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ON THE SOLUTION SET OF A LINEAR EQUATION WITH THE RIGHT-HAND SIDE AND OPERATOR GIVEN BY INTERVALS

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Introduction

While solving problems in various areas of knowledge by mathematical methods, the problem is often encountered of processing approximate data. The approximate character can be expressed, in particular, by interval representation of the results of the corresponding measurements. For solving problems of the type, research is intensively conducted in recent years within the so-called interval mathematics [1]. One of the problems, which can be ranked as classical, is that of solving a system of linear equations with the coefficients and the right-hand side given by intervals; i.e., finding the set of solutions x to an equation

$$\mathbb{A}x = \mathbb{B}, \tag{1}$$

where $\mathbb{A} \in I(R^{m\times n})$ is a real interval $m\times n$-matrix, $\mathbb{B} \in I(R^m)$ is an m-dimensional interval vector, and $x \in R^n$ [1]. The solution set of system (1) can be defined in different ways depending on the quantifiers that bound the coefficients of the matrix and the right-hand side [2]. In the present article we consider the following solution sets of equation (1), which are most frequent in the literature:

$\mathcal{R}_1 = \{x \in R^n \mid \exists C \in \mathbb{A}\ \exists c \in \mathbb{B}\ Cx = c\}$ is the total solution set [3–5],

$\mathcal{R}_2 = \{x \in R^n \mid \forall C \in \mathbb{A}\ \exists c \in \mathbb{B}\ Cx = c\}$ is the admissible solution set [6–8],

$\mathcal{R}_3 = \{x \in R^n \mid \forall c \in \mathbb{B}\ \exists C \in \mathbb{A}\ Cx = c\}$ is the set first considered while solving an interval modal control problem [9],

$\mathcal{R}_4 = \{(x \in R^n \mid \forall C \in \mathbb{A}\ \exists c \in \mathbb{B}\ Cx = c)\&(\forall d \in \mathbb{B}\ \exists D \in \mathbb{A}\ Dx = d)\}$ is the set of all pointwise algebraic interval solutions [10, 11].

The purpose of the present article is to describe these sets in terms of the endpoints of the intervals $\mathbb{A}$ and $\mathbb{B}$; moreover, we consider an analog of equation (1) in arbitrary ordered vector spaces.

§ 1. Description of the Solution Sets

We will introduce necessary notions and notation, basically following the terminology stemming from [12, 13].

Throughout the sequel, by F we mean some ordered vector space over R^1 and by E, some vector lattice. We shall denote the order relation in E and F (as well as in other ordered and preordered spaces introduced below) by the same symbol $\leq$. As usual, E_+ and F_+ are the cones of positive elements in E and F. If $x, y \in E$ then we denote by $x \vee y$ and $x \wedge y$ the supremum and the infimum of the elements x and y; by x^+ and x^-, the positive and negative parts of x; and by $|x| = x^+ + x^-$, the modulus of the element x. If F_1 is another ordered vector space then the space $L(F_1, F)$ of linear operators from F_1 into F is assumed to be preordered by means of the cone $L^+(F_1, F)$ of monotonic operators. If $a, b \in F$ and $a \leq b$ ($A, B \in L(F_1, F)$ and $A \leq B$) then, as usual, the set $[a, b] = \{c \in F \mid a \leq c \leq b\}$ ($[A, B] = \{C \in L(F_1, F) \mid A \leq C \leq B\}$) is an interval in F (in $L(F_1, F)$) and $I(F)$ $\big(I(L(F_1, F))\big)$ is the set of all nonempty intervals in F (in $L(F_1, F)$, respectively).

We denote by $\Lambda(F) \in I(L(F, F))$ the set of multipliers in F; i.e., $\Lambda(F) = [O_F, I_F]$, where O_F and I_F are the zero and identity operators in F. If $F_1, F_2 \subseteq F$ and $\lambda_1, \lambda_2 \in R^1$ then, as usual, $\lambda_1 F_1 + \lambda_2 F_2 = \{\lambda_1 x_1 + \lambda_2 x_2 \mid x_1 \in F_1, x_2 \in F_2\}$.

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We consider equation (1) in the case when $\mathbb{A} = [A,B] \in I(L(E,F))$ and $\mathbb{B} = [a,b] \in I(F)$. In this case, if we denote $[A,B]x = \{Cx \mid C \in [A,B]\}$ for $x \in E$ then the sets $\mathcal{R}_i$, $i = \overline{1,4}$, can be represented as

$$\begin{gathered}\mathcal{R}_1 = \{x \in E \mid ([A,B]x) \cap [a,b] \neq \varnothing\}, \quad \mathcal{R}_2 = \{x \in E \mid [A,B]x \subseteq [a,b]\}, \\ \mathcal{R}_3 = \{x \in E \mid [A,B]x \supseteq [a,b]\}, \quad \mathcal{R}_4 = \{x \in E \mid [A,B]x = [a,b]\}.\end{gathered} \tag{2}$$

The following relations between $\mathcal{R}_i$, $i = \overline{1,4}$, are obvious:

$$\mathcal{R}_1 \supseteq \mathcal{R}_2 \cup \mathcal{R}_3 \supseteq \mathcal{R}_4 = \mathcal{R}_2 \cap \mathcal{R}_3.$$

We will introduce notions that will be needed for describing these sets.

DEFINITION 1. Say that an ordered vector space F *possesses property* C (compressibility) if, for all $x, y \in F_+$, $x \le y$, there exists a multiplier $\alpha \in \Lambda(F)$ that takes y into x, i.e. such that $\alpha y = x$. We denote by $\mathcal{K}(C)$ the class of ordered vector spaces possessing property C.

DEFINITION 2. Say that a vector lattice E *possesses property* D (disjointness) if, for all $x, y \in E_+$, $x \wedge y = 0$, there exists a multiplier $\alpha \in \Lambda(E)$ such that $\alpha x = x$ and $\alpha y = 0$. We denote by $\mathcal{K}(D)$ the class of vector lattices possessing property D.

A characterization of the classes $\mathcal{K}(C)$ and $\mathcal{K}(D)$ is contained in the following lemmas.

Lemma 1. *For every ordered vector space, the following conditions are equivalent:*

(a) $F \in \mathcal{K}(C)$;

(b) *for every* $x \in F_+$, *the equality* $\Lambda(F)x = [0,x]$ *holds;*

(c) *for every ordered vector space* F_1 *and every* $[A,B] \in I(L(F_1,F))$, $y \in F_{1+}$, *the equality* $[A,B]y = [Ay, By]$ *holds.*

PROOF. Equivalence of conditions (b) and (a) is immediate from Definition 1.

We will show equivalence of assertions (a) and (c). Suppose that $F \in \mathcal{K}(C)$, $[A,B] \in I(L(F_1,F))$ and $y \in F_{1+}$. The inclusion $[A,B]y \subseteq [Ay,By]$ is obvious. Let $x \in [Ay,By]$. Then $0 \le x - Ay \le (B-A)y$ and, consequently, there exists an $\alpha \in \Lambda(F)$ such that $\alpha(B-A)y = x - Ay$. By letting $C = A + \alpha(B-A)$, we obtain $C \in [A,B]$ and $Cy = x$, i.e., $x \in [A,B]y$. Consequently, $[Ay,By] \subseteq [A,B]y$, and (c) is valid. Now, let condition (c) be satisfied for F. Assign $F_1 = F$, $A = O_F$, and $B = I_F$. Then, for each $x \in F_+$, the equality $[O_F, I_F]x = [0,x]$ holds; therefore, $F \in \mathcal{K}(C)$.

Lemma 2. *For every vector lattice* E, *the following conditions are equivalent:*

(a) $E \in \mathcal{K}(D)$;

(b) *for every* $x \in E$, *there exists an* $\alpha \in \Lambda(E)$ *such that the equality* $\alpha x = x^+$ *holds;*

(c) *for every ordered vector space* F *and every* $[A,B] \in I(L(E,F))$, $x \in E$, *the equality* $[A,B]x = [A,B]x^+ - [A,B]x^-$ *holds;*

(d) *for every* $x \in E$, *the equality* $\Lambda(E)x = \Lambda(E)x^+ - \Lambda(E)x^-$ *holds.*

PROOF. Equivalence of conditions (b) and (a) is immediate from Definition 2.

Show that (b) implies (c). Suppose that F is an ordered vector space, $[A,B] \in I(L(E,F))$, and $x \in E$. The inclusion $[A,B]x \subseteq [A,B]x^+ - [A,B]x^-$ is obvious. Test the reverse inclusion. Let $y \in [A,B]x^+ - [A,B]x^-$. Then $y = C_1x^+ - C_2x^-$ for some $C_1, C_2 \in [A,B]$. Take $\alpha \in \Lambda(E)$ such that $\alpha x = x^+$. By putting $C = C_1\alpha + C_2(I_E - \alpha)$, we obtain $Cx = y$ and $C \in [A,B]$, i.e., $y \in [A,B]x$.

To prove the fact that (c) implies (d), it suffices to assign $F = E$, $A = O_E$, and $B = I_E$ in the statement of condition (c).

The fact that (d) implies (b) is obvious, since always $x^+ \in \Lambda(E)x^+ - \Lambda(E)x^-$. The lemma is proven.

It is interesting to note that we may define the class $\mathcal{K}(D)$ without explicitly using the notions of supremum and infimum, as is seen from the next proposition:

Proposition 1. *For every ordered vector space* F, *the following conditions are equivalent:*

(a) F *is a vector lattice and* $F \in \mathcal{K}(D)$;

(b) *for every* $x \in F$, *there exist* $x_1, x_2 \in F_+$ *and* $\alpha \in \Lambda(F)$ *such that* $x = x_1 - x_2$, $\alpha x_1 = x_1$, *and* $\alpha x_2 = 0$.

PROOF. The fact that condition (a) implies (b) is immediate from the definition of the class $\mathcal{K}(D)$. We will show the converse. Let F satisfy condition (b). It is known [12] that if $x \vee 0$ exists for each $x \in F$ then F is a vector lattice. Take $x \in F$, $x_1, x_2 \in F_+$, and $\alpha \in \Lambda(F)$ such that $x = x_1 - x_2$, $\alpha x_1 = x_1$, and $\alpha x_2 = 0$. Demonstrate that $x_1 = x \vee 0$. Indeed, obviously, $x_1 \geq 0$ and $x_1 \geq x$, and if $y \geq 0$ and $y \geq x$ then $y \geq \alpha y \geq \alpha x = \alpha x_1 - \alpha x_2 = x_1$. Consequently, $x_1 = x \vee 0$, and F is a vector lattice. Moreover, from the equality $x_1 = x \vee 0 = x^+$ it follows that $\alpha x = x^+$ and condition (b) of Lemma 2 is satisfied. Therefore, $F \in \mathcal{K}(D)$.

As is known, the sum of two intervals in any vector lattice is an interval too. It turns out that this assertion is also valid in any ordered vector space that possesses property C.

Lemma 3. *Suppose that $F \in \mathcal{K}(C)$, $[a,b], [c,d] \in I(F)$. Then $[a,b] + [c,d] = [a+c, b+d]$.*

PROOF. The inclusion $[a,b] + [c,d] \subseteq [a+c, b+d]$ is obvious. We will show the reverse one. Suppose that $x \in [a+c, b+d]$. Then $0 \leq x - (a+c) \leq (b+d) - (a+c)$ and there exists an $\alpha \in \Lambda(F)$ such that $\alpha(b+d-a-c) = x - a - c$. Denote $x_1 = a + \alpha(b-a)$ and $x_2 = c + \alpha(d-c)$. Then it is obvious that $x_1 \in [a,b]$, $x_2 \in [c,d]$, and $x_1 + x_2 = x$, i.e., $x \in [a,b] + [c,d]$.

Significance of the introduced classes $\mathcal{K}(C)$ and $\mathcal{K}(D)$ is explained by the fact that the following theorem is valid for them, which allows one to obtain a description for the sets $\mathcal{R}_i$, $i = \overline{1,4}$.

Theorem 1. *Suppose that $E \in \mathcal{K}(D)$, $F \in \mathcal{K}(C)$, $[A,B] \in I(L(E,F))$, and $x \in E$. Then $[A,B]x \in I(F)$ and the following formula holds:*

$$[A,B]x = [Ax^+ - Bx^-, Bx^+ - Ax^-]. \tag{3}$$

The proof results from applying assertions (c) of Lemmas 1 and 2 and Lemma 3.

REMARK. In the case when $E = R^n$ and $F = R^m$ with coordinatewise order, the theorem essentially repeats the known Oettli-Prager theorem [14].

From Theorem 1 we immediately obtain a description for $\mathcal{R}_i$, $i = \overline{2,4}$.

Theorem 2. *Suppose that $E \in \mathcal{K}(D)$, $F \in \mathcal{K}(C)$, $[A,B] \in I(L(E,F))$, and $[a,b] \in I(F)$. Then*

$$\begin{aligned} \mathcal{R}_2 &= \{x_1 - x_2 \mid x_1 \geq 0,\ x_2 \geq 0,\ Ax_1 - Bx_2 \geq a,\ Bx_1 - Ax_2 \leq b\}, \\ \mathcal{R}_3 &= \{x_1 - x_2 \mid x_1 \wedge x_2 = 0,\ Ax_1 - Bx_2 \leq a,\ Bx_1 - Ax_2 \geq b\}, \\ \mathcal{R}_4 &= \{x_1 - x_2 \mid x_1 \wedge x_2 = 0,\ Ax_1 - Bx_2 = a,\ Bx_1 - Ax_2 = b\}. \end{aligned} \tag{4}$$

PROOF. Denote the sets in the right-hand side of formulas (4) by $\overline{\mathcal{R}}_i$, $i = \overline{2,4}$. Demonstrate that $\mathcal{R}_i = \overline{\mathcal{R}}_i$. The inclusion $\mathcal{R}_i \subseteq \overline{\mathcal{R}}_i$ for $i = \overline{2,4}$ results from the representation of any $x \in E$ in the form $x = x^+ - x^-$, formula (3), and the following obvious observation: if $[a_1, b_1], [a_2, b_2] \in I(F)$ then $[a_1, b_1] \subseteq [a_2, b_2]$ if and only if $a_1 \geq a_2$ and $b_1 \leq b_2$. The reverse inclusion for $i = 3, 4$ follows from the fact that if $x = x_1 - x_2$ and $x_1 \wedge x_2 = 0$ then $x_1 = x^+$, $x_2 = x^-$, and formula (3) is again applicable. Show that $\overline{\mathcal{R}}_2 \subseteq \mathcal{R}_2$. Suppose that $x = x_1 - x_2 \in \overline{\mathcal{R}}_2$. Denote $\bar{x}_1 = x_1 - x_1 \wedge x_2$, $\bar{x}_2 = x_2 - x_1 \wedge x_2$. It is obvious that $x = \bar{x}_1 - \bar{x}_2$, $\bar{x}_1 \wedge \bar{x}_2 = 0$; consequently, $\bar{x}_1 = x^+$, $\bar{x}_2 = x^-$ and, in addition, $A\bar{x}_1 - B\bar{x}_2 = Ax_1 - Bx_2 + (B-A)x_1 \wedge x_2 \geq a$ and $B\bar{x}_1 - A\bar{x}_2 = Bx_1 - Ax_2 - (B-A)x_1 \wedge x_2 \leq b$, since $B - A \geq 0$ and $x_1 \wedge x_2 \geq 0$. Therefore, by formula (3), $[A,B]x \subseteq [a,b]$ and $x \in \mathcal{R}_2$. The theorem is proven.

In order to obtain an analogous description for the set $\mathcal{R}_1$, at least in the case when $E \in \mathcal{K}(D)$ and $F \in \mathcal{K}(C)$, it is necessary in virtue of Theorem 1 to have some criterion for intervals in F to meet. It is easy to observe that if $[a_1, b_1], [a_2, b_2] \in I(F)$ and $[a_1, b_1] \cap [a_2, b_2] \neq \varnothing$ then $a_1 \leq b_2$ and $a_2 \leq b_1$. We will indicate a class of ordered vector spaces, for which the converse is valid as well, i.e., if $a_1 \leq b_2$ and $a_2 \leq b_1$ then $[a_1, b_1] \cap [a_2, b_2] \neq \varnothing$.

Introduce some relation $\prec$ on nonempty subsets of F as follows: if $F_1, F_2 \subseteq F$ then $F_1 \prec F_2$ if and only if $x \leq y$ for all $x \in F_1$, $y \in F_2$.

DEFINITION 3. We say that an ordered vector space F is a *K_0-space* if, for all nonempty subsets $F_1, F_2 \subseteq F$, the following condition is satisfied:

$$F_1 \prec F_2 \rightarrow (\exists \alpha \in F)\, F_1 \prec \{\alpha\} \prec F_2; \tag{5}$$

i.e., if $F_1 \prec F_2$ then there is an element $\alpha \in F$ serving as an upper bound for F_1 and a lower bound for F_2.

Observe that, using condition (5), we can also define other well-known classes of ordered spaces. For instance, if this condition holds for all nonempty subsets at least one of which is finite, then we obtain the definition of a conditional vector lattice; and if F is a vector lattice and condition (5) is satisfied for all nonempty F_1 and F_2 (one of which is at most countable), then we obtain the definition of a K-space (a K_σ-space).

Lemma 4. *For an ordered vector space F to be a K_0-space, it is necessary and sufficient that one of the following conditions be satisfied*:

(a) *for all $a_1, a_2, b_1, b_2 \in F$, if $a_i \leq b_j$, $i, j = 1, 2$ then there exists an element $c \in F$ such that $a_i \leq c \leq b_j$ (i.e., condition (5) is satisfied for two-element sets F_1 and F_2)*;

(b) *we have the following criterion for any two intervals, $[a_1, b_1], [a_2, b_2] \in I(F)$, to meet: $[a_1, b_1] \cap [a_2, b_2] \neq \varnothing$ if and only if $a_1 \leq b_2$, $a_2 \leq b_1$.*

The proof is straightforward from Definition 3.

Obviously, the class of K_0-spaces contains that of vector lattices and is also closed under direct products, direct sums, and the passage to subspaces whose intersection with the cone of positive elements is normal (= an order ideal) in the latter.

With the help of Lemma 4, for K_0-spaces, we obtain a description for the set $\mathcal{R}_1$.

Theorem 3. *If $E \in \mathcal{K}(D)$, $F \in \mathcal{K}(C)$, $[A, B] \in I(L(E, F))$, $[a, b] \in I(F)$, and F is a K_0-space, then*

$$\mathcal{R}_1 = \{x_1 - x_2 \mid x_1 \wedge x_2 = 0,\ Ax_1 - Bx_2 \leq b,\ Bx_1 - Ax_2 \geq a\}. \tag{6}$$

The proof is similar to that of Theorem 2.

In the case when the vector space F is a vector lattice, by using the centrally symmetric representation of intervals, we obtain the following description for the sets $\mathcal{R}_i$:

Theorem 4. *Suppose that $E \in \mathcal{K}(D)$, $F \in \mathcal{K}(C)$, $[A, B] \in I(L(E, F))$, $[a, b] \in I(F)$, and F is a vector lattice. Assign*

$$C = \frac{1}{2}(A + B), \quad D = \frac{1}{2}(B - A), \quad c = \frac{1}{2}(a + b), \quad d = \frac{1}{2}(b - a).$$

Then

$$\begin{aligned} \mathcal{R}_1 &= \{x \mid D|x| + d \geq |Cx - c|\}, & \mathcal{R}_2 &= \{x \mid d \geq D|x| + |Cx - c|\}, \\ \mathcal{R}_3 &= \{x \mid D|x| \geq d + |Cx - c|\}, & \mathcal{R}_4 &= \{x \mid Cx = c,\ D|x| = d\}. \end{aligned} \tag{7}$$

PROOF. We will prove formula (7) for $\mathcal{R}_1$ (for the remaining $\mathcal{R}_i$'s, the proof is similar). Since F is a vector lattice; F is a K_0-space and, therefore, $\mathcal{R}_1$ is representable as (6). Moreover, by using equalities $x^+ = \frac{1}{2}(|x| + x)$ and $x^- = \frac{1}{2}(|x| - x)$, we obtain:

$$\begin{aligned} Ax^+ - Bx^- &= \frac{1}{2}(A + B)x - \frac{1}{2}(B - A)|x| = Cx - D|x|, \\ Bx^+ - Ax^- &= \frac{1}{2}(A + B)x + \frac{1}{2}(B - A)|x| = Cx + D|x|. \end{aligned} \tag{8}$$

Next, as has already been mentioned, if $x = x_1 - x_2$ and $x_1 \wedge x_2 = 0$ then $x_1 = x^+$ and $x_2 = x^-$; therefore, from (6) it follows that $x \in \mathcal{R}_1$ if and only if

$$Ax^+ - Bx^- \leq b, \quad Bx^+ - Ax^- \geq a. \tag{9}$$

Substituting equalities (8) into these formulas and taking it into account that $a = c - d$ and $b = c + d$, we conclude that (9) is equivalent to $Cx - D|x| \leq c + d$ and $Cx + D|x| \geq c - d$, i.e. to

$$D|x| + d \geq Cx - c \geq -(D|x| + d). \tag{10}$$

Since F is a vector lattice, for any elements $y, z \in F$ the inequality $-z \leq y \leq z$ is equivalent to the inequality $|y| \leq z$; therefore, (10) is equivalent to $D|x| + d \geq |Cx - c|$ and the proof is complete.

Observe that the representation of $\mathcal{R}_4$ in the form (7) is also valid without the supposition that F is a vector lattice.

For the case $E = R^n$ and $F = R^m$, both ordered coordinatewise, a description for the set $\mathcal{R}_1$ in the form (7) was obtained in the paper [14] and a description for $\mathcal{R}_2$ in the form (7), as well as in the form (8), in the paper [8].

REMARK. The nonlinear condition $x_1 \wedge x_2 = 0$ in the description for the sets $\mathcal{R}_i$, $i = 1, 3, 4$, together with the positiveness restriction $x_1, x_2 \in E_+$, is bilinear in some cases.

In particular, if the lattice E is Archimedean then, on making use of an embedding of E into the space $C_\infty(Q)$ of continuous functions on Q with values in $R^1 \cup \{+\infty, -\infty\}$ for some extremally disconnected compact space Q and the multiplication operation for functions in $C_\infty(Q)$, we conclude that there exists a symmetric bilinear mapping $\varphi : E \times E \to C_\infty(Q)$ such that, for $x_1, x_2 \in E_+$, the conditions $x_1 \wedge x_2 = 0$ and $\varphi(x_1, x_2) = 0$ are equivalent. If, in addition, E has finite dimension n then, as is known, E is algebraically and latticially isomorphic to R^n with coordinatewise order and, consequently, we may take the scalar product as φ. More precisely, there exists a symmetric bilinear mapping $\varphi : E \times E \to R^1$ such that the quadratic form generated by it is positive definite and, for $x_1, x_2 \in E_+$, the conditions $x_1 \wedge x_2 = 0$ and $\varphi(x_1, x_2) = 0$ are equivalent. Thus, in the case when $E = R^m$ and $F = R^n$ both ordered coordinatewise, we obtain the following description for the sets $\mathcal{R}_i$:

Corollary. *Suppose that* $[A, B] \in I(R^{m \times n})$ *and* $[a, b] \in I(R^m)$. *Then*

$$\mathcal{R}_1 = \{x_1 - x_2 \mid x_1 \geq 0,\ x_2 \geq 0,\ (x_1, x_2) = 0,\ Ax_1 - Bx_2 \leq b,\ Bx_1 - Ax_2 \geq a\},$$
$$\mathcal{R}_2 = \{x_1 - x_2 \mid x_1 \geq 0,\ x_2 \geq 0,\ Ax_1 - Bx_2 \geq a,\ Bx_1 - Ax_2 \leq b\},$$
$$\mathcal{R}_3 = \{x_1 - x_2 \mid x_1 \geq 0,\ x_2 \geq 0,\ (x_1, x_2) = 0,\ Ax_1 - Bx_2 \leq a,\ Bx_1 - Ax_2 \geq b\},$$
$$\mathcal{R}_4 = \{x_1 - x_2 \mid x_1 \geq 0,\ x_2 \geq 0,\ (x_1, x_2) = 0,\ Ax_1 - Bx_2 = a,\ Bx_1 - Ax_2 = b\},$$

where $(\cdot, \cdot)$ *is the inner product in* R^n.

§ 2. Studying Properties of the Classes $\mathcal{K}(C)$ and $\mathcal{K}(D)$

We now turn to settling the following question: which of the known classes of ordered vector spaces possess properties C or D? First, we will indicate some structural properties of the classes $\mathcal{K}(C)$ and $\mathcal{K}(D)$ which will show, in particular, that the classes are rather wide. We use conventional methods and constructions [12].

Proposition 2. (a) *The classes* $\mathcal{K}(C)$ *and* $\mathcal{K}(D)$ *are closed under direct products and direct sums.*

(b) *If* $E \in \mathcal{K}(D)$ *and* E_1 *is a normal subspace of* E, *then* $E_1 \in \mathcal{K}(D)$.

(c) *If* $F \in \mathcal{K}(C)$ *and* F_1 *is a subspace of* F *such that the cone* $F_1 \cap F_+$ *is normal in* F_+, *then* $F_1 \in \mathcal{K}(C)$.

(d) *If* $E \in \mathcal{K}(D)$, E_0 *is a linearly ordered vector space, and* $E_0 \circ E$ *is the lexicographic product of* E_0 *and* E [15], *then* $E_0 \circ E \in \mathcal{K}(D)$.

(e) *If* $F \in \mathcal{K}(C)$ *then* $R^1 \circ F \in \mathcal{K}(C)$.

PROOF. Assertion (a) is immediate from the definitions.

To prove (b), it is sufficient to observe that if E_1 is normal in E and $\alpha \in \Lambda(E)$, then $\alpha(E_1) \subseteq E_1$ and, consequently, the restriction of α onto E_1 belongs to $\Lambda(E_1)$.

Suppose that $F \in \mathcal{K}(C)$, F_1 is a subspace of F, and $F_{1+} = F_1 \cap F_+$ is normal in F_+. Denote by $P : F \to F_1$ any projection from F onto F_1 and assign $\alpha_P = P\alpha$ for $\alpha \in \Lambda(E)$. Then it is easy to show that $\alpha_P(F) \subseteq F_1$ and $\alpha_P x = \alpha x$ for every $x \in F_{1+}$. Therefore, the restriction of α_P onto F_1 belongs to $\Lambda(F_1)$ and if $x, y \in F_{1+}$, $x \leq y$, and $\alpha \in \Lambda(E)$ is such that $\alpha y = x$, then $\alpha_P y = \alpha y = x$. Consequently, $F_1 \in \mathcal{K}(C)$.

We will now prove (d). By the definition of lexicographic product, $E_0 \circ E = \{(x_0, x) \mid x_0 \in E_0, x \in E\}$ and $(E_0 \circ E)_+ = \{(x_0, x) \mid x_0 > 0 \vee (x_0 = 0 \ \ \&\ \ x \geq 0)\}$. In view of Lemma 2, it suffices to show that, for every $z \in E_0 \circ E$, there is an $\alpha \in \Lambda(E_0 \circ E)$ such that $\alpha z = z^+$. Take a $z = (x_0, x) \in E_0 \circ E$. Since E_0 is linearly ordered, for x_0 we have $(x_0 > 0) \vee (x_0 < 0) \vee (x_0 = 0)$. If $x_0 > 0$ then $z^+ = z$ and $\alpha = I$. If $x_0 < 0$ then $z^+ = 0$ and $\alpha = 0$. If $x_0 = 0$ then $z^+ = (0, x^+)$ and, since $E \in \mathcal{K}(D)$, there exists an $\alpha_0 \in \Lambda(E)$ such that $\alpha_0 x = x^+$. In this case, it is easy to show that, by letting $\alpha : E_0 \circ E \to E_0 \circ E$, $\alpha(y_0, y) = ((1/2)y_0, \alpha_0 y)$, we obtain $\alpha \in \Lambda(E_0 \circ E)$ and, in addition, $\alpha z = z^+$. Consequently, $E_0 \circ E \in \mathcal{K}(D)$.

Let us prove (e). Take $F \in \mathcal{K}(C)$ and $z_1, z_2 \in (R^1 \circ F)_+$, $z_1 \leq z_2$. We need to show that there is an $\alpha \in \Lambda(R^1 \circ F)$ such that $\alpha z_2 = z_1$. Suppose that $z_i = (\lambda_i, x_i)$, $\lambda_i \in R^1$, $x_i \in F$, $i \in 1, 2$. Then $0 \leq \lambda_1 \leq \lambda_2$. Consider three cases separately: $\lambda_1 = \lambda_2 = 0$, $\lambda_1 = \lambda_2 > 0$, and $\lambda_1 > \lambda_2 \geq 0$.

If $\lambda_1 = \lambda_2 = 0$ then $0 \leq x_1 \leq x_2$ and, since $F \in \mathcal{K}(C)$, there exists an $\alpha_0 \in \Lambda(F)$ such that $\alpha_0 x_2 = x_1$. Then, by letting $\alpha(\lambda, x) = ((1/2)\lambda, \alpha_0 x)$ for all $(\lambda, x) \in R^1 \circ F$, we obtain $\alpha \in \Lambda(R^1 \circ F)$ and $\alpha z_2 = z_1$.

If $\lambda_1 = \lambda_2 > 0$ then $x_1 \leq x_2$. Assign $\alpha(\lambda, x) = \left(\lambda, x + \frac{\lambda}{\lambda_2}(x_1 - x_2)\right)$ for $(\lambda, x) \in R^1 \circ F$. It is easy to show that, in this case, $\alpha \in \Lambda(R^1 \circ F)$ and $\alpha z_2 = z_1$.

If $\lambda_2 > \lambda_1 \geq 0$ then, for $(\lambda, x) \in R^1 \circ F$, assign $\alpha(\lambda, x) = \frac{\lambda}{\lambda_2} z_1$. In this case, it is also easy to show that $\alpha \in \Lambda(R^1 \circ F)$ and $\alpha z_2 = z_1$.

So, in each of the cases, there is an $\alpha \in \Lambda(R^1 \circ F)$ such that $\alpha z_2 = z_1$ and, consequently, $R^1 \circ F \in \mathcal{K}(C)$. Proposition 2 is proven.

With the help of Proposition 2, starting with given spaces that belong to the classes $\mathcal{K}(C)$ or $\mathcal{K}(D)$, we can obtain new spaces in the same classes.

The following theorem answers the question that is formulated at the beginning of Section 2.

Theorem 5. *The class $\mathcal{K}(C) \cap \mathcal{K}(D)$ contains*
(a) *all finite-dimensional vector lattices;*
(b) *all vector lattices that are K_σ-spaces.*

PROOF. (a) By Theorem XV.4 of [15], every finite-dimensional vector lattice E is either the direct sum of vector lattices of lesser dimension or the lexicographic product $E = R^1 \circ E_0$ of the set R^1 of reals and a vector lattice E_0 of dimension less than that of E by one. Since, obviously, $R^1 \in \mathcal{K}(C) \cap \mathcal{K}(D)$, the desired assertion can be obtained by induction on the dimension with the help of Proposition 2.

Let us prove (b). Let E be a K_σ-space. We will show that $E \in \mathcal{K}(D)$. Take $x, y \in E_+$ such that $x \wedge y = 0$. Consider the principal band E_x generated by x [13]:

$$E_x = \{z \in E \mid \forall u \in E \ |u| \wedge x = 0 \to |u| \wedge |z| = 0\}.$$

By Theorem IV.3.4 of [13], for every $z \in E$, there exists an orthogonal projection $\operatorname{Pr} z$ of it onto E_x and

$$\operatorname{Pr} z = \sup_n (z \wedge nx) \tag{11}$$

for $z \in E_+$. Moreover, the operator Pr is linear. From (11) it is clear that $0 \leq \operatorname{Pr} z \leq z$ for $z \in E_+$; therefore, $\operatorname{Pr} \in \Lambda(E)$. In addition, it is obvious that $\operatorname{Pr} x = x$ and $\operatorname{Pr} y = 0$; consequently, $E \in \mathcal{K}(D)$.

We will now demonstrate that $E \in \mathcal{K}(C)$. Take $x, y \in E_+$, $y \leq x$. Once again, consider the principal band E_x and the orthogonal projection operator $\operatorname{Pr} : E \to E_x$. The space E_x, as a normal subspace (= an order ideal) of E, is itself a K_σ-space and, by Theorem IV.3.6 of [13], x is an order unit in E_x (i.e., for every $z \in E_x$, if $z > 0$ then $x \wedge z > 0$). Moreover, it is obvious that $y \in E_x$. Next, by Theorem V.8.1 of [13], we can define in E_x a multiplication operation (some function $\varphi : (E_x \times E_x) \to E_x$ defined not everywhere in general) so that E_x becomes a generalized ordered commutative ring with a unit (see Definition V.8.2 in [13]), in which x serves as a unit for the multiplication; i.e., for every $z \in E_x$, the values $\varphi(z, x)$ and $\varphi(x, z)$ are defined and $\varphi(z, x) = \varphi(x, z) = z$. In addition, from the definition of a generalized ordered commutative ring it follows that, for every $z \in E_x$, the value $\varphi(y, z)$ is defined, since $0 \leq y \leq x$ and $0 \leq \varphi(y, z) \leq \varphi(x, z) = z$ whenever $z \geq 0$. Moreover, from the same definition it follows that the function $\varphi(y, \cdot) : E_x \to E_x$ (as a function of the second argument) is a linear operator in E_x. Now, assign $\alpha z = \varphi(y, \operatorname{Pr} z)$ for each $z \in E$. Then $\alpha z = \varphi(y, \operatorname{Pr} z) \geq 0$ for $z \in E_+$, since $\operatorname{Pr} z \geq 0$, $\alpha z = \varphi(y, \operatorname{Pr} z) \leq \operatorname{Pr} z \leq z$ and

$\alpha x = \varphi(y, \Pr x) = \varphi(y, x) = y$; i.e., $\alpha \in \Lambda(E)$ and $\alpha x = y$. Consequently, $E \in \mathcal{K}(C)$. The theorem is proven.

We will give examples of an ordered vector space not possessing properties C or D.

EXAMPLE 1. Let F_0 be a 3-dimensional vector space with basis a_1, a_2, a_3. Order it with the cone

$$F_{0+} = \{\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 \mid \alpha_i \in R^1,\ \alpha_i \geq 0,\ i = \overline{1,3},\ \alpha_1 + \alpha_2 \geq \alpha_3\}.$$

Let E_0 be a 3-dimensional vector lattice with coordinatewise order corresponding to some basis $b_1, b_2, b_3 \in E_0$, i.e., $E_{0+} = \{\alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 \mid \alpha_i \in R^1,\ \alpha_i \geq 0,\ i = \overline{1,3},\}$. We will demonstrate that Theorem 1 fails for the pair F_0, E_0. Take an $A_0 \in L(E_0, F_0)$ that is defined on the basis of E_0 as follows: $A_0 b_1 = a_1$, $A_0 b_2 = a_2$, $A_0 b_3 = \frac{1}{2}a_1 + \frac{1}{2}a_2 + a_3$. It is obvious that A_0 is a positive operator; therefore, $[\mathbb{O}, A_0] \in I(L(E_0, F_0))$. In addition, $A \in [\mathbb{O}, A_0]$ if and only if there exist numbers $\lambda_1, \lambda_2 \in [0,1]$ and $\beta_1, \beta_2 \in [0, 1/2]$ such that $Ab_1 = \lambda_1 a_1$, $Ab_2 = \lambda_2 a_2$, and $Ab_3 = \beta_1 a_1 + \beta_2 a_2 + (\beta_1 + \beta_2)a_3$. Take an $x \in E_{0+}$, $x = 2b_1 + 2b_3$. It is obvious that $y = a_1 + a_2 \leq A_0 x = 3a_1 + a_2 + 2a_3$ and $y \geq 0$ in F_0, i.e., $y \in [0, A_0 x]$. At the same time, if the equality $Ax = y$ was valid for some $A \in [\mathbb{O}, A_0]$ then we would have $2(\lambda_1 + \beta_1)a_1 + 2\beta_2 a_2 + 2(\beta_1 + \beta_2)a_3 = a_1 + a_2$ and, consequently, $\beta_2 = 1/2$, $\beta_1 = -\beta_2 = -1/2 \notin [0, 1/2]$. Therefore, $y \notin [\mathbb{O}, A_0]x$ and $[\mathbb{O}, A_0]x \neq [0, A_0 x]$; i.e., formula (3) is not valid in this case. In particular, it follows that $F_0 \notin \mathcal{K}(C)$.

We also point out that F_0 is not a K_0-space.

EXAMPLE. Let F_1 be a 2-dimensional vector space with basis e_1, e_2 and let $F_{1+} = \{\alpha_1 e_1 + \alpha_2 e_2 \mid (\alpha_1 = \alpha_2 = 0) \vee (\alpha_1 > 0 \& \alpha_2 > 0)\}$. It is easy to show that F_1 is a K_0-space, $F_1 \in \mathcal{K}(C)$, and, since F_1 is not a vector lattice, $F_1 \notin \mathcal{K}(D)$.

EXAMPLE 3. Consider the vector lattice $C[0,1]$ of continuous functions from the interval $[0,1]$ into R^1 in natural order; i.e., we assume $f_1 \leq f_2$ whenever $f_1(t) \leq f_2(t)$ for all $t \in [0,1]$. It is known [12, p. 181] that each multiplier $\alpha \in \Lambda(C[0,1])$ is representable as multiplication by a continuous function from $[0,1]$ into $[0,1]$; i.e., $\alpha \in \Lambda(C[0,1])$ if and only if there exists a continuous $\varphi : [0,1] \to [0,1]$, φ such that $(\alpha f)(t) = \varphi(t) f(t)$ for all $f \in C[0,1]$ and $t \in [0,1]$. We will demonstrate that $C[0,1] \notin \mathcal{K}(C) \cup \mathcal{K}(D)$. Indeed, define $f_1(t) = 1 - 2t$ for $t \in [0, 1/2]$, $f_1(t) = 0$ for $t \in (1/2, 1]$, $f_2(t) = 0$ for $t \in [0, 1/2]$, and $f_2(t) = 2t - 1$ for $t \in (1/2, 1]$. It is clear that $f_1 \geq 0$, $f_2 \geq 0$, and $f_1 \wedge f_2 = 0$. If $\alpha \in \Lambda(C[0,1])$ is such that $\alpha f_1 = f_1$ and $\alpha f_2 = 0$, then, for the function φ that corresponds to α, we have $\varphi(t) f_1(t) = f_1(t)$ and $\varphi(t) f_2(t) = 0$ for all $t \in [0,1]$. But then $\varphi(t) = 1$ for $t \in [0, 1/2)$ and $\varphi(t) = 0$ for $t \in (1/2, 1]$. Therefore, φ cannot be continuous and, consequently, $C[0,1] \notin \mathcal{K}(D)$. Next, define $f_3(t) = t^2 \sin^2(1/t)$ for $t \in (0,1]$ and $f_3(0) = 0$ and $f_4(t) = t^2$ for $t \in [0,1]$. Then $0 \leq f_3 \leq f_4$, and if $\alpha f_4 = f_3$ for some $\alpha \in \Lambda(C[0,1])$ then, for the function φ that corresponds to α, we obtain $\varphi(t) f_4(t) = f_3(t)$ for all $t \in [0,1]$. But then $\varphi(t) = \sin^2(1/t)$ for $t \in (0,1]$, and φ cannot be continuous at the point $t = 0$. Consequently, $C[0,1] \notin \mathcal{K}(C)$.

From the given examples it is clear that an arbitrary ordered vector space need neither possess property C nor be a K_0-space; there are ordered vector spaces possessing property C and presenting K_0-spaces but failing to be a vector lattice; an Archimedean vector lattice need possess neither property C nor property D.

§ 3. Algorithmic Complexity

Consider the problem of solvability of equation (1), i.e. the problem of checking whether the sets $\mathcal{R}_i$, $i = \overline{1,4}$, are nonempty, from the viewpoint of computational complexity [16]. We shall restrict ourselves to the case of $E = R^m$ and $F = R^n$ both ordered coordinatewise and integer matrices A, B and vectors a, b. More precisely, for a fixed $i = \overline{1,4}$, by the problem $\mathcal{R}_i \neq \varnothing$ we mean the following:

PROBLEM $\mathcal{R}_i$. *Given are integer $m \times n$-matrices A and B, $A \leq B$, and integer m-dimensional vectors a and b, $a \leq b$. Does there exist an n-dimensional real vector x such that $x \in \mathcal{R}_i$?*

It is easy to observe that, for every $i = \overline{1,4}$, the problem $\mathcal{R}_i \neq \varnothing$ belongs to the class $\mathcal{NP}$ [16]. Indeed, an indeterminate algorithm solving the problem $\mathcal{R}_i \neq \varnothing$ in polynomial time is constructed as follows. First, we "guess" at the signs of the coordinates of the vector x. With the signs fixed, by Theorem 4 the problem $\mathcal{R}_i \neq \varnothing$ now transforms into that of solvability for a system of linear inequalities which is polynomially solvable [17]. In addition, since by Theorem 2 the set $\mathcal{R}_2$ can be described as a system of linear inequalities, the problem $\mathcal{R}_2 \neq \varnothing$ is in the class $\mathcal{P}$ (polynomially solvable). As regards the problems $\mathcal{R}_i \neq \varnothing$ for $i = 1, 3, 4$, we have the following:

Theorem 6. *The problems $\mathcal{R}_i \neq \varnothing$, for $i = 1, 3, 4$, are $\mathcal{NP}$-complete.*

PROOF. We will show that the following $\mathcal{NP}$-complete problem PARTITION [16] is polynomially reducible to the problems $\mathcal{R}_i \neq \varnothing$.

PROBLEM PARTITION. *Let q positive integer numbers $s_1, \ldots, s_q$ $(q \geq 1)$ be given. Is it possible to choose signs $\varepsilon_i \in \{-1, 1\}$, $i = \overline{1, q}$, for s_i so that $\varepsilon_1 s_1 + \cdots + \varepsilon_q s_q = 0$; i.e., is the set*

$$\mathcal{R}_0 = \left\{ (\varepsilon_1, \ldots, \varepsilon_q)^T \,\middle|\, \sum_{i=1}^{q} \varepsilon_i s_i = 0,\ \varepsilon_i \in \{-1, 1\},\ i = \overline{1, q} \right\}$$

nonempty, where $(\cdot)^T$ is the taking of transpose?

While reducing problems, we make use of the description for $\mathcal{R}_i$ obtained in Theorem 4 and the routine matrix-vector notation. In particular, we assume $\mathbb{O}_q$ to be the zero $q \times q$-matrix and E_q to be the identity $q \times q$-matrix; moreover, $0_q = (0, \ldots, 0)^T$, $e_q = (1, \ldots, 1)^T$, $\varepsilon = (\varepsilon, \ldots, \varepsilon)^T$, and $s = (s_1, \ldots, s_q)^T$ are q-dimensional vector-columns. We also point out that, in the case under consideration (a finite-dimensional vector space with coordinatewise order), the moduli of a vector x is the vector constituted by the modulus of the coordinates of x.

We will show that problem PARTITION is reducible to the problem $\mathcal{R}_1 \neq \varnothing$.

Let a q-dimensional vector $s = (s_1, \ldots, s_q)^T$ be given. Assign $m = 2q + 1$, $n = q$, and define $(2q + 1) \times q$-matrices C and D and $(2q + 1)$-dimensional vectors d and c as follows:

$$D = \begin{pmatrix} 0_q^T \\ E_q \\ \mathbb{O}_q \end{pmatrix}, \quad C = \begin{pmatrix} s^T \\ \mathbb{O}_q \\ E_q \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 0_q \\ e_q \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ e_q \\ 0_q \end{pmatrix};$$

moreover, $A = C - D$, $B = C + D$, $a = c - d$, and $b = c + d$ are integer and, obviously, $A \leq B$ and $a \leq b$. Show that, with these A, B, a, and b, the set $\mathcal{R}_1$ coincides with $\mathcal{R}_0$.

Indeed, by Theorem 4, $x \in \mathcal{R}_1$ if and only if $D|x| + d \geq |Cx - c|$, which in our case amounts to the system of inequalities

$$0 \geq |s^T x|, \quad |x| \geq e_q, \quad e_q \geq |x|$$

or, equivalently, $|x| = e_q$, $s^T x = 0$; i.e., if and only if $x \in \mathcal{R}_0$. Since, obviously, the matrices A and B and the vectors a and b are constructed from the vector s in polynomial time, problem PARTITION is polynomially reducible to the problem $\mathcal{R}_1 \neq \varnothing$.

The fact that problem PARTITION is reducible to the problem $\mathcal{R}_4 \neq \varnothing$ can be proven similarly, if we assign $m = q + 1$, $n = q$,

$$D = \begin{pmatrix} 0_q^T \\ E_q \end{pmatrix}, \quad C = \begin{pmatrix} s^T \\ 0_q \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ e_q \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0_q \end{pmatrix}.$$

Now, consider the problem $\mathcal{R}_3 \neq \varnothing$. Assign $m = 3q+1$, $n = 2q$, and define $(3q+1) \times 2q$-matrices C and D and $(3q + 1)$-dimensional vectors c and d as follows:

$$D = \begin{pmatrix} 0_q^T & 0_q^T \\ \mathbb{O}_q & E_q \\ \mathbb{O}_q & E_q \\ E_q & \mathbb{O}_q \end{pmatrix}, \quad C = \begin{pmatrix} s^T & 0_q^T \\ \mathbb{O}_q & 2E_q \\ E_q & \mathbb{O}_q \\ \mathbb{O}_q & E_q \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ e_q \\ 0_q \\ 0_q \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 2e_q \\ 0_q \\ 0_q \end{pmatrix};$$

moreover, as before, $A = C - D$, $B = C + D$, $a = c - d$, and $b = c + d$. We shall write solutions to the corresponding interval problem as $\begin{pmatrix} x \\ y \end{pmatrix}$, where x and y are q-dimensional vector-columns. Then, by Theorem 4, $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{R}_3$ is equivalent to

$$D\begin{pmatrix} |x| \\ |y| \end{pmatrix} \geq d + \left| C\begin{pmatrix} x \\ y \end{pmatrix} - c \right|$$

or to the following system of inequalities:

$$0 \geq |s^T x|, \quad |y| \geq e_q + |2y - 2e_q|, \quad |y| \geq |x|, \quad |x| \geq |y|,$$

which is clearly equivalent to the system

$$s^T x = 0, \quad 2|y - e_q| - |y| + e_q \leq 0, \quad |x| = |y|. \tag{12}$$

Since, obviously,

$$|y - e_q| = 2|y - e_q| - |y - e_q| \leq 2|y - e_q| - (|y - e_q|) = 2|y - e_q| - |y| + e_q,$$

the second inequality of system (12) is equivalent to $y = e_q$ and, consequently, (12) can be written as

$$s^T x = 0, \quad y = e_q, \quad |x| = e_q. \tag{13}$$

From (13) it follows that if $x \in \mathcal{R}_0$ then $\begin{pmatrix} x \\ e_q \end{pmatrix} \in \mathcal{R}_3$ and, conversely, if $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{R}_3$ then $y = e_q$ and $x \in \mathcal{R}_0$. Since it is clear that the matrices A and B and the vectors a and b are constructed from the vector s in polynomial time, problem PARTITION is polynomially reducible to the problem $\mathcal{R}_3 \neq \varnothing$. The theorem is proven.

Thus, if $\mathcal{P} \neq \mathcal{NP}$ then, while solving the problem $\mathcal{R}_i \neq \varnothing$ for $i = 1, 3, 4$, it is impossible to eliminate exponential exhaustion.

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