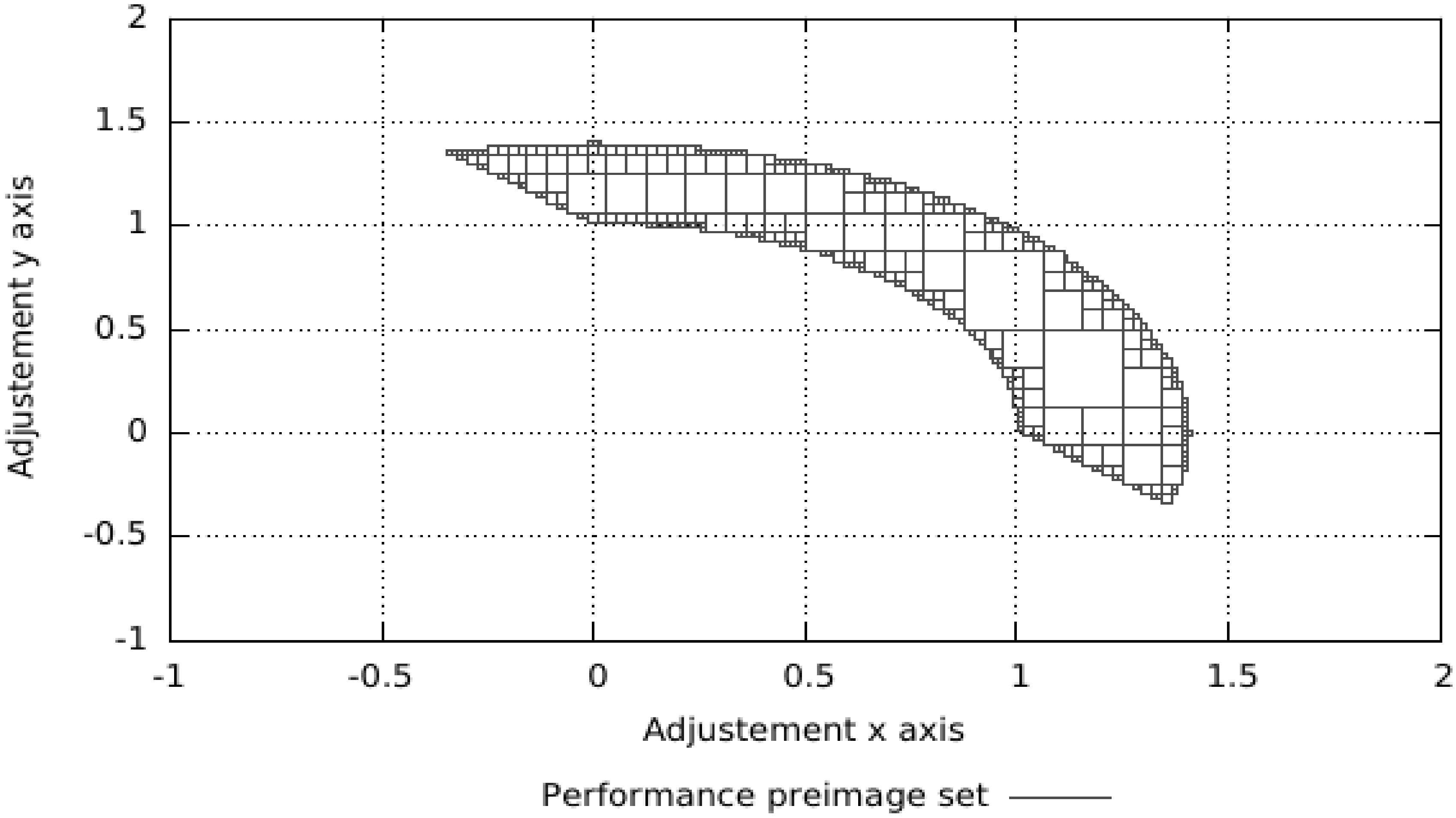


Figure 3: ψ-algorithm for f_1.

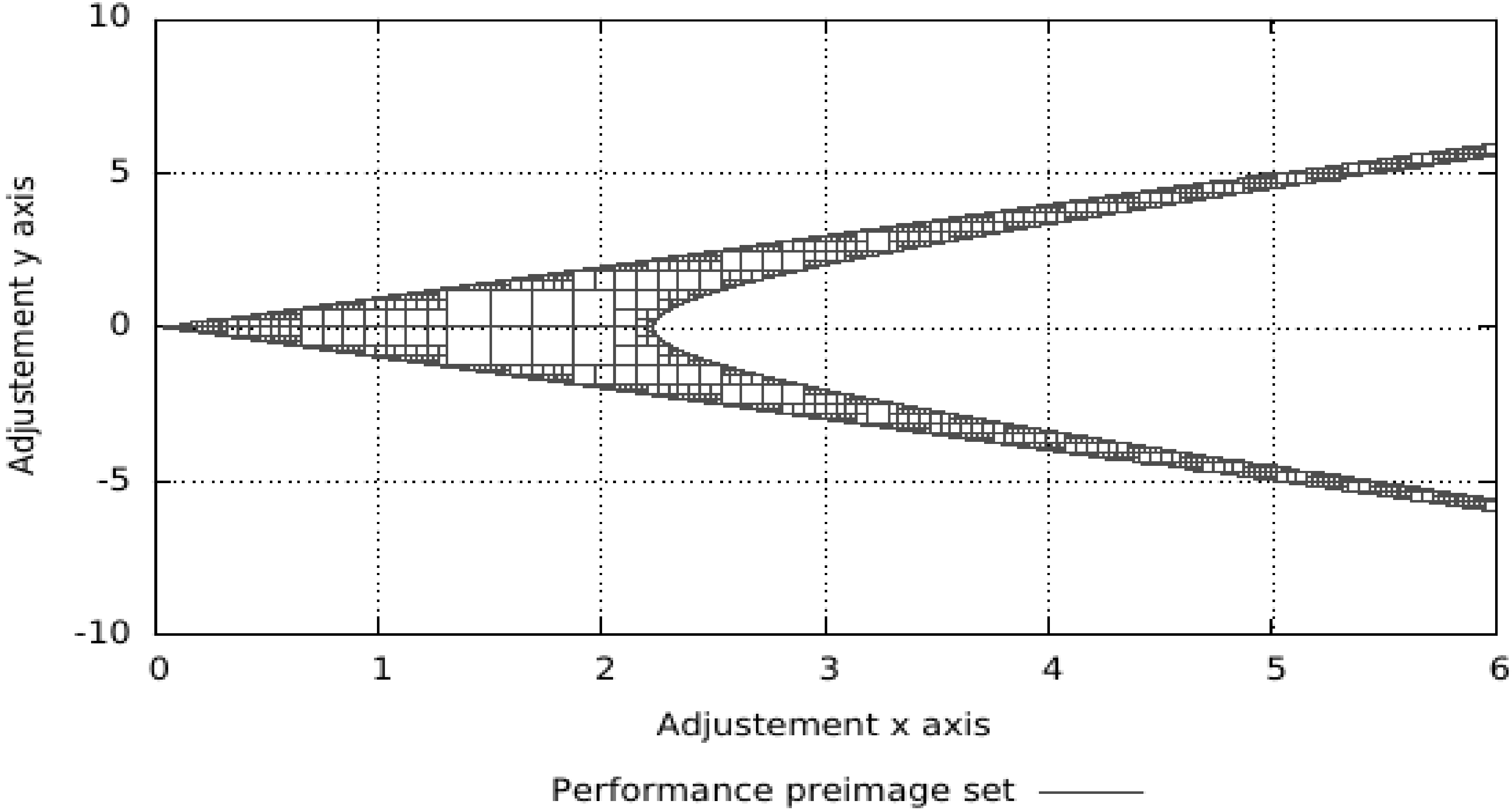


Figure 4: ψ-algorithm for f_2.

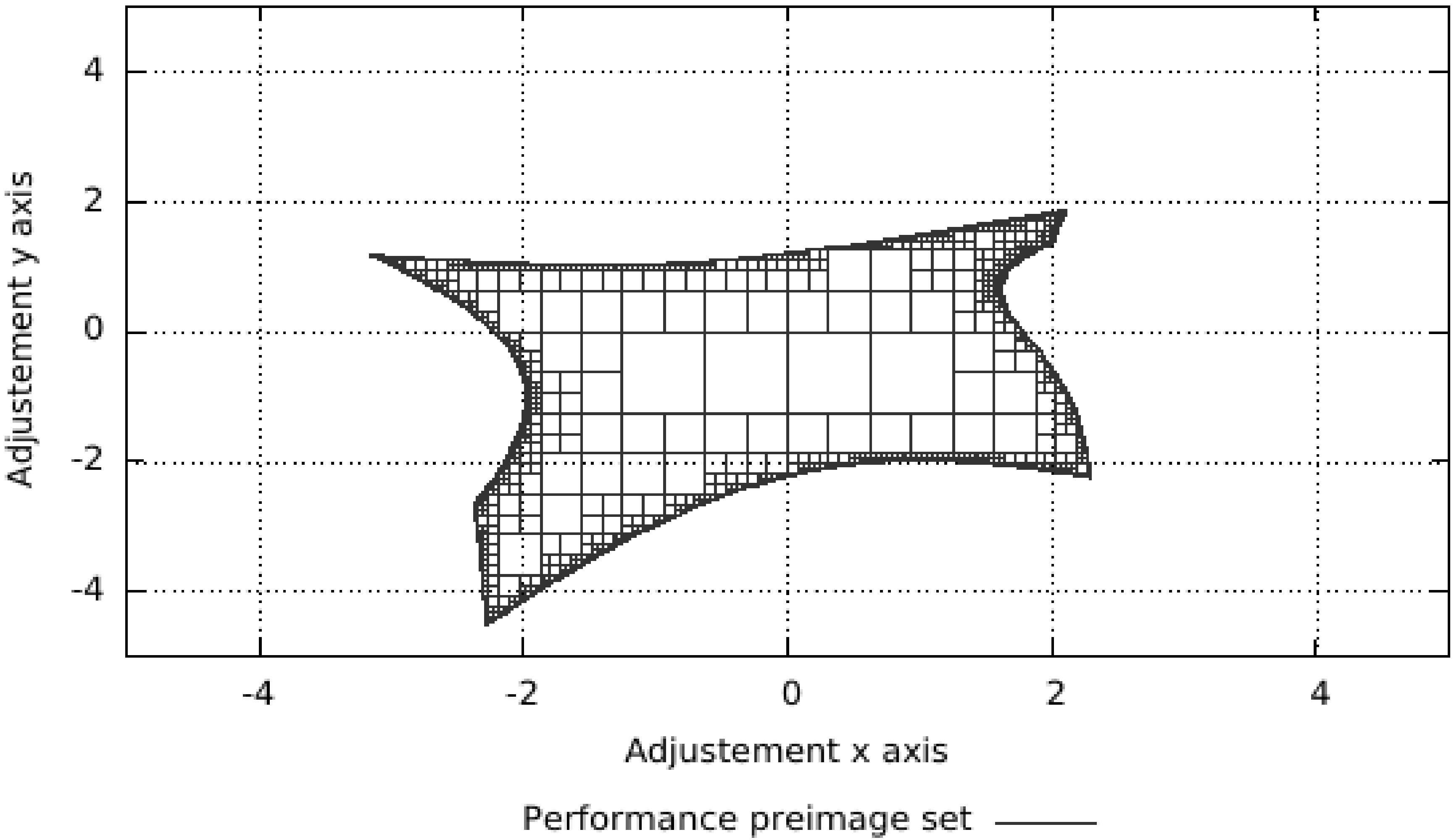


Figure 5: ψ-algorithm for f_3.

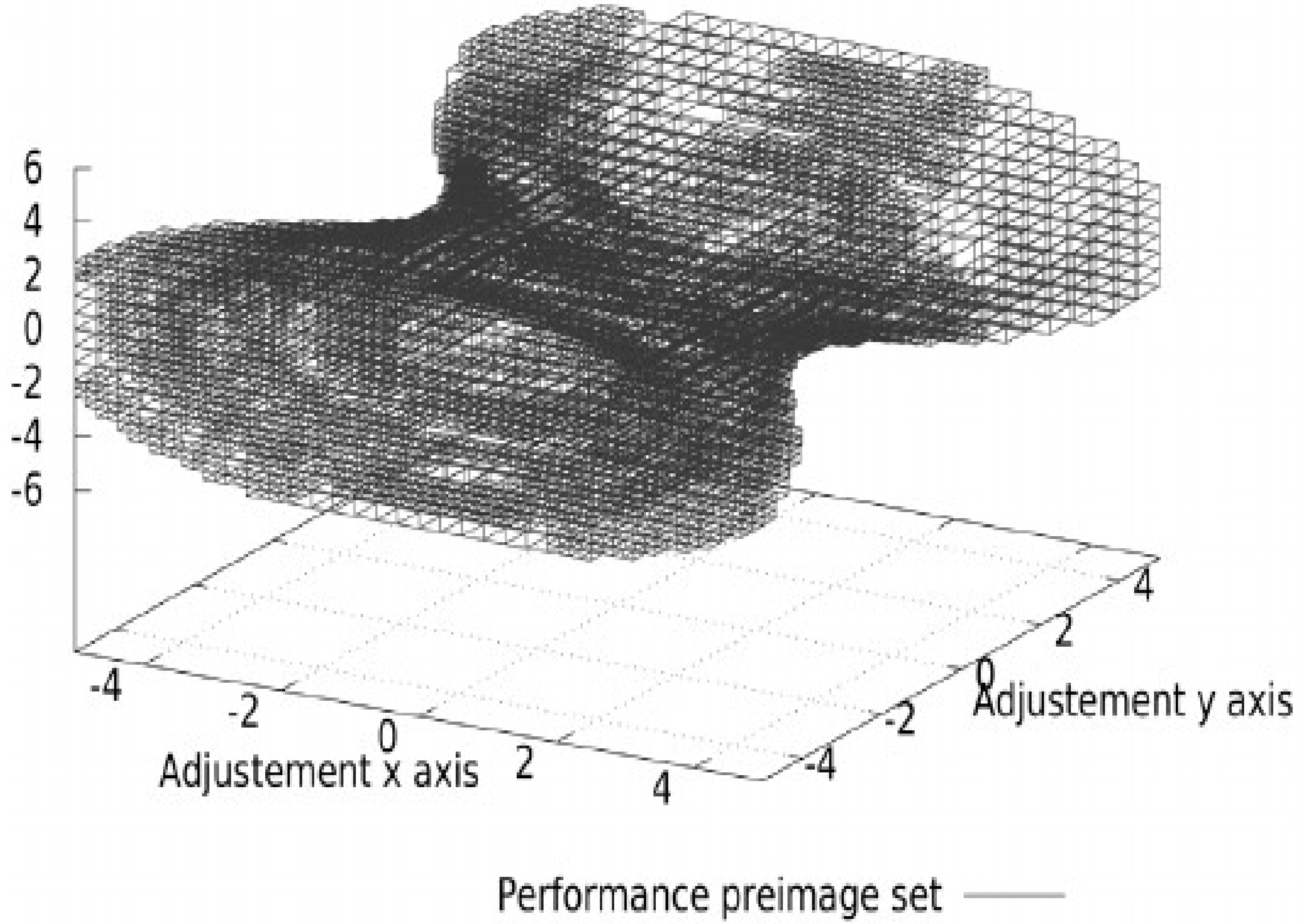


Figure 6: ψ-algorithm for $f_4(x, y, z) = (x, y, z, x^2 - y^2 + z^2)$ with $\mathcal{R}_4 = [-5, 5]^3$ and $\mathcal{P}_4 = [-10, 10]^4$.

Improvements of PSI

Due to the bisection, the algorithm computational complexity is exponential according to the iterations N, and it is not improved compared to SIVIA one. In our scheme, computational time is defined as

$$T_{comp} = \mathcal{O}(N) = k \cdot 2^N.$$

- **Adaptive mesh, with bisection spanned only in the space directions where the derivative magnitude is larger than a certain fixed value, because it is not useful to bisect in flat directions.**
- **Random and/or Monte-Carlo-like methods.**

Спасибо Большое !

Merci beaucoup !