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NP-Hard Classes of Linear Algebraic Systems with Uncertainties

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Abstract. For a system of linear equations $Ax = b$, the following natural questions appear:

- does this system have a solution?
- if it does, what are the possible values of a given objective function $f(x_1, \dots, x_n)$ (e.g., of a linear function $f(x) = \sum c_i x_i$) over the system's solution set?

We show that for several classes of linear equations with uncertainty (including interval linear equations) these problems are NP-hard. In particular, we show that these problems are NP-hard even if we consider only systems of $n + 2$ equations with n variables, that have integer positive coefficients and finitely many solutions.

1. Brief Informal Introduction

It is known that algorithms for solving interval linear algebraic equations and linear programming problems with interval uncertainties are often very time-consuming (for the latest algorithms, see, e.g., [12], [14], [15]). This is partially explained by the result (proven in [6]–[9]) that in general, the problem of solving interval linear systems is NP-hard (as well as other related problems, such as finding the largest possible value of a given linear function on a set of all solutions). In this paper, we improve the results from [6], [7].

2. Formulation of the Problem. NP-Hard Problems

DENOTATIONS.

- In this paper, we will use standard denotations $\mathbf{R}$, $\mathbf{Q}$, $\mathbf{Z}$, $\mathbf{N}$, for the sets of real, rational, integer, and natural numbers, respectively.
- $\mathbf{N}_+ = \mathbf{N} \setminus \{0\}$ will denote the set of all positive integers.

- By $\mathbf{R}^n$ ($\mathbf{Q}^n, \mathbf{Z}^n, \ldots$), we will denote the set of n-dimensional vectors with coordinates from $\mathbf{R}$ (correspondingly, from $\mathbf{Q}, \mathbf{Z}, \ldots$). In matrix operations, these vectors will be treated as columns.
- A^T will denote a transposition of A.
- For every $i \le j$, $\pi_{i,j} : \mathbf{R}^j \to \mathbf{R}^i$ will denote a *projection*, i.e., a function that transforms a j-dimensional vector into the vector consisting of its first i coordinates: $\pi_{i,j}(x_1, \ldots, x_i, \ldots, x_j) = (x_1, \ldots, x_i)$.
- By $\mathbf{R}^{m\times n}$ (correspondingly, $\mathbf{Q}^{m\times n}, \mathbf{Z}^{m\times n}, \ldots$), we denote the set of all $(m \times n)$-matrices with elements from $\mathbf{R}$ (correspondingly, from $\mathbf{Q}$, from $\mathbf{Z}, \ldots$).

DEFINITION 2.1.

- Let m, n, and s be non-negative integers. By a *system of m linear equations with n unknowns and s-parametric uncertainty* (or a *linear system*, for short), we mean a triple $\Omega = (D, A, b)$, where $D \subseteq \mathbf{R}^s$ is a nonempty set, and $A : D \to \mathbf{R}^{m\times n}$ and $b : D \to \mathbf{R}^m$ are mappings from the set D into, correspondingly, the sets $\mathbf{R}^{m\times n}$ and $\mathbf{R}^m$. This system will also be denoted by

$$A(d)x = b(d). \tag{2.1}$$

- By a (*united*) *solution set* $\Sigma(\Omega)$ of a linear system Ω, we mean the set

$$\Sigma(\Omega) = \Sigma(D, A, B) = \{x \in \mathbf{R}^n \mid \exists d \in D : \ A(d)x = b(d)\}.$$

EXAMPLE. An interval $m \times n$ linear system $\mathbf{A}x = \mathbf{b}$ with an interval matrix $\mathbf{A}$ and an interval vector $\mathbf{b}$ can be represented as a linear system in the sense of Definition 2.1, if we take:

- $s = m \cdot n + m$. For this s, $\mathbf{R}^s = \mathbf{R}^{m\times n} \times \mathbf{R}^m$, so, every s-dimensional vector p can be represented as a pair (A, b) of an $(m \times n)$-matrix A and an m-dimensional vector b.
- As D, we take the set of all vectors $d = (A, b)$ for which $A \in \mathbf{A}$ and $b \in \mathbf{b}$.
- As $A(d)$, we take a first (matrix) component of d; as $b(d)$, we take the second (vector) component.

In this case, the solution set as defined by Definition 2.1 coincides with the united solution set as defined, e.g., in [12], [14], [15]:

$$\Sigma(\Omega) = \Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbf{R}^n \mid \exists A \in \mathbf{A}\ \exists b \in \mathbf{b} : \ Ax = b\}.$$

Comment. For a linear system $A(d)x = b(d)$, the following problems naturally appear:

- Does this system have a solution, i.e., is $\Sigma(\Omega)$ nonempty?
- If a linear system has solutions, what are the possible values of a given objective function f (e.g., of a linear function $f(x) = \sum c_i x_i$)?

In this paper, we will show that for several reasonable classes of linear systems, these problems are in general computationally intractable, or NP-hard.

For exact definitions of this notion, see, e.g., [3]. Intuitively, a problem is NP-hard if whenever we can solve this problem in polynomial time, we will be able to solve a huge class of different real-life problems in polynomial time; this class (called NP) is so huge that it is usually considered to be impossible to have an algorithm that solves all problems from this class in polynomial time. Therefore, we can conclude that no polynomial time algorithm can solve a given NP-hard problem.

Let us give an intuitive idea of the class NP. This class contains problems with the following property (that many real-life problems have): each of these problems may be difficult to solve, but if we have a candidate for a solution, then it is easy to check that this is indeed a solution. For example:

- if a problem consists of solving a system of equations, then it is easy to substitute the candidate into all these equations and check;
- if a problem consists of proving a theorem, then it is easy to check the proof step-by-step (if it is indeed a detailed proof, and not just an idea of the proof);
- if a problem consists of finding the dependency that explains the results of the experiments, then we can simply substitute the data into the candidate dependency and check whether all the data satisfy it;
- in design problems, we can usually easily check whether the given candidate design satisfies all the requirements.

Such problems in which checking a candidate solution is easy (i.e., can be done in polynomial time) are called *problems from the class NP*. For such problems, if we guessed correctly, we will check the guess and get a solution. A problem is called NP-hard if, whenever we can solve this problem in polynomial time, we can solve *all* problems from the class NP in polynomial time.

DEFINITION 2.2. Let S be a class of linear systems. Then, the following problems can be defined:

- *Problem $N(S)$*:

  Instance: a system $\Omega \in S$.

  Question: Is it true that $\Sigma(\Omega) \neq \emptyset$?

  The set of all systems $\Omega \in S$ for which this is true will be denoted by NS.

- *Problem $M(NS)$:*

  Instance: a system $\Omega \in NS$ (i.e., $\Omega \in S$ with $\Sigma(\Omega) \neq \emptyset$) with n unknowns $x_1, \ldots, x_n$, and $n+1$ integers $c_0, c_1, \ldots, c_n$.

  Question: Is it true that

$$\max \left\{ \sum_{i=1}^{n} c_i x_i \mid x = (x_1, \ldots, x_n)^T \in \Sigma(\Omega) \right\} \geq c_0?$$

Sometimes, it is important to solve these problems not for all problems from the class S, but only for problems that are in some sense "regular": e.g., for which the solution set is bounded, or finite, etc. Such "regular" problems form subclasses of the class S. Let us introduce the denotations for the corresponding subsets:

DENOTATIONS.

- By BS, we denote the class of all problems $\Omega \in S$ for which the set $\Sigma(\Omega)$ is bounded.
- By RS, we denote the class of all problems $\Omega \in S$ for which the set $\Sigma(\Omega)$ is *regular*, i.e., non-empty and bounded.
- By FS, we denote the class of all problems $\Omega \in S$ for which the set $\Sigma(\Omega)$ is finite.
- By DS, we denote the class of all problems $\Omega \in S$ for which the set $\Sigma(\Omega)$ is *discrete*, i.e., nonempty and finite.

Comment. Due to this definition, the following relations hold:

$$RS = NS \cap BS = NBS = BNS \quad \text{and} \quad DS = NS \cap FS = NFS = FNS.$$

DENOTATIONS. Let S be a class of linear systems. Then:

- $S^{m,n}$, $m, n > 0$, denotes the class of all systems from S that consist of m equations with n unknowns.
- For every natural number $k \in N$, $S^{(k)} = \bigcup_{n>0} S^{n+k,n}$ and $S^{(-k)} = \bigcup_{m>0} S^{m,m+k}$.

Comments.

- It is easy to see that $S^{(k_1)} \cap S^{(k_2)} = \emptyset$ for $k_1 \neq k_2$, and that $S = \bigcup_{k \in \mathbf{Z}} S^{(k)}$.
- To solve a "maximization problem" $M(NS)$, i.e., to find out whether there exists a solution $x \in \Sigma(\Omega)$ for which $\sum c_i x_i \geq c_0$, it is often useful to first find out whether there is a solution $x \in \Sigma(\Omega)$ for which the equality $\sum c_i x_i = c_0$ holds. This problem is equivalent to checking non-emptiness of the set of solutions for the following *auxiliary* linear system:

DENOTATION. Let $\Omega = (D, A, b)$ be a linear system with n unknowns, and let $c = (c_0, c_1, \ldots, c_n)^T \in \mathbf{R}^{n+1}$ be a sequence of real numbers. By $\Omega_c = (D, A_c, b_c)$, we will denote the following linear system:

$$\begin{cases} \sum_{i=1}^{n} c_i x_i = c_0, \\ A(d)x = b(d), \;\; d \in D. \end{cases} \tag{2.2}$$

This system can obviously be represented in the form (2.1) with naturally defined maps $A_c : D \to \mathbf{R}^{(m+1)\times n}$ and $b_c : D \to \mathbf{R}^{m+1}$.

3. Classes that Represent Boolean Vectors and the Main Lemma

NP-hard problems are usually formulated in terms of *discrete* systems, i.e., systems, in which we have n variables $x_1, \ldots, x_n$, each of which takes only two values. These two values can be, e.g., interpreted as “true” and “false,” in which case, each variable becomes *Boolean*, and the vector $(x_1, \ldots, x_n)$ becomes a *Boolean vector*. If, e.g., these values are 1 and -1, then the set of possible values of $x = (x_1, \ldots, x_n)$ is $\{-1, 1\}^n$. Therefore, a natural way to prove that a problem is NP-hard for a class S of linear systems is to prove that such Boolean vectors can be represented in S. This leads us to the following definition:

DEFINITION 3.1. We say that a class S of linear systems *describes Boolean vectors* if there exists a polynomial-time algorithm $\mathcal{U}$ that for every integer $\eta > 0$, constructs a system $\Omega^\eta \in S$ with $n(\eta) \geq \eta$ variables for which $\pi_{\eta,n}(\Sigma(\Omega^\eta)) = \{-1, 1\}^\eta$, and $\Omega^\eta_a \in S$ for all $a \in \mathbf{N} \times \mathbf{N}^\eta_+$.

Comment. Obviously, if a class S describes Boolean vectors, and $S \subseteq S'$, then S' also describes Boolean vectors. Therefore, in the further results, we will try to prove the property of “describing Boolean vectors” for classes that are as small as possible.

MAIN LEMMA. *If a class S describes Boolean vectors, then the problems $N(S)$ and $M(NS)$ are NP-hard.*

Proof.

1. We will prove this result by showing that for such classes S, we can reduce the problem Partition (known to be NP-hard [3]) to each of the problems $N(S)$ and $M(NS)$. Thus, if we could solve the problems $N(S)$ or $M(NS)$ in polynomial time, we would be able to solve Partition in polynomial time, and therefore, we would have P = NP.

Partition problem is defined as follows:

Problem Partition:

Instance: $\mu > 0$ positive integers $v_1, \ldots, v_\mu$.

Question: Does there exist a sequence of signs $x_1, \ldots, x_\mu \in \{-1, 1\}$ such that $\sum_{i=1}^{\mu} x_i v_i = 0$?

Let us show how to reduce every instance of this problem to $N(S)$ and $M(NS)$. Let a vector $v = (v_1, \ldots, v_\mu)$ be given. Then the question of Partition is to check whether the set $\Sigma_R(v) = \{x \in \{-1, 1\}^\mu \mid \sum_{i=1}^{\mu} x_i v_i = 0\}$ is non-empty.

2. Reducing this problem to $N(S)$ is easy: Let us take the system Ω^μ, whose existence is guaranteed by the property that the class S describes Boolean vectors, and take Ω^μ_a, where $a_0 = 0$ and $a_i = v_i$ for $i = 1, \ldots, \mu$. Then, $\Omega^\mu_a \in S$, and clearly, $\pi_{\mu,n(\mu)}(\Sigma(\Omega^\mu_a)) = \Sigma_R(v)$; therefore, $\Sigma_R(v) \neq \emptyset$ iff $\Sigma(\Omega^\mu_a) \neq \emptyset$. It is clear that if we know v, then we can construct the system $\Omega^\mu_a \in S$ in polynomial time.

3. Let us now show that Partition can be also reduced to $M(NS)$. Take $\eta = \mu + 1$. Let Ω^η be a system whose existence is guaranteed by Definition 3.1, and let $a_i = 2v_i$

for $1 \le i \le \mu$, and $a_0 = a_{\mu+1} = \sum_{i=1}^{\mu} v_i$. Then, the first equation of the extended system Ω_a^η will have the following form:

$$2\left(\sum_{i=1}^{\mu} v_i x_i\right) + \left(\sum_{i=1}^{\mu} v_i\right) x_{\mu+1} = \sum_{i=1}^{\mu} v_i. \tag{3.1}$$

The vector $(1, \ldots, 1, -1)^T \in \{-1, 1\}^\eta$ satisfies the equation (3.1). By the choice of Ω^η, this vector belongs to $\pi_{\eta,n}(\Sigma(\Omega^\eta))$ and therefore, it can be extended to a vector $x^* = (1, \ldots, 1, -1, x^*_{\eta+1}, \ldots, x^*_n)^T \in \Sigma(\Omega^\eta)$. Hence, $x^* \in \Sigma(\Omega_a^\eta)$, and consequently, $\Omega_a^\eta \in NS$.

Let us now take $c_0 = c_{\mu+1} = 1$ and $c_i = 0$ for all other $i \le n$, and consider the corresponding maximization problem. This problem is solvable iff there exists a solution $x \in \Sigma(\Omega_a^\eta)$ for which $\sum c_i x_i \ge c_0$. For our choice of coefficients, this inequality is equivalent to $x_{\mu+1} \ge 1$. Since Ω_a^η was obtained by adding an equation to Ω^η, every solution x of Ω_a^η is also a solution of Ω^η, i.e., $\Sigma(\Omega_a^\eta) \subseteq \Sigma(\Omega^\eta)$. Therefore, due to our choice of Ω^η, the first $\eta = \mu + 1$ components of each solution x are equal to ± 1. In particular, $x_{\mu+1} \in \{-1, 1\}$. So, the only case when $x_{\mu+1} \ge 1$ is when $x_{\mu+1} = 1$. In this case, the equation (3.1) is equivalent to $\sum_{i=1}^{\mu} v_i x_i = 0$. Since $x_i \in \{-1, 1\}$ for $i \le \mu$, if such a solution exists, then the Partition problem also has a solution. Vice versa, if a Partition problem has a solution $x_1, \ldots, x_\mu$, then adding $x_{\mu+1} = 1$ and extending the resulting vector $\in \{-1, 1\}^\eta$ leads to an element of $\Sigma(\Omega^\eta)$ that also satisfies the corresponding instance of the problem $M(NS)$. So, Partition is indeed reducible in polynomial time to $M(NS)$ and therefore, the problem $M(NS)$ is NP-hard. □

4. Auxiliary Result: Reduction to the Case When Different Parameters Describe Uncertainty in A and in b

In general, one and the same parameter d_i can influence both the matrix A and the right-hand side b. In the above interval example, some coordinates of the vector d are responsible only for A and some only for b. A natural question is: if we impose this restriction on linear systems, will we then restrict the class of linear systems? The answer is "no": every linear system can be thus reformulated.

DEFINITION 4.1. We say that a linear system has *separated parameters* if the following three statements are true:

- $s = t + l$ (so that $D \subseteq \mathbf{R}^s = \mathbf{R}^t \times \mathbf{R}^l$);
- A depends only on the first t parameters p of the vector $d \in D$; and
- b depends only on the last l parameters q of d.

DENOTATION. We will denote a system with separated parameters by

$$A(p)x = b(q). \tag{4.1}$$

PROPOSITION 4.1. *For every linear system* Ω, *there exists a linear system* Ω' *with separated parameters that is equivalent to* Ω *in the sense that* $\Sigma(\Omega) = \Sigma(\Omega')$.

Proof. Before we start proving this result, let us remark that every $(m \times n)$-matrix A can be represented as a $(m \cdot n)$-dimensional vector

$$(A_{11}, A_{12}, \ldots, A_{1n}, A_{21}, \ldots, A_{2n}, \ldots, A_{m1}, \ldots, A_{mn}).$$

This is exactly how the matrix A is represented inside the computer in many programming languages. An m-dimensional vector b can be described as $(b_1, \ldots, b_m)$. If in the computer, the description of the matrix A is followed by the description of a vector b, then we get the following sequence of real numbers (or, in more mathematical terms, the following $(m \cdot n + m)$-dimensional vector):

$$(A_{11}, \ldots, A_{mn}, b_1, \ldots, b_m).$$

In the remaining part of this proof, we will denote this vector by $\langle A, b\rangle$. Now, we are ready for the proof itself.

Let us take $\Omega' = (D', A', b')$, where:

- $m' = m$;
- $n' = m$;
- $S' = m \cdot n + m$;
- the set D' is the set of all possible S'-dimensional vectors $d' = \langle A(d), b(d)\rangle$ that correspond to different values $d \in D$ (i.e., $D' = \{\langle A(d), b(d)\rangle \mid d \in D\}$);
- the new mapping $A'(d')$ is defined, for $d' = (d'_1, \ldots, d'_{m \cdot n+m})$, as

  $$A'(d'_1, \ldots, d'_{m \cdot n}, d'_{m \cdot n+1}, \ldots, d'_{m \cdot n+m}) = (d'_1, \ldots, d'_{m \cdot n});$$

  and
- the new mapping $b'(d')$ is defined as

  $$b'(d'_1, \ldots, d'_{m \cdot n}, d'_{m \cdot n+1}, \ldots, d'_{m \cdot n+m}) = (d'_{m \cdot n+1}, \ldots, d'_{m \cdot n+m}).$$

From this construction, it is clear that this new linear system has separated parameters: Indeed:

- the value $A'(d')$ depends only on the first $m \cdot n$ parameters of the vector d', while
- the value $b'(d')$ depends only on the last m parameters of the vector d'.

Let us show that a vector x is a solution of the new linear system Ω' iff x is a solution of the original linear system Ω':

- If x is a solution of the new linear system, this means that for some $d' \in D'$, we have $A'(d')x = b'(d')$. By definition of the set D', every element of this set is of the form $\langle A(d), b(d)\rangle$ for some $d \in D$. So, $d' = \langle A(d), b(d)\rangle$ for some $d \in D$. For this d', we can apply the definitions of A' and b' given above, and get the following formulas: $A'(d') = A'(\langle A(d), b(d)\rangle) = A(d)$ and $b'(d') = b'(\langle A(d), b(d)\rangle) = b(d)$. Therefore, from $A'(d')x = b'(d')$, we conclude that $A(d)x = b(d)$ for some $d \in D$, i.e., that x is a solution of the original linear system Ω.

- Vice versa, let x be a solution of the old linear system. This means that for some $d \in D$, we have $A(d)x = b(d)$. By definition of the set D', the vector $d' = \langle A(d), b(d)\rangle$ belongs to this set D'. By definition of the mappings A' and b', for this vector d', we have $A'(d') = A'(\langle A(d), b(d)\rangle) = A(d)$ and $b'(d') = b'(\langle A(d), b(d)\rangle) = b(d)$. Hence, from $A(d)x = b(d)$, we conclude that $A'(d')x = b'(d')$ for some $d' \in \Omega'$, i.e., that x is solution of the new system. □

Comments.

- To avoid misunderstanding, it is important not to confuse two somewhat similar notions:
  - our notion of *separated parameters*, and
  - a similar notion of a system in which the matrix A and the vector b are *independent* in the following sense:
    if

    a matrix A is possible (i.e., $A = A'(d')$ for some $d' \in D'$),

    and

    a vector b is possible (i.e., $b = b'(d'')$ for some $d'' \in D'$),

    then

    the pair (A, b) is also possible (i.e., $A = A'(\tilde{d})$ and $b = b'(\tilde{d})$ for some $\tilde{d} \in D'$).

  A system (D', A', b') with separated parameters is definitely independent (in this sense) if the corresponding set $D' \subseteq \mathbf{R}^t \times \mathbf{R}^l$ can be represented as a Cartesian product $D' = D_1 \times D_2$, where $D_1 \subseteq \mathbf{R}^t$ and $D_2 \subseteq \mathbf{R}^l$; otherwise, the matrix A and the vector b are *dependent*, and their dependence is described by the set D'.
- Our Main Lemma result is not mathematically complicated, and we believe that it may have been known before; however, since we did not find it in the literature and since we believe it to be important, we included this result (with a detailed proof) in our paper.

5. Classes of Linear Systems That Describe Boolean Vectors

Due to Proposition 4.1, we can (without loss of generality) consider only systems with separated parameters. Therefore, in this section, we will only consider systems of type (4.1). To describe the classes, we will need the following denotations:

DENOTATIONS. Let integers m and n be given.

- By $e^{(j)}$, we will denote an m-dimensional vector whose j-th element is 1 and all other elements are 0. Vectors $e^{(j)}$ will be called *vector units.*
- By $E^{(i,j)}$, we will denote an $(m \times n)$-matrix whose (i, j)-th element is 1, and all other elements are zeros. Matrices $E^{(i,j)}$ will be called *matrix units.*

- For every $r \in [1,+\infty] = [1,+\infty) \cup \{\infty\}$, we define l_r-norm $\|p\|_r$ of a vector $p \in \mathbf{R}^n$ as follows:
$$\|p\|_r = \left(\sum_{i=1}^{n} |p_i|^r\right)^{1/r} \quad \text{if} \quad r < \infty, \qquad \|p\|_\infty = \max |p_i|.$$
- Let us now define the following three sets (we will call them *sets of indices*):
$$\Lambda_1 = \{(r, r_1, r_2) \mid r \in (1,+\infty),\ r_1 \in [r,+\infty],\ r_2 \in [\sigma(r),+\infty]\},$$
where
$$\sigma(r) = \frac{r}{r-1},$$
$$\Lambda_2 = \{(r,\tau) \mid r,\tau \in [1,+\infty]\},$$
$$\Lambda = \Lambda_1 \cup \Lambda_2.$$
- For each $\lambda \in \Lambda$ and for each $n \in \mathbf{N}_+$, we define the set $D_{n,\lambda} \subseteq \mathbf{R}^n \times \mathbf{R}^n$ as follows:
  - If $\lambda = (r, r_1, r_2) \in \Lambda_1$, then
$$D_{n,\lambda} = \Big\{(p,q) \in \mathbf{R}^n \times \mathbf{R}^n \mid \frac{1}{r} n^{-r/r_1}(\|p\|_{r_1})^r + \frac{1}{\sigma(r)} n^{-\sigma(r)/r_2}(\|q\|_{r_2})^{\sigma(r)} \le 1\Big\}.$$
  - If $\lambda = (r,\tau) \in \Lambda_2$, then $D_{n,\lambda} = P_{n,r} \times Q_{n,\tau}$, where
$$P_{n,r} = \{p \in \mathbf{R}^n \mid \|p\|_r \le n^{1/r}\}, \quad Q_{n,\tau} = \{q \in \mathbf{R}^n \mid \|q\|_\tau \le n^{1/\tau}\}.$$

Now, we can introduce the following classes S_λ, $\lambda \in \Lambda$, of linear systems:

DEFINITION 5.1. Let $\lambda \in \Lambda$ be given. We say that a system $\boldsymbol{\Omega} = (D, A, b)$ of the form (4.1) with m equations of n variables belongs to the class S_λ if $s = l = n$, $D = D_{n,\lambda}$, and the mappings A and b are of the form

$$A(p) = A^{(0)} + \sum_{i=1}^{n} p_i \varepsilon_i A^{(i)}, \quad b(q) = b^{(0)} + \sum_{i=1}^{n} q_i \delta_i b^{(i)}, \tag{5.1}$$

where:

- $A^{(0)} \in \mathbf{Z}^{m\times n}$ and $b^{(0)} \in \mathbf{Z}^m$ (i.e., the matrix $A^{(0)}$ and the vector $b^{(0)}$ have integer components);
- for every $i = 1, \ldots, n$, $A^{(i)}$ is a matrix unit and $b^{(i)}$ is a vector unit;
- $\varepsilon_i, \delta_i \in \{0, 1\}$;
- if $\varepsilon_i = \varepsilon_j = 1$ and $i \ne j$, then $A^{(i)} \ne A^{(j)}$; if $\delta_i = \delta_j = 1$, then $b^{(i)} \ne b^{(j)}$.

Comment. When $\lambda \in \Lambda_1$, this means, crudely speaking, that the vector formed by the differences between the nominal $A^{(0)}$, $b^{(0)}$ and actual values of the coefficients A

and b is bounded in the l^p sense. When $\lambda \in \Lambda_2$, this means, crudely speaking, that the difference vector that corresponds to A and the difference vector that corresponds to b are bounded. In particular, for $r_1 = r_2 = r = \tau = \infty$, l^r-norm turns into max, and our definitions turn into a definition of a usual interval linear system. The possibility $\varepsilon_i, \delta_i = 0$ enables us to consider linear systems that depend on $s < n$ and $l < n$ parameters.

We will prove that for every $\lambda \in \Lambda$, the class FS_λ (of those systems for which the solution set is finite) describes Boolean vectors.

PROPOSITION 5.1. *For every $\lambda \in \Lambda$, the class FS_λ describes Boolean vectors.*

Proof. To prove this result, we will consider, for each $n > 0$ and for each $p, q \in \mathbf{R}^n$, the following system $\Omega^{n,\lambda} = (D_{n,\lambda}, A^n, b^n)$ of the type (5.1): $m = 2n$,

$$A^n(p) = \sum_{i=1}^{n} E^{(n+i,i)} + \sum_{i=1}^{n} p_i E^{(i,i)}, \quad b^n(q) = \sum_{i=1}^{n} e^{(i)} + \sum_{i=1}^{n} q_i e^{(n+i)}. \tag{5.2}$$

The corresponding system of $2n$ equations with n unknowns $A^n(p)x = b^n(q)$ is as follows:

$$\begin{cases} p_i x_i = 1, & i = \overline{1,n}, \\ x_i = q_i, & i = \overline{1,n}. \end{cases} \tag{5.3}$$

LEMMA 5.1. *For every $\lambda \in \Lambda$ and $n > 0$, $\Sigma(\Omega^{n,\lambda}) = \{-1, 1\}^n$.*

To prove this lemma, we will use two inequalities: a well-known inequality (which is a particular case of Young's inequality [1]) and the inequality between l_r-norms, which can be easily proven (either from the general statement of monotonicity of weighted sums [1], or directly).

LEMMA 5.2 (Young's inequality [1]). *Let $\tau, \theta \in (1, \infty)$, $1/\tau + 1/\theta = 1$. Then, for arbitrary $a, c \geq 0$,*

$$ac \leq \frac{1}{\tau} a^\tau + \frac{1}{\theta} c^\theta;$$

this inequality turns into an equality iff $a^\tau = c^\theta$.

LEMMA 5.3. *For every τ and θ such that $1 \leq \theta \leq \tau \leq +\infty$, and for every $p \in \mathbf{R}^n$,*

$$\|p\|_\theta \leq n^{1/\theta - 1/\tau} \|p\|_\tau; \tag{5.4}$$

this inequality becomes an equality iff all the values $|p_i|$, $1 \leq i \leq n$, are equal.

Proof of Lemma 5.1. Let $n \in \mathbf{N}_+$ and $\lambda \in \Lambda$. The inclusion $\{-1, 1\}^n \subseteq \Sigma(\Omega^{n,\lambda})$ is obvious, since any vector $x \in \{-1, 1\}^n$ is a solution of the system (5.3) for $p = q = x$, and $\{-1, 1\}^n \times \{-1, 1\}^n \subseteq D_{n,\lambda}$.

Let us prove the converse inclusion $\Sigma(\Omega^{n,\lambda}) \subseteq \{-1,1\}^n$. Select some $x \in \Sigma(\Omega^{n,\lambda})$. By definition, this means that there exist such $(p,q) \in D_{n,\lambda}$ for which the equations (5.3) are satisfied, i.e., for which $x = q$ and $p_i q_i = 1$ for all $i = \overline{1,n}$.

Since $\lambda \in \Lambda = \Lambda_1 \cup \Lambda_2$, we have two possibilities:

- $\lambda \in \Lambda_1$, and
- $\lambda \in \Lambda_2$.

Let us prove the desired inclusion for both cases.

1) First, we will consider the cased when $\lambda \in \Lambda_1$, i.e., when $\lambda = (r, r_1, r_2)$, where $r \in (1,+\infty)$, $r_1 \in [r,+\infty]$, $r_2 \in [\sigma,+\infty]$, and $\sigma = r/(r-1)$. Using Lemmas 5.2, 5.3, and the relations $p_i q_i = 1$, and taking into account that $(p,q) \in D_{n,\lambda}$ we obtain the following chain of equalities and inequalities:

$$1 \geq \frac{1}{r} n^{-r/r_1} \left(\|p\|_{r_1}\right)^r + \frac{1}{\sigma} n^{-\sigma/r_2} \left(\|q\|_{r_2}\right)^\sigma$$

(since $(p,q) \in D_{n,\lambda}$)

$$\geq \frac{1}{r} n^{-r/r_1} \cdot (n^{\frac{1}{r_1}-\frac{1}{r}} \|p\|_r)^r + \frac{1}{\sigma} n^{-\sigma/r_2} (n^{\frac{1}{r_2}-\frac{1}{\sigma}} \|q\|_\sigma)^\sigma$$

(due to Lemma 5.3)

$$= \frac{1}{n} \left[\frac{1}{r}(\|p\|_r)^r + \frac{1}{\sigma}(\|q\|_\sigma)^\sigma\right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left(\frac{1}{r}|p_i|^r + \frac{1}{\sigma}|q_i|^\sigma\right)$$

$$\geq \frac{1}{n} \sum_{i=1}^{n} |p_i| \cdot |q_i|$$

(due to Lemma 5.2)

$$= 1$$

(since $p_i \cdot q_i = 1$). Since this chain of inequalities starts and ends with 1, all inequalities in this chain are equalities. In particular, we have

$$\sum_{i=1}^{n} \left(\frac{1}{r}|p_i|^r + \frac{1}{\sigma}|q_i|^\sigma\right) = \sum_{i=1}^{n} |p_i| \cdot |q_i|.$$

Due to Lemma 5.2, for each i, we have the inequality

$$\left(\frac{1}{r}|p_i|^r + \frac{1}{\sigma}|q_i|^\sigma\right) \geq |p_i| \cdot |q_i|.$$

Therefore, the only case when the sum of the left-hand sides is equal to the sum of the right-hand sides is when each left-hand side is equal to the corresponding right-hand side. According to Lemma 5.2, this is possible only when $|p_i|^r = |q_i|^\sigma$ for all i. We already know that $p_i = 1/q_i$. Therefore, we have $|q_i|^{-r} = |q_i|^\sigma$, and $|q_i| = 1$. Therefore, $q_i = \pm 1$, and $x = q \in \{-1, 1\}^n$.

2) Let us now consider the case when $\lambda \in \Lambda_2$, i.e., when $\lambda = (r, \tau)$ for some $r, \tau \in [1, +\infty]$.

In this case, from the inequality (5.4) it follows that if $1 \leq \theta < \tau$, then $P_{n,\theta} \supseteq P_{n,\tau}$. Therefore, $P_{n,r} \supseteq P_{n,1}$, $Q_{n,\tau} \supseteq Q_{n,1}$, and consequently, $D_{n,\lambda} \supseteq D_{n,\lambda_0}$, where by λ_0, we denoted $\lambda_0 = (1, 1) \in \Lambda_2$. Since in this case $\Sigma(\Omega^{n,\lambda}) \subseteq \Sigma(\Omega^{n,\lambda_0})$, it is sufficient to prove that $\Sigma(\Omega^{n,\lambda_0}) \subseteq \{-1, 1\}^n$.

So, let us assume that $r = \tau = 1$.

In this case, the condition $q \in Q_{n,1}$ may be written in the form

$$\frac{1}{n}\sum_{i=1}^{n} |q_i| \leq 1,$$

and the condition $p \in P_{n,1}$ in the form

$$\frac{1}{n}\sum_{i=1}^{n} |p_i| \leq 1.$$

Since $p_i = 1/q_i$, we can rewrite the second condition as follows:

$$\frac{1}{n}\sum_{i=1}^{n} \frac{1}{|q_i|} \leq 1.$$

This inequality, in its turn, can be rewritten as

$$\left(\frac{1}{n}\sum_{i=1}^{n} \frac{1}{|q_i|}\right)^{-1} \geq 1.$$

Using the well-known inequality between the arithmetic and harmonic averages [1], we arrive at the following chain of inequalities:

$$1 \leq \left(\frac{1}{n}\sum_{i=1}^{n} \frac{1}{|q_i|}\right)^{-1} \leq \frac{1}{n}\sum_{i=1}^{n} |q_i| \leq 1.$$

Hence, all the inequalities in this chain are actually equalities. In particular, the arithmetic average of the values $|q_i|$ is equal to the harmonic average of these values. It is known [1] that this equality is possible only in one case: when all the values $|q_i|$ are equal, i.e., when there exists a λ such that $|q_i| = \lambda$ for all i. Then, from the equality

$$\frac{1}{n}\sum_{i=1}^{n} |q_i| = 1,$$

we conclude that $\lambda = 1$. Hence, $|q_i| = 1$, $q_i = \pm 1$, $x_i = q_i = \pm 1$, and $x \in \{-1, 1\}^n$. The lemma is proven. □

Proof of Proposition 5.1. Clearly, if $\Omega^\eta \in S_\lambda$, then $\Omega^\eta_a \in S_\lambda$ (the only coefficients that we have added when going from Ω^η to Ω^η_a are added to the "nominal" part, the part depending on p_i and q_j is unchanged). Therefore, the proof of Lemma 5.1 proves Proposition 5.1. □

THEOREM 5.1. *For every $\lambda \in \Lambda$, the problems $N(FS_\lambda)$ and $M(DS_\lambda)$ are NP-hard.*

Proof. This proof follows from the Main Lemma, Proposition 5.1, and from the fact that $NFS_\lambda = DS_\lambda$. □

Comment. We have actually proved that not only the class FS_λ itself describes Boolean vectors, but also that its certain proper *subclass* is a one:

DENOTATION. Let us denote

$$\tilde{S}_\lambda = \bigcup_{n>0} S_\lambda^{2n+1,\,n}.$$

COROLLARY. *For every $\lambda \in \Lambda$, the class $F\tilde{S}_\lambda$ describes Boolean vectors.*

Proof. Indeed, in the above construction, the number of equations of the systems $\Omega^{n,\lambda}$ is twice as large as the number of the variables.

THEOREM 5.2. *For every $\lambda \in \Lambda$, the problems $N(F\tilde{S}_\lambda)$ and $M(D\tilde{S}_\lambda)$ are NP-hard.*

Comment. Theorem 5.2 says that if we restrict ourselves to linear systems with finite number of solutions, in which the number of equations is no more than *twice larger* than the number of variables, then even for such systems, the solution problem $N(S)$ is NP-hard. A natural question is: what if we impose a stronger restriction on the number of equations, i.e., what if we require that for some given $k > 0$, the number of equations does not exceed the number of unknowns plus k; will this problem still be NP-hard? We already have a denotation for this restriction: we have denoted the corresponding class of linear equations by $S^{(k)}$. Using this denotation, we can reformulate the problem as follows: is the problem $N(FS_\lambda^{(k)})$ NP-hard for all $\lambda \in \Lambda$? The answer to this question is currently (1996) unknown. However, if we do not require finiteness (i.e., go from $FS^{(k)}$ to $S^{(k)}$), then the answer is "yes."

PROPOSITION 5.2. *For every $\lambda \in \Lambda$ and for every $k \in \mathbf{Z}$, the class $S_\lambda^{(k)}$ describes Boolean vectors.*

Proof. To prove this proposition, we will show how to equalize the numbers of equations and unknowns by introducing "fictitious" variables.

DENOTATIONS.

- For a given matrix $M \in \mathbf{R}^{m\times n}$, we will use the following denotations:
  - By $V(M)$, we denote an $m \times (n+1)$-matrix obtained from M by adding $(n+1)$-st zero column.
  - By $E(M)$, we denote an $(m+1) \times n$-matrix obtained from M by adding $(m+1)$-st zero row.
- Corresponding changes in the linear equations will be denoted as follows:
  - An operator $\mathbf{V} : S_\lambda \to S_\lambda$ (called *adding a variable*) transforms a system

$$\Omega = (D_{n,\lambda},\ A, b) \in S_\lambda^{m,n}$$

of the form (5.3) into a system

$$\mathbf{V}(\Omega) = \big(D_{n+1,\lambda}, \mathbf{V}(A), \mathbf{V}(b)\big) \in S_\lambda^{m,n+1},$$

where for $p, q \in \mathbf{R}^{n+1}$:

$$\begin{aligned} [\mathbf{V}(A)](p) &= V(A^{(0)}) + \sum_{i=1}^{n} p_i \varepsilon_i V(A^{(i)}) + p_{n+1}\varepsilon_{n+1}E^{(1,1)}, \quad \varepsilon_{n+1} = 0,\\ [\mathbf{V}(b)](q) &= b^{(0)} + \sum_{i=1}^{n} q_i \delta_i b^{(i)} + q_{n+1}\delta_{n+1}e^{(1)}, \qquad \delta_{n+1} = 0. \end{aligned}$$

  - An operator $\mathbf{E} : S_\lambda \to S_\lambda$ (called *adding an equation*) transforms a system

$$\Omega = (D_{n,\lambda}, A, b) \in S_\lambda^{m,n}$$

of the form (5.3) into a system

$$\mathbf{E}(\Omega) = \big(D_{n,\lambda}, \mathbf{E}(A), \mathbf{E}(b)\big) \in S_\lambda^{m+1,n},$$

where for $p, q \in \mathbf{R}^{n}$:

$$\begin{aligned} [\mathbf{E}(A)](p) &= E(A^{(0)}) + \sum_{i=1}^{n} p_i \varepsilon_i E(A^{(i)}),\\ [\mathbf{E}(b)](q) &= E(b^{(0)}) + \sum_{i=1}^{n} q_i \delta_i E(b^{(i)}). \end{aligned}$$

LEMMA 5.4. *For an arbitrary system $\Omega \in S_\lambda$, we have:*

$$\Sigma(\mathbf{V}(\Omega)) = \Sigma(\Omega) \times \mathbf{R}^1; \tag{5.5}$$

$$\Sigma(\mathbf{E}(\Omega)) = \Sigma(\Omega). \tag{5.6}$$

Proof is evident.

Proof of Proposition 5.2. Let $\lambda \in \Lambda$, $k \in \mathbf{Z}$ be fixed. Select some $\eta \in \mathbf{N}_+$. We need to construct a system Ω^η for which $\pi_{\eta,n}(\Sigma(\Omega^\eta)) = \{-1, 1\}^\eta$ and $\Omega_a^\eta \in S_\lambda^{(k)}$ for all $a \in \mathbf{N} \times \mathbf{N}_+^\eta$. To construct such a system, we will use two different constructions, depending on whether $\eta \le k-1$ or $\eta > k-1$.

If $\eta \le k-1$, then take $n = \eta$ and consider the system $\Omega^\eta = \mathbf{E}^{k-\eta-1}(\Omega^{\eta,\lambda})$. Since $\Omega^{\eta,\lambda} \in S_\lambda^{2\eta,\eta}$, we have $\Omega^\eta \in S_\lambda^{\eta+k-1,\eta}$, and consequently, for every $a \in \mathbf{N}^{\eta+1}$, for the extended system, we have $\Omega_a^\eta \in S_\lambda^{\eta+k,\eta}$. From (5.6), it follows that $\Sigma(\Omega^\eta) = \Sigma(\Omega^{\eta,\lambda}) = \{-1,1\}^\eta$.

If $\eta > k-1$, then take $n = 2\eta - k + 1 \ge \eta$, and consider a system $\Omega^\eta = \mathbf{V}^{\eta-k+1}(\Omega^{\eta,\lambda})$. Then, $\Omega^\eta \in S_\lambda^{2\eta,2\eta-k+1} = S_a^{n+k-1,n}$, and hence, $\Omega_a^\eta \in S_\lambda^{n+k,n}$. From (5.5), it follows that $\Sigma(\Omega^\eta) = \Sigma(\Omega^{\eta,\lambda}) \times \mathbf{R}^{n-\eta} = \{-1,1\}^\eta \times \mathbf{R}^{n-\eta}$; hence, $\pi_{\eta,n}(\Sigma(\Omega^\eta)) = \{-1,1\}^\eta$.

It is obvious that the computations that lead from η to the coefficients of Ω^η take polynomial time; therefore, by Definition 3.1, the class $S_\lambda^{(k)}$ describes Boolean vectors. □

THEOREM 5.3. *For every $\lambda \in \Lambda$, the problems $N(S_\lambda^{(k)})$ and $M(NS_\lambda^{(k)})$ are NP-hard.*

Proof directly follows from the Main Lemma and from Proposition 5.2.

Comment. Let us show that the systems constructed in this proof do not, in general, have finitely many solutions (moreover, for these systems, the set of solutions is not even always bounded). Indeed, if $\eta > \max\{1, k-1\}$, and if for some vector $v \in \mathbf{N}_+^\eta$, the set $\Sigma_R(v)$, constructed in the process of proving the Main Lemma, is nonempty, then for $a = (0, v_1, \ldots, v_\eta) \in \mathbf{N} \times \mathbf{N}_+^\eta$ and for the system Ω^η, we obtain $\Sigma(\Omega_a^\eta) = \Sigma_R(v) \times \mathbf{R}^{\eta-k+1}$. This set is not bounded; hence, $\Omega_a^\eta \notin BS_\lambda^{(k)}$. Hence, the above arguments do not prove that the classes $BS_\lambda^{(k)}$ and $FS_\lambda^{(k)}$ describes Boolean vectors for *all* k and λ. We will, however, be able to prove that these classes *do* describes Boolean vectors for *some* k and λ.

DENOTATION. By $\Lambda_0 \subseteq \Lambda_2$, we will denote the following set:

$$\Lambda_0 = \left\{(r,\tau) \mid r,\tau \in (1,+\infty],\ \frac{1}{r} + \frac{1}{\tau} < 1\right\}.$$

PROPOSITION 5.3. *For every $\lambda \in \Lambda_0$, and for every $k \ge 2$, the class $FS_\lambda^{(k)}$ describes Boolean vectors.*

Proof. To prove this proposition, for each n and $m = n+1$, we will consider a linear system $\Delta^{n,\lambda} = (D_{n,\lambda}, \tilde{A}^n, \tilde{b}^n)$ with $\tilde{A}^n : \mathbf{R}^n \to \mathbf{R}^{(n+1)\times n}$ and $\tilde{b}^n : \mathbf{R}^n \to \mathbf{R}^{n+1}$ defined as follows:

$$\tilde{A}^n(p) = \sum_{i=1}^n E^{(i+1,i)} + \sum_{i=1}^n p_i E^{(1,i)}, \quad \tilde{b}^n(q) = ne^{(1)} + \sum_{i=1}^n q_i e^{(i+1)}. \tag{5.7}$$

This system of $n + 1$ equations with n unknowns has the form:

$$\begin{cases} \sum_{j=1}^{n} p_j x_j = n, \\ x_i = q_i, \qquad i = \overline{1, n}. \end{cases} \tag{5.8}$$

LEMMA 5.5. *For every $\lambda \in \Lambda_0$ and $n > 0$, $\Sigma(\Delta^{n,\lambda}) = \{-1, 1\}^n$.*

Proof of Lemma 5.5. Let $\lambda \in \Lambda_0$ and $n > 0$. The fact that $\lambda \in \Lambda_0$ means that $\lambda = (r, \tau)$ for some $r, \tau \in (1, +\infty]$ for which $1/r + 1/\tau < 1$. The inclusion $\{-1, 1\}^n \subseteq \Sigma(\Delta^{n,\lambda})$ is obvious, since any vector $x \in \{-1, 1\}^n$ is the solution of (5.8) for $p = q = x$, and $\{-1, 1\}^n \times \{-1, 1\}^n \subseteq D_{n,\lambda}$.

To complete the proof of the lemma, it is thus sufficient to prove the converse inclusion $\Sigma(\Delta^{n,\lambda}) \subseteq \{-1, 1\}^n$. Indeed, let $x \in \Sigma(\Delta^{n,\lambda})$. This means that for some $p \in P_{n,r}$ and $q \in Q_{n,\tau}$, the equations (5.8) hold, i.e., $x = q$ and $\sum_{i=1}^{n} p_i q_i = n$.

Let us denote $\sigma = r/(r-1)$ (if $r = \infty$, we take $\sigma = 1$). Then, $1/r + 1/\sigma = 1$. Since $\tau \in (1, +\infty]$, $1/r + 1/\tau < 1$, and $1/r + 1/\sigma = 1$, we have $\sigma < \tau$. Using Hoelder's inequality ([1], Lemma 3) and the fact that $(p, q) \in D_{n,\lambda}$, we obtain the following sequence of equalities and inequalities:

$$\begin{aligned} n = \sum_{i=1}^{n} p_i q_i \;&\le\; \sum_{i=1}^{n} |p_i| \cdot |q_i| \le \|p\|_r \|q\|_\sigma \le \|p\|_r n^{\frac{1}{\sigma} - \frac{1}{\tau}} \|q\|_\tau \\ &\le\; n^{\frac{1}{r}} n^{\frac{1}{\sigma} - \frac{1}{\tau}} n^{\frac{1}{\tau}} = n. \end{aligned}$$

Since the first and the last expressions in this sequence coincide, all these inequalities are actually equalities. In particular, $\|q\|_\sigma = n^{\frac{1}{\sigma} - \frac{1}{\tau}} \|q\|_\tau$. Hence, from Lemma 5.3, we conclude that all the values $|q_i|$ are equal, i.e., that $|q_i| = \lambda > 0$ for all $i = \overline{1, n}$. Let us show that $\lambda = 1$. Indeed:

- From the equality $\|p\|_r \|q\|_\sigma = n$, and from the condition $p \in P_{n,r}$, we conclude that

  $$n = \|p\|_r \|q\|_\sigma = \|p\|_r \lambda n^{\frac{1}{\sigma}} \le n^{\frac{1}{r}} \lambda n^{\frac{1}{\sigma}} = \lambda n,$$

  i.e., $\lambda \ge 1$.
- From the condition $q \in Q_{n,\tau}$, we (similarly) conclude that $\lambda \le 1$.

So, $\lambda = |q_i| = 1$, $q_i = \pm 1$, and $x = q \in \{-1, 1\}^n$. □

Proof of Proposition 5.3. From the above lemma, it follows that for $\lambda \in \Lambda_0$, $\Delta^{n,\lambda} \in FS_\lambda^{n+1,n}$; for any $a \in \mathbf{N}^{n+1}$ the extended system $\Delta_a^{n,\lambda} \in FS_\lambda^{n+2,n}$; hence, the class $FS_\lambda^{(2)}$ describes Boolean vectors.

For $k > 2$, take $\Omega^\eta = \mathbf{E}^{k-2}(\Delta^{\eta,\lambda})$. From the above lemma and from the property (5.6), we can conclude that Ω^η represents $\{-1, 1\}^n$, and hence, that the class $FS_\lambda^{(k)}$ indeed describes Boolean vectors. □

Comment. We have proved Proposition 5.3 for all λ from a *subclass* of the class Λ_2. The proof itself cannot be generalized to *all* $\lambda \in \Lambda_2$: For example, if $\lambda = (r, \tau)$ for $r, \tau \in (1, +\infty)$ and $1/r + 1/\tau = 1$, then

$$\Sigma(\Delta^{n,\lambda}) = \partial Q_{n,s} = \{x \in \mathbf{R}^n \mid \|x\|_\sigma = n^{\frac{1}{\sigma}}\} \neq \{-1, 1\}^n.$$

THEOREM 5.4. *For every $\lambda \in \Lambda_0$ and for every $k \geq 2$, the problems $N(FS_\lambda^{(k)})$ and $M(DS_\lambda^{(k)})$ are NP-hard.*

6. Classes of $n \times n$ Linear Systems That Do Not Describe Boolean Vectors

DEFINITION 6.1. We say that a linear system $\mathbf{\Omega} = (D, A, b)$ of the form (2.1) *belongs to the class C* if the following two conditions hold:

- the set $D \subseteq \mathbf{R}^s$ is compact and connected; and
- the mappings A and b are continuous on D.

Comment. In accordance with the denotations introduced in Section 1, we will also consider classes $C^{(k)}$, $k \in \mathbf{Z}$, and their subclasses $NC^{(k)}$, $BC^{(k)}$, $RC^{(k)}$, $FC^{(k)}$, and $DC^{(k)}$.

Similarly to the definition of a regular interval matrix (see, e.g., [12]), we can introduce the following notion:

DEFINITION 6.2. Let $D \subseteq \mathbf{R}^s$. A mapping $A : \mathbf{R}^s \to \mathbf{R}^{n\times n}$ is called *regular* on D if for every $p \in D$, the matrix $A(p)$ is non-singular.

Comments.

- In [12], an interval matrix $\mathbf{A}$ is called *regular* if all matrices $A \in \mathbf{A}$ are non-singular. One can easily see that an interval matrix is regular in the sense of Definition 6.2 iff it is regular in the sense of [12].
- The notion of regularity introduced in this definition is related to the class $RC^{(0)}$ (and this relationship is the reason why we called the class *RS regular*):

PROPOSITION 6.1. *Let $D \subseteq \mathbf{R}^s$ be a compact connected set, and let $A : \mathbf{R}^s \to \mathbf{R}^{n\times n}$ be a mapping whose restriction to D is continuous. Then, the following two statements hold:*

1) *if for some $b : D \to \mathbf{R}^n$, the system $\mathbf{\Omega} = (D, A, b)$ belongs to $RC^{(0)}$, then A is regular on D;*
2) *if A is regular on D, then for every continuous mapping $b : D \to \mathbf{R}^n$, the system $\mathbf{\Omega} = (D, A, b)$ belongs to $RC^{(0)}$.*

Proof.

1) Let $\Omega = (D, A, b) \in RC^{(0)}$. This means, that the solution set $\Sigma(\Omega)$ is nonempty and bounded. That this set is non-empty means that there exist $p^{(0)} \in D$ and $x^{(0)} \in \mathbf{R}^n$ for which $A(p^{(0)})x^{(0)} = b(p^{(0)})$. Let us prove (by reduction to a contradiction) that this matrix $A(p^{(0)})$ is nonsingular. Indeed, if the matrix $A(p^{(0)})$ is singular, then we can find a vector $v \in \mathbf{R}^n$, $v \neq 0$, for which $A(p^{(0)})v = 0$. But then, for every $\lambda \in \mathbf{R}$, we have $A(p^{(0)})(x^{(0)} - \lambda v) = b(p)$ and therefore, $x^{(0)} - \lambda v \in \Sigma(\Omega)$. This conclusion contradicts to the fact that the solution set $\Sigma(\Omega)$ is bounded. Hence, the matrix $A(p^{(0)})$ is nonsingular.

Let us now show (also by reduction to a contradiction) that the matrix $A(p)$ is non-singular *for all* p (i.e., that A is a regular mapping). Indeed, assume that there exists $p^{(1)} \in D$ for which the matrix $A(p^{(1)})$ is singular. Since the set D is connected, we can find a continuous function $\varphi : [0, 1] \to D$ such that $\varphi(0) = p^{(0)}$ and $\varphi(1) = p^{(1)}$. Then, the function $\Phi : [0, 1] \to \mathbf{R}^{n \times n}$ defined as $\Phi(t) = A(\varphi(t))$ is continuous. For $t = 0$, the matrix $\Phi(0) = A(p^{(0)})$ is nonsingular; for $t = 1$, $\Phi(1) = A(p^{(1)})$ is singular. Let us denote by t_0 the infimum of the set of all t for which $\Phi(t)$ is singular:

$$t_0 = \inf\{\tau \in [0, 1] \mid \Phi(\tau) \text{ is singular}\}.$$

Then, $\Phi(t)$ is non-singular for $t < t_0$.

A matrix A is singular iff $\det(A) = 0$. Since t_0 is an infimum of the values t for which $\det(\Phi(t)) = 0$, and Φ is continuous, we can conclude that $\det(\Phi(t_0)) = 0$, and hence, the matrix $\Phi(t_0)$ is singular. Since we already know that the matrix $\Phi(0)$ is non-singular, we can thus conclude that $t_0 > 0$.

Let us now define a sequence $t_k = t_0(1 - 1 / k) \in [0, t_0)$, $k = 1, 2, \ldots$ For each k, $t_k < t_0$, and therefore, the matrix $\Phi(t_k)$ is nonsingular. Hence,

$$x^{(k)} = [\Phi(t_k)]^{-1} b(\varphi(t_k)) = [A(\varphi(t_k))]^{-1} b(\varphi(t_k)) \in \Sigma(\Omega).$$

Since the set $\Sigma(\Omega)$ is bounded, from the sequence $x^{(k)}$, $k \in \mathbf{N}_+$ it is possible to choose a convergent subsequence $x^{(k_l)}$, $l \in \mathbf{N}_+$:

$$\lim_{l \to \infty} x_{k_l} = x^*.$$

Because of our definition of $x^{(n)}$, we have $A(\varphi(t_{k_l}))x^{(k_l)} = b(\varphi(t_{k_l}))$. Because of our choice of t_k, we have $t_k \to t_0$ and therefore, $t_{k_l} \to t_0$. Since the functions A, b, and φ are continuous, we can tend l to ∞ and arrive at the following equation: $A(\varphi(t_0))x^* = b(\varphi(t_0))$. If we denote $p^* = \varphi(t_0)$, we conclude that $A(p^*)x^* = b(p^*)$. In the first part of this proof, we have already shown that the existence of such a solution leads to a conclusion that the matrix $A(p^*)$ is non-singular. However, we already proven that for $p^* = \varphi(t_0)$, the matrix $A(p^*) = \Phi(t_0)$ *is* singular. This contradiction shows that our assumption (that A is not a regular mapping) was false, and A *is* a regular mapping.

2) Let us now show that if A and b are continuous on D, and if A is regular on D, then $\Omega = (D, A, b) \in RC^{(0)}$.

Indeed, since A is regular, for every $p \in D$, there exists an inverse matrix $(A(p))^{-1}$, and for any fixed $p \in D$ the equation (2.1) has a unique solution $x = (A(p))^{-1}b(p)$. So,

$$\Sigma(\Omega) = \{(A(p))^{-1}b(p) \mid p \in D\}. \tag{6.1}$$

From this equality, we can immediately conclude that $\Sigma(\Omega) \neq \emptyset$.

Let us now prove that $\Sigma(\Omega)$ is bounded. Indeed, it is well known (see, for example, [5]) that the mapping from $\mathbf{R}^{n\times n}$ into $\mathbf{R}^{n\times n}$ that transforms a matrix $M \in \mathbf{R}^{n\times n}$ into its inverse M^{-1} is continuous on its domain of definition. Therefore, the function $c(p) = (A(p))^{-1}b(p)$ is continuous on D, and therefore, the set $\Sigma(\Omega) = c(D)$ is bounded (as an image of the compact set under a continuous mapping). □

Comment. From the representation of the solution set $\Sigma(\Omega)$ in the form (6.1), from the continuity of the function c, and from the fact that the image of the connected set under a continuous function is connected, we obtain the following proposition:

PROPOSITION 6.2.

1) *For every system $\Omega \in RC^{(0)}$, the solution set $\Sigma(\Omega)$ is connected.*
2) *If $\Omega \in DC^{(0)}$, then the mappings A and b are constant on D, and $\Sigma(\Omega)$ is a one-element set.*

Comment. The classes $RC^{(-k)}$ with $k > 0$ are easy to describe, because it turns out that for $\Omega \in C^{(-k)}$, the solution set $\Sigma(\Omega)$ is either empty or unbounded.

PROPOSITION 6.3. *For every $k > 0$, $RC^{(-k)} = \emptyset$.*

Proof. We will prove this result by reduction to a contradiction. Let us assume that there exists $\Omega = (D, A, b) \in RC^{(-k)}$. Above, we have defined an operator $\mathbf{E}$; this definition can be easily extended to the class C. By applying this extended operator $\mathbf{E}$ to the system Ω k times, and using (5.6), we conclude that $\mathbf{E}^k(\Omega) \in RC^{(0)}$. But for every p, the matrix $\mathbf{E}^k(A)(p)$ has k zero rows, and is, therefore, singular. Hence, our conclusion contradicts Proposition 6.1. This contradiction proves that $RC^{(-k)} = \emptyset$. □

Let us now prove the main proposition of this section.

PROPOSITION 6.4. *The class $BC^{(0)}$ does not describe Boolean vectors.*

Proof. We will prove (by reduction to a contradiction) that the representation from Definition 3.1 is impossible for $\eta \geq 2$. Indeed, let us assume that for some system Ω^η, we have $\pi_{\eta,n}(\Sigma(\Omega^\eta)) = \{-1, 1\}^\eta$, and $\Omega^\eta_a \in BC^{(0)}$ for all $a \in \mathbf{N} \times \mathbf{N}^\eta_+$. Let us choose a vector $v \in \mathbf{N}^\eta_+$ for which the set

$$\Sigma_R(v) = \left\{x \in \{-1, 1\}^\eta \mid \sum_{i=1}^{\eta} x_i v_i = 0\right\}$$

(introduced in the proof of the Main Lemma) is nonempty. For example, we can take $v_1 = v_2 = \cdots = v_{\eta-1} = 1$, $d_\eta = \eta - 1$; then, $x = (1, \ldots, 1, -1)^T \in \Sigma_R(v)$ and therefore, $\Sigma_R(v) \neq \emptyset$. If $x \in \Sigma_R(v)$, then $-x \in \Sigma_R(v)$, and hence, the set $\Sigma_R(v)$ is *not connected*.

On the other hand, as we have noticed in the proof of the Main Lemma, $\Sigma_R(v) = \pi_{\eta,n}(\Sigma(\Omega_a^\eta))$, where $a = (0, v_1, \ldots, v_\eta)^T \in \mathbf{N} \times \mathbf{N}_+^\eta$. Due to our assumption that $BC^{(0)}$ describes Boolean vectors, we have $\Omega_a^\eta \in BC^{(0)}$. Then, due to Proposition 6.2, the solution set $\Sigma(\Omega_a^\eta)$, and therefore, its projection $\Sigma_R(v) = \pi_{\eta,n}(\Sigma(\Omega_a^\eta))$ is also *connected*. The resulting contradiction shows that our assumption is false and thence, the class $BC^{(0)}$ does not describe Boolean vectors. □

Comment. A word of warning: In our proof of this proposition, we refer several times to auxiliary algebraic statements proved while proving the Main Lemma. To avoid confusion, we must notice that there is an important difference in our use of these statements in the proof of the Main Lemma and in the proof of Proposition 6.4:

- In the proof of the Main Lemma, we used these auxiliary algebraic statements to show that the problem Partition (that is known to be NP-hard) can be reduced to the problems $N(S)$ and $M(S)$, and that therefore, the problems $N(S)$ and $M(S)$ are also NP-hard.
- In the proof of Proposition 6.4, we simply use these statements as algebraic statements, without any reference to the Partition problem (the proof of these statements did not use the Partition problem).

Proposition 6.4 leads to the following result:

PROPOSITION 6.5.

- *For* $k \geq 2$, *the class* $FC^{(k)}$ *describes Boolean vectors.*
- *For* $k \leq 0$, *the class* $FC^{(k)}$ *does not describe Boolean vectors.*

Proof. For $k = 0$, this result follows from Proposition 6.4: since $FC^{(0)}$ is a subclass of $BC^{(0)}$, and $BC^{(0)}$ does not describe Boolean vectors, a subclass $FC^{(0)}$ does not describe Boolean vectors either.

Let us consider the case $k < 0$. In this case, due to Proposition 6.3, for each linear system from $C^{(k)}$, every solution set is either unbounded or empty. By definition, for every system $\Omega \in BC^{(k)} \subseteq C^{(k)}$, the solution set is bounded; therefore, it is empty. Hence, a projection of this solution set is also empty, and therefore, this class does not describe Boolean vectors.

For $k \geq 2$, the class $FC^{(k)}$ contains the class $FS_\lambda^{(k)}$ for every $\lambda \in \Lambda_0$, and the class $FS_\lambda^{(k)}$ describes Boolean vectors according to Proposition 5.3. Hence, the class $FC^{(k)}$ also describes Boolean vectors. □

Comments.

- For $k = 1$, we do not know whether the class $FC^{(1)}$ describes Boolean vectors or not.
- Since the class $BC^{(0)}$ does not describe Boolean vectors, it is natural to assume that the problem $N(BC^{(0)})$ is not difficult (i.e., not NP-hard), but easy (i.e., solvable in polynomial time). Basically, this assumption turns out to be correct. However, literally speaking, it is *not* true for all $BC^{(0)}$ because first, we must compute $A(p)$ and $b(q)$ at least for some p and q; since in our definition of this class, we did not restrict the computational complexity of computing these values, it could happen that the corresponding instance of the problem $N(BC^{(0)})$ requires a lot of computation time simply because computing $A(p)$ and $b(q)$ takes too long. If we prohibit such situations, we arrive at the following result:

DEFINITION 6.3. We say that a class S of linear systems is *algorithmically inhabited* (or *inhabited*, for short) if there exists a polynomial time algorithm that for every system $(D, A, b) \in S$, computes the following two things:

- the values $p^{(0)} \in D$ of the parameters for which all the components of the matrix $A(p^{(0)})$ are rational numbers (i.e., $A(p^{(0)}) \in \mathbf{Q}^{n\times n}$), and
- the components of the corresponding matrix $A(p^{(0)})$.

THEOREM 6.1. *If S is an inhabited subclass of $BC^{(0)}$, then the problem $N(S)$ is solvable in polynomial time.*

Comment. If a system belongs to the class $BC^{(0)}$, then (by definition of BS) its solution set is *bounded*. This does not automatically mean that this solution set is non-empty, because an empty set is also bounded. For example, if D is a one-point set, $A = A(p)$ is a singular matrix, and the equation $Ax = b$ has no solutions, then this system belongs to $BC^{(0)}$ and at the same time does not have a solution.

Proof. The corresponding algorithm is as follows. Let $\Omega = (D, A, b) \in S^{n\times n}$ be given.

- Since S is inhabited, we can (in time, bounded by a polynomial of n) compute the parameters $p^{(0)}$ and the matrix $A(p^{(0)}) \in \mathbf{Q}^{n\times n}$.
- Next, we check whether this matrix $A(p^{(0)})$ is singular or not. This can be done by computing its determinant $\det(A(p^{(0)}))$ and comparing it with 0. It is known that computing a determinant takes the same asymptotic time as matrix multiplication, so this computation can be done in time $O(n^{2.376})$ (see, e.g., [2], Chapter 31).
- If the matrix $A(p^{(0)})$ is nonsingular, then we claim that $\Sigma(\Omega) \neq \emptyset$, else, that $\Sigma(\Omega) = \emptyset$.

This algorithm clearly requires polynomial time. Let us show that this algorithm is correct.

- If the matrix $A(p^{(0)})$ is nonsingular, then $x = [A(p^{(0)})]^{-1}b(p^{(0)}) \in \Sigma(\Omega)$, and therefore, $\Sigma(\Omega) \neq \emptyset$.
- If the matrix $A(p^{(0)})$ is singular, then $\Sigma(\Omega) = \emptyset$, since otherwise, as we have proved in the proof of Proposition 6.1, the solution set $\Sigma(\Omega)$ is unbounded, which is in contradiction with $\Omega \in BC^{(0)}$. □

The following is a simple corollary of Theorem 6.1:

THEOREM 6.2. *For every $\lambda \in \Lambda$, the problem $N(BS_\lambda^{(0)})$ is solvable in polynomial time.*

Proof. Clearly, $BS_\lambda^{(0)} \in BC^{(0)}$; so, to apply Theorem 6.1, we must show that the class $BS_\lambda^{(0)}$ is inhabited. Indeed, we can take $p^{(0)} = (0, \ldots, 0)^T \in D_\lambda$; then, $A(p^{(0)}) \in \mathbf{Z}^{n\times n} \subset \mathbf{Q}^{n\times n}$. □

7. Interval Systems

We have already remarked that as a particular case of linear system (as defined by Definition 2.1), we get standard interval linear systems; such systems will be analyzed in this section.

DENOTATIONS.

- In this section, we will use a standard denotation $\mathbf{IR}$ for the set of all intervals $[\underline{a}, \overline{a}]$.
- By $\mathbf{IR}^n$, we will denote the set of n-dimensional *interval vectors*, i.e., n-dimensional vectors (columns) with coordinates from $\mathbf{IR}$.
- By $\mathbf{IR}^{m\times n}$, we denote the set of all $m \times n$ *interval matrices*, i.e., $(m \times n)$-matrices with elements from $\mathbf{IR}$.

DEFINITION 7.1. By an *interval linear system*, we mean a system of the type

$$\mathbf{A}x = \mathbf{b}, \tag{7.1}$$

where:

- $\mathbf{A} = [\underline{A}, \overline{A}] \in \mathbf{IR}^{m\times n}$ is an interval $(m \times n)$-matrix, and
- $\mathbf{b} = [\underline{b}, \overline{b}] \in \mathbf{IR}^m$ is an interval m-vector.

Comment. We have already shown that interval linear systems can be reformulated as a particular case of linear systems in the sense of Definition 2.1. Another reformulation is described in [11]. In this section, we will use the third reformulation:

DEFINITION 7.2. Let an interval linear system (7.1) be given. By its *reformulation*, we mean a linear system $\Omega = (D, A, b)$, where:

- $t = m \cdot n$, and $A : \mathbf{R}^{m \cdot n} \to \mathbf{R}^{m \times n}$ is defined as

$$A(p) = \frac{1}{2}(\underline{A} + \overline{A}) + \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{2}p_{n(i-1)+j}(\overline{a}_{ij} - \underline{a}_{ij})E^{(i,j)}, \tag{7.2}$$

  where $\overline{a}_{ij}, \underline{a}_{ij}$ are coefficients of matrices $\overline{A}$ and $\underline{A}$, respectively (matrices $E^{(i,j)}$ were defined in Section 5);
- $l = m$, and $b : \mathbf{R}^m \to \mathbf{R}^m$ is defined as

$$b(q) = \frac{1}{2}(\underline{b} + \overline{b}) + \sum_{i=1}^{m}\frac{1}{2}q_i(\overline{b}_i - \underline{b}_i)e^{(i)}, \tag{7.3}$$

  where $\overline{b}_i, \underline{b}_i$ are coordinates of the vectors $\overline{b}$ and $\underline{b}$, respectively (vectors $e^{(i)}$ were defined in Section 5);
- $D = P \times Q \subseteq \mathbf{R}^{mn} \times \mathbf{R}^m$, where

$$P = \{p \in \mathbf{R}^{m \cdot n} \mid \|p\|_\infty \le 1\}, \quad \text{and} \quad Q = \{q \in \mathbf{R}^m \mid \|q\|_\infty \le 1\}. \tag{7.4}$$

PROPOSITION 7.1. *If $\Omega = (D, A, b)$ if a reformulation of the interval linear system (7.1), then $\Sigma(\Omega) = \Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b})$ and*

$$\mathbf{A} = \{A(p) \mid p \in D\}, \quad \mathbf{b} = \{b(q) \mid q \in G\}. \tag{7.5}$$

Proof is evident.

Let us now define the classes of interval linear systems.

DENOTATION.

- By J, we will denote the class of all interval linear systems.
- By I, we will denote the class of all interval linear systems with integer coefficients (i.e., for which $\underline{A}, \overline{A} \in \mathbf{Z}^{m \times n}$ and $\underline{b}, \overline{b} \in \mathbf{Z}^m$).

Comment. Using denotations introduced in Section 1, we can define classes NI, BI, RI, FI, DI, $I^{(k)}$, etc. (similarly, NJ, ...). In particular:

- $I^{(k)}$ is the class of all interval linear system with integer-valued matrices $\mathbf{A}$ and right-hand side $\mathbf{b}$ that consist of $n + k$ equations with n unknowns;
- $FI^{(k)}$ is the class of all systems from $I^{(k)}$ that have finitely many solutions.

PROPOSITION 7.2.

- *For every integer k, the class $I^{(k)}$ describes Boolean vectors.*
- *For every $k \ge 2$, the class $FI^{(k)}$ describes Boolean vectors.*

Proof. Since the map of the form (5.1) is a particular case of maps (7.2)–(7.3), we can conclude that for $\lambda_0 = (\infty, \infty) \in \Lambda_0$, the class S_{λ_0} defined by Definition 5.1 is contained in I; as a result, $S^{(k)}_{\lambda_0} \subseteq I^{(k)}$, and $FS^{(k)}_{\lambda_0} \subseteq FI^{(k)}$ for every $k \in \mathbf{Z}$. According to Propositions 5.2 and 5.3, the classes $S^{(k)}_{\lambda_0}$ (for all k) and $FS^{(k)}_{\lambda_0}$ (for all $k \geq 2$) describes Boolean vectors; hence, the classes $I^{(k)}$ and $FI^{(k)}$ also describe Boolean vectors. □

THEOREM 7.1.

- *For every integer k, the problems $N(I^{(k)})$ and $M(NI^{(k)})$ are NP-hard.*
- *For every $k \geq 2$, the problems $N(FI^{(k)})$ and $M(DI^{(k)})$ are NP-hard.*

Comment. The definition of NP-hardness given above is based on the definition of the class NP of problems for which checking a candidate solution is easy (i.e., can be done in polynomial time). For problems from the class NP, if we guessed correctly, we will check the guess and get a solution. Some NP-hard problems have this property (they are called *NP-complete*), other do not. It is interesting to know whether our NP-hard problems are NP-complete or not. The answer to this question follows from the following proposition:

PROPOSITION 7.3. *Problems $N(I)$ and $M(NI)$ belong to the class NP (consequently, this is true for every subclass of the class I).*

Proof. Let an interval system $(\mathbf{A}, \mathbf{b}) \in I^{m,n}$ be given, with $\mathbf{A} = [\underline{A}, \overline{A}]$ and $\mathbf{b} = [\underline{b}, \overline{b}]$. For every vector $y \in \{-1, 1\}^n$, let us denote by $\mathbf{R}^n_y$ the orthant of $\mathbf{R}^n$ corresponding to the sequence of signs y, i.e., the set

$$\mathbf{R}^n_y = \{x \in \mathbf{R}^n \mid y_i \cdot x_i \geq 0, \quad i = \overline{1, n}\}.$$

According to Oettli-Prager's theorem [10], the set of solutions for the system (7.1) can be rewritten as

$$\mathbf{\Sigma}_{\exists\exists}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbf{R}^n \mid |A_c x - b_c| \leq \Delta |x| + \delta\},$$

where

$$A_c = \frac{1}{2}(\overline{A} + \underline{A}), \quad \Delta = \frac{1}{2}(\overline{A} - \underline{A}), \quad b_c = \frac{1}{2}(\overline{b} + \underline{b}), \quad \delta = \frac{1}{2}(\overline{b} - \underline{b}),$$

and the absolute value $|.|$ of the vector and the relation $\leq$ for the vectors are understood coordinate-wise.

For $x \in \mathbf{R}^n_y$, we have $|x| = \Delta_y x$, where $D(y) = \mathrm{diag}(y_1, \ldots, y_n)$ is a diagonal matrix with elements y_i. So, the intersection of $\mathbf{\Sigma}_{\exists\exists}(\mathbf{A}, \mathbf{b})$ and $\mathbf{R}^n_y$ can be expressed as follows:

$$\begin{aligned}
&\mathbf{\Sigma}_{\exists\exists}(\mathbf{A}, \mathbf{b}) \cap \mathbf{R}^n_y \\
&= \{x \in \mathbf{R}^n \mid -\Delta \cdot D(y)x - \delta \leq A_c x - b_c \leq \Delta \cdot D(y)x + \delta, \ D(y)x \geq 0\} \\
&= \{x \in \mathbf{R}^n \mid (A_c + \Delta \cdot D(y))x \geq b_c - \delta, \ (A_c - \Delta \cdot D(y))x \leq b_c + \delta, \ D(y)x \geq 0\}.
\end{aligned}$$

In other words, this intersection is the set of solutions for the system of $(2m + n)$ linear inequalities in n unknowns.

There exist polynomial time algorithms that check whether such a system has a solution (see, e.g., [4]). Therefore, to check whether a given interval system has a solution, we can do the following:

- first, we guess the orthant;
- second, we check if the intersection of the solution set with this orthant is indeed non-empty.

Hence, the problem $N(I)$ belongs to the class NP.

Similarly, to solve the problem $M(NI)$, we can do the following:

- first, guess an orthant in which the given linear form $\sum c_i x_i$ attains its maximum;
- for the guessed orthant, the problem of maximizing $\sum c_i x_i$ under given constraints is a linear programming problem; so, we apply a polynomial time algorithm from [4], and compare the result of this application with c_0.

Hence, the problem $M(NI)$ also belongs to the class NP. □

As a corollary, we get the following result:

THEOREM 7.2.

- *For every integer k, the problems* $N(I^{(k)})$ *and* $M(NI^{(k)})$ *are NP-complete.*
- *For every* $k \geq 2$*, the problems* $N(FI^{(k)})$ *and* $M(DI^{(k)})$ *are NP-complete.*

For some other classes, we will describe polynomial time algorithms. For that, we will need the following auxiliary results:

PROPOSITION 7.4. *Let* $\mathbf{A} \in \mathbf{IR}^{n \times n}$*. Then:*

1) *if for some interval vector* $\mathbf{b} \in \mathbf{IR}^n$*, the solution set* $\Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b})$ *is nonempty and bounded, then the interval matrix* $\mathbf{A}$ *is regular;*
2) *if an interval matrix* $\mathbf{A}$ *is regular, then for every interval vector* $\mathbf{b} \in \mathbf{IR}^n$*, the solution set* $\Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b})$ *is nonempty and bounded.*

Proof directly follows from Proposition 6.1.

Comment. In other words, Proposition 7.4 says that the class $RJ^{(0)}$ is the class of all interval linear systems systems with a regular interval $(n \times n)$-matrix $\mathbf{A}$.

PROPOSITION 7.5. *For every interval linear system* $(\mathbf{A}, \mathbf{b}) \in RJ^{(0)}$*, the solution set* $\Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b})$ *is connected.*

Proof directly follows from Proposition 6.2.

THEOREM 7.3. *The problem* $N(BI^{(0)})$ *is solvable in polynomial time.*

Proof. To prove this result, we will use Theorem 6.1 according to which the problem $N(S)$ is solvable in polynomial time for every inhabited subclass of $BC^{(0)}$. The class $BI^{(0)}$ is a subclass of the class $BC^{(0)}$, and it is inhabited: we can take $p^{(0)} = (0, \ldots, 0)^T \in D$ and, correspondingly, $A(p^{(0)}) = (1/2) \cdot (\overline{A} + \underline{A}) \in \mathbf{Q}_{n,n}$. □

Comments.

- Due to Theorem 7.2, the problem $N(FI^{(k)})$ is NP-hard for $k \geq 2$; from Theorem 7.3, it follows that for $k = 0$, this problem is solvable in polynomial time. We do not know whether the problem $N(FI^{(1)})$ is NP-hard or not.
- For a similar question for interval linear systems with *bounded* sets of solutions instead of *finite* ones, the answer is known:

THEOREM 7.4. *For every $k \geq 0$, the problem $M(RI^{(k)})$ is NP-hard.*

Proof. The proof of this theorem uses the following result proven by Rohn and Kreinovich in [13]. To describe this result, let us denote, for every interval linear system (7.1) and for every $i = \overline{1, n}$,

$$\overline{x}_i = \max\{x_i \mid x \in \Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b})\}.$$

In [13], the following theorem was proved:

THEOREM R [13]. *The following problem is NP-hard:*
Given:

- *$(n \times n)$-matrices $\underline{A}, \overline{A} \in \mathbf{Z}^{n\times n}$ such that $\underline{A} \leq \overline{A}$ and the interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ is regular;*
- *vectors $\underline{b}, \overline{b} \in \mathbf{Z}^n$ for which $\underline{b} \leq \overline{b}$;*
- *a number $c \in \mathbf{Z}$,*

To check: whether $\overline{x}_1 \geq c$.

The statement of our theorem for $k = 0$ follows directly from Theorem R and Proposition 7.4. For $k \geq 1$, it is sufficient to note that we can transform the problem $M(RI^{(0)})$ into particular cases of the problem $M(RI^{(k)})$ if we apply the operator $\mathbf{E}$ k times (i.e., if we add k zero equations). □

Comment. For bounded sets of solutions, the situation is slightly different:

THEOREM 7.5.

- *For every $k \geq 1$, the problem $N(BI^{(k)})$ is NP-hard.*
- *The problem $N(BI^{(0)})$ is solvable in polynomial time.*

Proof. The second part has already been proven (in Theorem 7.3), so it is sufficient to prove the first part. To prove the first part, we will show how to transform each instance of the problem $M(RI^{(k)})$ into an instance of the problem $N(BI^{(k+1)})$; since $M(RI^{(k)})$ is NP-hard, this reduction will prove that the problem $N(BI^{(k+1)})$ is NP-hard for all $k \geq 0$.

To describe an instance of the problem $M(RI^{(k)})$, we must define an interval system $(\mathbf{A}, \mathbf{b}) \in RI^{n+k, n}$ and a vector

$$c = (c_0, c_1, \ldots, c_n)^T \in \mathbf{Z}^{(n+1)}.$$

Let us denote by $(\widetilde{\mathbf{A}}, \widetilde{\mathbf{b}})$ the following system of $(n + k + 2)$ equations with $(n + 1)$ unknowns:

$$\begin{cases} \mathbf{A}x = \mathbf{b}, \\ \sum_{i=1}^{n} c_i x_i - x_{n+1} = c_0 - 1, \\ [0, 1]x_{n+1} = [1, 1]. \end{cases} \tag{7.6}$$

If $\begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in \Sigma_{\exists\exists}(\widetilde{\mathbf{A}}, \widetilde{\mathbf{b}})$ then, obviously, $x \in \Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b})$ and

$$x_{n+1} = \sum_{i=1}^{n} c_i x_i - c_0 + 1.$$

So, boundedness of the solution set of the original system $\Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b})$ implies that the solution set of the new system $\Sigma_{\exists\exists}(\widetilde{\mathbf{A}}, \widetilde{\mathbf{b}})$ is also bounded (or empty). Therefore, $(\widetilde{\mathbf{A}}, \widetilde{\mathbf{b}}) \in BI^{(k+1)}$.

One can easily see that the last equation of the system (7.6) is equivalent to the inequality $x_{n+1} \geq 1$; hence, the last two equations of the system (7.6) are equivalent to the inequality

$$\sum_{i=1}^{n} c_i x_i \geq c_0.$$

Hence, the solvability of the system (7.6) is equivalent to existence of a vector $x \in \Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b})$ for which

$$\sum_{i=1}^{n} c_i x_i \geq c_0,$$

i.e., to the solvability of the original instance of the problem $M(BI^{(k)})$.

The algorithm that constructs the system (7.6) from the original system clearly takes polynomial time; therefore, if we were able to solve all instances $N(BI^{(k+1)})$ in polynomial time, we would thus be able to solve all instance of $M(BI^{(k)})$ in polynomial time, and we already know that the problem $M(BI^{(k)})$ is NP-hard. Therefore, the problem $N(BI^{(k+1)})$ is also NP-hard. □

8. Positive Interval Linear Systems

S. P. Shary attracted the authors' attention to interval systems with positive coefficients. Several interesting properties of such systems are presented in his paper [16].

DEFINITION 8.1. We say that an interval linear system is *strongly positive* if the lower bounds of all coefficients are ≥ 1, i.e., if $\underline{a}_{ij} \geq 1$ and $\underline{b}_j \geq 1$ for all i and j.

Comments.

- If we restrict ourselves to integer coefficients, then strongly positive means simply that all coefficients are positive.
- It is convenient to describe these component-wise inequalities in matrix and vector forms. For that, we will introduce the following denotations:

DENOTATIONS. Let n and m be positive integers.

- By e^n, we denote the vector $e^n = (1, \ldots, 1)^T \in \mathbf{R}^n$.
- By $E^{m,n}$, we denote the $(m \times n)$-matrix $E^{m,n} = e^m (e^n)^T$ whose all components are equal to one.

Comment. In these terms, an interval linear system is strongly positive iff $\underline{A} \geq E^{m,n}$ and $\underline{b} \geq e^m$.

DENOTATIONS.

- By PJ, we will denote the class of all strongly positive interval linear systems.
- By PI, we will denote the class of all strongly positive interval linear systems with integer coefficients (i.e., for which $0 < \underline{A} \leq \overline{A} \in \mathbf{Z}^{m\times n}$ and $0 < \underline{b} \leq \overline{b} \in \mathbf{Z}^m$).

We will show that the problems that are NP-hard for general interval linear systems remain NP-hard if we restrict ourselves to strongly positive systems only. To prove that, we will show that solving a general interval linear system can be reduced to solving a strongly positive one. For that, we will need the following transformation:

DEFINITION 8.2. Let us define an operator $\mathcal{P} : J \to J$ as follows: it maps a system $(\mathbf{A}, \mathbf{b})$, where $\mathbf{A} = [\underline{A}, \overline{A}] \in \mathbf{IR}^{m\times n}$ and $\mathbf{b} = [\underline{b}, \overline{b}] \in \mathbf{IR}^m$, into a new system $\mathcal{P}(\mathbf{A}, \mathbf{b}) = (\mathbf{A}_p, \mathbf{b}_p)$, where $\mathbf{A}_p = [\underline{A}_p, \overline{A}_p] \in \mathbf{IR}^{(m+1)\times(n+1)}$ and $\mathbf{b}_p = [\underline{b}_p, \overline{b}_p] \in \mathbf{IR}^{m+1}$ are defined as follows:

$$\overline{A}_p = \begin{pmatrix} \overline{A} + L & e^m \\ \gamma (e^n)^T & 1 \end{pmatrix}, \qquad \underline{A}_p = \begin{pmatrix} \underline{A} + L & e^m \\ \gamma (e^n)^T & 1 \end{pmatrix},$$

$$\overline{b}_p = \begin{pmatrix} \overline{b} + \delta e^m \\ \delta \end{pmatrix}, \qquad \underline{b}_p = \begin{pmatrix} \underline{b} + \delta e^m \\ \delta \end{pmatrix},$$

where:

$$L = \gamma e^m (e^n)^T = \gamma E^{m,n}, \quad \gamma = \max\{1, 1 - \min_{i,j} \underline{a}_{ij}\}, \quad \text{and}$$
$$\delta = \max\{1, 1 - \min_i \underline{b}_i\}.$$

Comment. For the transformed system, $\Sigma_{\exists\exists}(\mathcal{P}(\mathbf{A}, \mathbf{b})) \subseteq \mathbf{R}^{n+1}$; for convenience, we will describe vectors from $\mathbf{R}^{n+1}$ in the form $\begin{pmatrix} x \\ x_{n+1} \end{pmatrix}$, where $x \in \mathbf{R}^n$ and $x_{n+1} \in \mathbf{R}$.

PROPOSITION 8.1. *For every* $\mathbf{A} \in \mathbf{IR}^{m \times n}$ *and* $\mathbf{b} \in \mathbf{IR}^m$, *we have* $\mathcal{P}(\mathbf{A}, \mathbf{b}) \in PJ^{m+1, n+1}$ *and*

$$\Sigma_{\exists\exists}(\mathcal{P}(\mathbf{A}, \mathbf{b})) = \left\{ \begin{pmatrix} x \\ \delta - \gamma(e^n)^T x \end{pmatrix} \mid x \in \Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b}) \right\}. \tag{8.1}$$

Proof.

First, let us show that $\mathcal{P}(\mathbf{A}, \mathbf{b}) \in PJ$.

Indeed, for every $i = \overline{1, m}$ and $j = \overline{1, n}$, the following inequalities hold:

$$\underline{a}_{ij} + \gamma \;\geq\; \underline{a}_{ij} + 1 - \min_{k,l} \underline{a}_{kl} \;\geq\; 1,$$
$$\underline{b}_i + \gamma \;\geq\; \underline{b}_i + 1 - \min_{k} \underline{b}_k \;\geq\; 1,$$

$\gamma \geq 1$, and $\delta \geq 1$. Hence, $\underline{A}_p \geq E^{m+1, n+1}$, $\underline{b}_p \geq e^{m+1}$, and therefore, $\mathcal{P}(\mathbf{A}, \mathbf{b}) \in PJ^{m+1, n+1}$.

Let us now prove the equality (8.1). Note that the matrix M_p belongs to $\mathbf{A}_p$ (correspondingly, the vector c_p belongs to $\mathbf{b}_p$) if and only if there exists $M \in \mathbf{A}$ (corr., $c \in \mathbf{b}$) such that $M_p = \begin{pmatrix} M + L & e^m \\ \gamma(e^n)^T & 1 \end{pmatrix}$ (correspondingly, $c_p = \begin{pmatrix} c + \delta e^m \\ \delta \end{pmatrix}$). Therefore, the equality $M_p \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} = c_p$ is equivalent to the following system of equations:

$$\begin{cases} Mx + \gamma e^m (e^n)^T x + x_{n+1} e^m = c + \delta e^m, \\ \gamma(e^n)^T x + x_{n+1} = \delta. \end{cases} \tag{8.2}$$

If we subtract from the first equation of the system (8.2) the second equation multiplied by e^m, we will conclude that the system (8.2) is equivalent to the following system:

$$\begin{cases} Mx = c, \\ \gamma(e^n)^T x + x_{n+1} = \delta. \end{cases}$$

Therefore, $\begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in \Sigma_{\exists\exists}(\mathcal{P}(\mathbf{A}, \mathbf{b}))$ if and only if $x \in \Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b})$ and $x_{n+1} = \delta - \gamma(e^n)^T x$. □

Comment. In particular, from this proposition, it follows that the operator $\mathcal{P}$ does not change the following properties of the system:

- nonemptiness, boundedness, and finiteness of the solution set;
- the difference between the number of equations and the number of unknowns;
- the fact that all coefficients are integers.

Therefore, from the above NP-hardness results for general interval linear systems, we can deduce the following conclusions about NP-hardness of the strongly positive interval linear systems:

THEOREM 8.1.

- *For every integer k, the problems $N(PI^{(k)})$ and $M(NPI^{(k)})$ are NP-complete.*
- *For every $k \geq 2$, the problems $N(FPI^{(k)})$ and $M(DPI^{(k)})$ are NP-complete.*
- *For every $k \geq 0$, the problem $M(RPI^{(k)})$ is NP-complete.*
- *For every $k \geq 1$, the problem $N(BPI^{(k)})$ is NP-complete.*
- *The problem $N(BPI^{(0)})$ is solvable in polynomial time.*

Comment. The last statement follows from the fact that there is a polynomial time algorithm that solves all problems from a larger class $N(BI^{(0)})$.

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