Linear Algebraic Equations in Kaucher Arithmetic

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***Abstract*—**
S. P. Shary [8] and L. V. Kupriyanova [3] have proven that an algebraic interval solution of linear equation in Kaucher's extended arithmetic allows us to obtain maximal interval estimates for various sets of solutions of interval linear algebraic systems (namely, for united, tolerable, controlled, and some other solution sets; for definitions, see [8]).

In this paper, we derive a special analytical representation of interval product in Kaucher's arithmetic [2, 1] in terms of lattice operations (i.e., positive and negative parts, module, least upper bound), and we use this representation to formulate numerical (non-interval) equations that are equivalent to linear interval equations.

For several interesting classes of interval linear equations, this reformulation enables us to reduce these equations to equations of the type $x = M|x| + a$ that have been studied earlier by J. Rohn [6, 4].

This reformulation also enable us to prove that for these classes, the problem of checking whether a given interval linear system has a solution is NP-complete.

I. Kaucher Arithmetic (Extended Interval Arithmetic): Definitions and Denotations

Let us first recall the basic notions of Kaucher arithmetic.

- The main objects of Kaucher arithmetic are pairs of real numbers that are called *extended intervals*. A pair consisting of real numbers a and b is denoted by $[a, b]$.
- The set of all such intervals is denoted by

$$I^*(R) = \{[\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in R\}$$

(in this paper, we will use notations from [1]).

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- Intervals for which $a \le b$ are called *proper intervals*. The set of all proper intervals is denoted by

$$I(R) = \{[\underline{x}, \overline{x}] \mid \underline{x} \le \overline{x},\ \underline{x}, \overline{x} \in R\}.$$

In this paper, we will (as usual in interval arithmetic) identify $x \in R$ with an interval $[x, x]$, and thus assume that

$$R \subset I(R).$$

The following are the basic operations of Kaucher arithmetic. By $*$, let us denote any operation from the set $\{+, -, \cdot\}$. Let $\mathrm{x} = [\underline{x}, \overline{x}] \in I^*(R)$, $\mathrm{y} = [\underline{y}, \overline{y}] \in I^*(R)$, and let $\{\mathrm{x}_i\}_{i \in J}$ is an bounded set of extended intervals $\mathrm{x}_i = [\underline{x}_i, \overline{x}_i] \in I^*(R)$. Then:

- $dual(\mathrm{x}) = [\overline{x}, \underline{x}]$
- $pro(\mathrm{x}) :=$ if x proper then x, else $dual(\mathrm{x})$;
- $$\mathbf{sup}\{\mathrm{x}_i \mid i \in J\} = [\inf\{\underline{x}_i \mid i \in J\}, \sup\{\overline{x}_i \mid i \in J\}];$$
- $$\mathbf{inf}\{\mathrm{x}_i \mid i \in J\} = [\sup\{\underline{x}_i \mid i \in J\}, \inf\{\overline{x}_i \mid i \in J\}];$$
- $\Omega^{\mathrm{x}}(\mathrm{t}) :=$ if x proper then t, else $dual(\mathrm{t})$;
- $\mathrm{x} * \mathrm{y} = \Omega^{\mathrm{x}}(\Omega^{\mathrm{y}}(\{x * y \mid x \in pro(\mathrm{x}), y \in pro(\mathrm{y})\}))$.

In this paper, we will also use the following denotations for operations with real numbers x and y:

- $x^+ = max\{x, 0\}$;
- $x^- = max\{-x, 0\}$;
- $x \vee y = max\{x, y\}$.

II. Interval Product in $I^*(R)$

THEOREM 1. *For any* $\mathrm{x} = [\underline{x}, \overline{x}] \in I^*(R)$ *and for any* $\mathrm{y} = [\underline{y}, \overline{y}] \in I^*(R)$, *the following equality holds:*

$$\mathrm{x} \cdot \mathrm{y} = [(\underline{x}^+\underline{y}^+) \vee (\overline{x}^-\overline{y}^-) - (\overline{x}^+\underline{y}^-) \vee (\underline{x}^-\overline{y}^+), (\overline{x}^+\overline{y}^+) \vee (\underline{x}^-\underline{y}^-) - (\underline{x}^+\overline{y}^-) \vee (\overline{x}^-\underline{y}^+)]. \quad (1)$$

If we know how many of the two intervals (if any) are proper, then we can simplify the formula (1):

THEOREM 2. *Let* $\mathrm{x}, \mathrm{y} \in I^*(R)$*. Then:*

1) if both intervals x *and* y *are proper, then*

$$\mathrm{x} \cdot \mathrm{y} = [\underline{x}^+\underline{y}^+ + \overline{x}^-\overline{y}^- - (\overline{x}^+\underline{y}^-) \vee (\underline{x}^-\overline{y}^+),$$
$$(\overline{x}^+\overline{y}^+) \vee (\underline{x}^-\underline{y}^-) - \underline{x}^+\overline{y}^- - \overline{x}^-\underline{y}^+], \qquad (2)$$

2) if neither x*, nor* y *is proper, then*

$$\mathrm{x} \cdot \mathrm{y} = [(\underline{x}^+\underline{y}^+) \vee (\overline{x}^-\overline{y}^-) - \overline{x}^+\underline{y}^- - \underline{x}^-\overline{y}^+,$$
$$\overline{x}^+\overline{y}^+ + \underline{x}^-\underline{y}^- - (\underline{x}^+\overline{y}^-) \vee (\overline{x}^-\underline{y}^+)], \qquad (3)$$

3) if one of the intervals x, y *is proper, and another one is not proper, then*

$$\mathrm{x} \cdot \mathrm{y} = [\underline{x}^+\underline{y}^+ + \overline{x}^-\overline{y}^- - \overline{x}^+\underline{y}^- - \underline{x}^-\overline{y}^+,$$
$$\overline{x}^+\overline{y}^+ + \underline{x}^-\underline{y}^- - \underline{x}^+\overline{y}^- - \overline{x}^-\underline{y}^+]. \qquad (4)$$

III. Linear Equations in $I^*(R^n)$

Let us fix integers m and n.

- By $I^*(R^n)$ and $I^*(R^{m\times n})$, we will denote the sets of n-dimensional vectors and of $m \times n$ matrixes with elements from $I^*(R)$,
- y $I(R^n)$ and $I(R^{m\times n})$ we denote the sets of n-dimensional vectors and of $m \times n$ matrixes with elements from $I(R)$.
- In the following text:
  - $\mathbf{A} = [\underline{A}, \overline{A}]$ will denote an (extended) interval matrix (from $I^*(R^{m\times n})$), and
  - $\mathbf{b} = [\underline{b}, \overline{b}]$ will denote an extended interval vector (from $I^*(R^m)$).

We consider the linear algebraic equation

$$\mathbf{Ax} = \mathbf{b}, \qquad (5)$$

where $\mathbf{x} = [\underline{x}, \overline{x}] \in I^*(R^n)$, and the product of the matrix $\mathbf{A}$ and the vector $\mathbf{x}$ is defined in the usual way, through the addition and multiplication in $I^*(R)$.

We will also need some denotations for real-valued vectors and matrices:

- On real-valued vectors and matrices, operations $+, -, ^+, \ ^-, \vee$, and $|\cdot|$ mean component-wise operations.

  For example, if A is a matrix with components A_{ij}, then $|A|$denotes the matrix with components $|A_{ij}|$.
- For an arbitrary vector x, $diag(x)$ will denote a diagonal matrix whose diagonal is the vector x.
- $e^n = (1, ..., 1) \in R^n$ will denote a vector with all components equal to 1.

THEOREM 3. *If* $\mathbf{Ax} = [\underline{\mathbf{Ax}}\ , \ \overline{\mathbf{Ax}}]$ *then*

$$\overline{\mathbf{Ax}} = (\overline{A}^+ diag(\overline{x}^+) \vee \underline{A}^- diag(\underline{x}^-) e^n -$$
$$(\underline{A}^+ diag(\overline{x}^-) \vee \overline{A}^- diag(\underline{x}^+))e^n, \qquad (6)$$
$$\underline{\mathbf{Ax}} = (\underline{A}^+ diag(\underline{x}^+) \vee \overline{A}^- diag(\overline{x}^-) e^n -$$
$$\overline{A}^+ diag(\underline{x}^-) \vee \underline{A}^- diag(\overline{x}^+))e^n. \qquad (7)$$

If we equate the expressions from the right-hand sides of equations (6) and (7) with $\overline{b}$ and $\underline{b}$ respectively, we will get the explicit form of the interval system of equations (5). If we replace $a \vee b$ by $(1/2)\cdot(a+b-|a-b|)$, and use matrix notations, then this representation can be reformulated as follows:

THEOREM 4. *The vector* $\mathbf{x} = [\underline{x}\ , \ \overline{x}] \in I^*(R^n)$ *is a solution to the equation (5) iff*

$$\begin{pmatrix} \overline{A}^+ & \underline{A}^- \\ \underline{A}^- & \overline{A}^+ \end{pmatrix} \begin{pmatrix} \overline{x} \\ -\underline{x} \end{pmatrix}^+ - \begin{pmatrix} \underline{A}^+ & \overline{A}^- \\ \overline{A}^- & \underline{A}^+ \end{pmatrix} \begin{pmatrix} \overline{x} \\ -\underline{x} \end{pmatrix}^- +$$
$$\left| \begin{pmatrix} \overline{A}^+ & -\underline{A}^- \\ -\underline{A}^- & \overline{A}^+ \end{pmatrix} \begin{pmatrix} diag(\overline{x}) \\ diag(-\underline{x}) \end{pmatrix}^+ \right| e^n -$$
$$\left| \begin{pmatrix} \underline{A}^+ & -\overline{A}^- \\ -\overline{A}^- & \underline{A}^+ \end{pmatrix} \begin{pmatrix} diag(\overline{x}) \\ diag(-\underline{x}) \end{pmatrix}^- \right| e^n =$$
$$2 \begin{pmatrix} \overline{b} \\ -\underline{b} \end{pmatrix}. \qquad (8)$$

If $\mathbf{A} \in I(R^{m\times n})$ or $dual(\mathbf{A}) \in I(R^{m\times n})$, then we can replace one of $\vee$ by $+$ in the formulas (6),(7), and get rid of one of the terms in (8) by using Theorem 2.

The system (8) can be further simplified if $dual(\mathbf{A}) \in I(R^{m\times n})$, and we are interested in a solution $\mathbf{x} \in I(R^n)$ (as shown in [8, 3], in this case $\mathbf{x}$ is an inner estimate of the united set of solutions of the corresponding interval system):

THEOREM 5. *If* $dual(\mathbf{A}) \in I(R^{m\times n})$*, then* $\mathbf{x} \in I(R^n)$ *is a solution of (5) iff*

$$\begin{pmatrix} \overline{A}^+ & \underline{A}^- \\ \underline{A}^- & \overline{A}^+ \end{pmatrix} \begin{pmatrix} \overline{x} \\ -\underline{x} \end{pmatrix}^+ - \begin{pmatrix} \underline{A}^+ & \overline{A}^- \\ \overline{A}^- & \underline{A}^+ \end{pmatrix} \begin{pmatrix} \overline{x} \\ -\underline{x} \end{pmatrix}^- =$$
$$\begin{pmatrix} \overline{b} \\ -\underline{b} \end{pmatrix}. \qquad (9)$$

By introducing denotations

$$C^+ = \frac{1}{2}(\overline{A}^+ + \underline{A}^+), \ \ C^- = \frac{1}{2}(\underline{A}^- + \overline{A}^-),$$

$$D^+ = \frac{1}{2}(\overline{A}^+ - \underline{A}^+), \ \ D^- = \frac{1}{2}(\underline{A}^- - \overline{A}^-),$$

we can represent the system (9) in the following form

$$\begin{pmatrix} C^+ & C^- \\ C^- & C^+ \end{pmatrix} \begin{pmatrix} \overline{x} \\ -\underline{x} \end{pmatrix} + \begin{pmatrix} D^+ & D^- \\ D^- & D^+ \end{pmatrix} \left| \begin{pmatrix} \overline{x} \\ -\underline{x} \end{pmatrix} \right| = \begin{pmatrix} \overline{b} \\ -\underline{b} \end{pmatrix}. \tag{10}$$

Such system have been thoroughly analyzed by J. Rohn [6] (see also [4]). In particular, by using the concept of "nonexpanding matrix" and Theorem 6.1.5 from [4], we can prove the following statement. Let us denote:

$$\tilde{C} = \begin{pmatrix} C^+ & C^- \\ C^- & C^+ \end{pmatrix}, \ \ \tilde{D} = \begin{pmatrix} D^+ & D^- \\ D^- & D^+ \end{pmatrix},$$

$$C = \frac{1}{2}(\overline{A} + \underline{A}), \ \ C_0 = \frac{1}{2}(|\underline{A}| + |\overline{A}|),$$

$$D = \frac{1}{2}(\overline{A} - \underline{A}), \ \ D_0 = \frac{1}{2}(|\overline{A}| - |\underline{A}|).$$

THEOREM 6. *Let $m = n$, C and C_0 be nonsingular matrixes (this condition is equivalent to nonsingularity of $\tilde{C}$), and let $\tilde{C}^{-1}\tilde{D}$ be nonexpanding. Then, the equation (10) has a unique solution for every $\overline{b}, \underline{b}$. If, in addition, $C_0^{-1}D \le \mathbf{0}$, and $\overline{b}, \underline{b}$ satisfy the inequality $C_0^{-1}\overline{b} \ge C_0^{-1}\underline{b}$, then this solution belongs to $I(R^n)$.*

We note that here,

$$\tilde{C}^{-1} = \frac{1}{2} \begin{pmatrix} C_0^{-1} + C^{-1} & C_0^{-1} - C^{-1} \\ C_0^{-1} - C^{-1} & C_0^{-1} + C^{-1} \end{pmatrix},$$

$$\tilde{C}^{-1}\tilde{D} = \frac{1}{2} \begin{pmatrix} C_0^{-1}D + C^{-1}D_0 & C_0^{-1}D - C^{-1}D_0 \\ C_0^{-1}D - C^{-1}D_0 & C_0^{-1}D + C^{-1}D_0 \end{pmatrix}.$$

It is interesting to mention that, in addition to the above-described case, the equation (8) can be transformed into an equivalent form (10) in the following two additional cases:

1) when $\mathbf{A} \in I(R^{m\times n})$ and $dual(\mathbf{x}) \in I(R^n)$;
2) when $\underline{a}_{ij}\overline{a}_{ij} \ge 0$ for all i, j (where $\underline{a}_{ij}$ and $\overline{a}_{ij}$ are the elements of the matrices $\underline{A}$ and $\overline{A}$).

In these cases, we can prove results similar to Theorem 6.

IV. NP-COMPLETENESS

Let us define a special class of interval equations (5). We will say that an equation (5) belong to the class E if all of the following conditions are satisfied:

- $m = n$;
- matrixes and vectors are integer-valued;
- $\underline{A} \ge \mathbf{0}$, $\overline{A} \ge \mathbf{0}$, and $\overline{b} \ge \underline{b}$;
- the matrix $C = \frac{1}{2}(\overline{A} + \underline{A})$ is nonsingular;
- the number of solutions of (5) is finite (e.g., is bounded by $\le 2^{n+1}$).

Due to the above conditions, the equation (5) is equvalent to (10) and, moreover,

$$C^+ = C, \ \ D^+ = D, \ \ C^- = D^- = \mathbf{0}.$$

Then, the equations (10) can be further simplified, into the following system:

$$\begin{cases} C\overline{x} + D|\overline{x}| = \overline{b}, \\ C\underline{x} = D|\underline{x}| + \underline{b}. \end{cases} \tag{11}$$

If we are looking for a solution that belongs to $I(R^n)$ (i.e., for which $\mathbf{x} = [\underline{x}, \overline{x}] \in I(R^n)$), then we must add the cooresponding inequality

$$\underline{x} \le \overline{x}. \tag{12}$$

to the equations (11).

THEOREM 7. *On the class E, the following problems are NP-complete:*

- *Checking whether a given system has an extended interval solution (i.e., a solution from the set $I^*(R^n)$).*
- *Checking whether a given system has a proper interval solution (i.e., a solution from the set $I(R^n)$).*

Both problems remain NP-complete if we restrict the class of problems by additionally imposing one of the following two conditions:

$$\underline{A} \ge \overline{A} \ge \mathbf{0}$$

or

$$\overline{A} \ge \underline{A} \ge \mathbf{0}.$$

Comments. It is interesting to notice that if we impose an additional condition $\overline{A} \ge \underline{A}$ (that makes an interval matrix $\mathbf{A}$ proper), and if we are looking looking for proper solutions $\mathbf{x} \in I(R^n)$, then the problem (5) becomes equivalent to to the problem of finding a solution of an linear interval system of equations [5] in classical interval arithmetic.

If, instead of the condition $\overline{A} \ge \underline{A}$, we impose the condition $\underline{A} \ge \overline{A}$, then the problem of finding a proper solution corresponds to a problem of finding, using the equation (5), of inner interval estimate of the united set of solution [8]. This problem is thus also proven to be NP-complete.

As a corollary, we obtain the following result:

COROLARY. *Solvability problem for equations of the form*

$$\tilde{C}\tilde{x} = \tilde{D}|\tilde{x}| + \tilde{b}$$

with a nonsingular matrix $\tilde{C}$ *and* $\tilde{C} \geq \tilde{D} \geq \mathbf{0}$ *is NP-complete.*

Comment. This corollary solves an open problem that has been previously formulated by J. Rohn [7].

Idea of the Proof of Theorem 7. The proof of results from Theorem 7 is done by formulating a polynomial reducibility of the problem PARTITION to the problems, formulated in this Theorem.

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