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Analytical descriptions of quantifier solutions to interval linear systems of relations

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We study systems of relations of the form $Ax\,\sigma\,b$, where σ is a vector of binary relations with the components "$=$", "$\geq$", and "$\leq$", while the parameters (elements of the matrix A and right-hand side vector b) are uncertain and can take values from prescribed intervals. What is considered to be the set of its solutions depends on which logical quantifier is associated with each interval-valued parameter and what is the order of the quantifier prefixes for specific parameters. For solution sets that correspond to the quantifier prefix of a general form, we present equivalent quantifier-free analytical descriptions in the classical interval arithmetic, in Kaucher complete interval arithmetic and in the usual real arithmetic.

Keywords: Interval linear equation; interval linear inequality; interval linear system of relations; quantifier solution; solution set; analytical description; quantifier elimination; classical interval arithmetic; Kaucher interval arithmetic.

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1. Introduction

1.1. *Quantifier solutions to interval linear systems*

Our work is devoted to interval linear systems in which equations and inequalities may be present together. We call them *interval linear systems of relations*. In recent years, such mathematical objects have become the subject of intensive study, since they adequately describe various static linear models with bounded uncertainty and ambiguity. Some results and concepts from this field also penetrate into the theory

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of fuzzy systems. Below, we widely use the technique of interval analysis (see e.g. [15, 16, 18, 21, 33]), and therefore there is a need to recall some of its concepts and facts.

An *interval* is a bounded connected and closed subset of the real line $\mathbb{R}$. For example, $[-1, 2]$, $[1000, 1234.56]$, and so on. The set of all real intervals, together with the interval operations defined on it, is known to be called *interval arithmetic*. According to the notation standard [9], we will denote interval objects in bold type, as $\boldsymbol{A}$, $\boldsymbol{B}, \ldots, \boldsymbol{y}$, $\boldsymbol{z}$, in contrast to usual point (non-interval) quantities that are not specifically distinguished.

We consider systems of linear equations and inequalities of the form

$$\begin{gathered} Ax\,\sigma\,b, \quad A \in \mathbb{R}^{m\times n}, \quad x \in \mathbb{R}^n, \\ b \in \mathbb{R}^m, \quad \sigma \in \{=, \geq, \leq\}^m, \quad m, n \in \mathbb{N}, \end{gathered}$$

where x is a vector of unknowns, σ is a vector of binary relations, with the components "$=$", "$\geq$", and "$\leq$", and every uncertain element of the matrix A or of the right-hand side b can take values within the prescribed eponymous interval. Symbolically, we shall denote such interval systems of relations as

$$\boldsymbol{A}x\,\sigma\,\boldsymbol{b} \tag{1}$$

with an interval $m \times n$-matrix $\boldsymbol{A}$, an n-vector of unknowns x, an interval m-vector $\boldsymbol{b}$, and a relation vector $\sigma \in \{=, \geq, \leq\}^m$. An example of such "mixed" systems is the following 3×2-system:

$$\begin{cases} [2, 3]\, x_1 - [3, 4]\, x_2 \leq [-2, -1], \\ [5, 8]\, x_1 + [-6, 1]\, x_2 = [7, 12], \\ [-1, 0]\, x_1 + [4, 8]\, x_2 \geq [5, 9]. \end{cases}$$

The interval system (1) will thus denote the set of all possible point systems of the same structure for which the elements of the matrix and components of the right-hand side vector can take values from the corresponding intervals. What will the "solution" to the interval system of relations (1) mean?

We proceed from the fact, first noted in [31], that the interval uncertainty has a dual character, i.e. there are two different uncertainty types related to any interval of values that interests us. This follows from the intuitively clear fact that, in practice, we usually consider intervals in connection with a certain property, say $P(u)$, that can be either fulfilled or not fulfilled for the point members u of the intervals. For instance, the property P may have the form "to be a solution to an equation", "to be a solution to a problem" with some parameters that can take values from prescribed intervals, and so on. Then the following different situations may occur:

(1) either the property $P(u)$ holds for *all* members u from the given interval $\boldsymbol{u}$, or
(2) the property $P(u)$ holds only for *some* members u from the interval $\boldsymbol{u}$, not necessarily all, or even for a single value from $\boldsymbol{u}$.

Formally, the above distinction can be expressed by logical quantifiers (see e.g. [10] or any other textbook on mathematical logic):

(1) in the first case, we write "$(\forall\, u \in \boldsymbol{u})\, P(u)$"
and speak of *interval A-uncertainty* or *interval uncertainty of A-type*;

(2) in the second case, we write "$(\exists\, u \in \boldsymbol{u})\, P(u)$"
and speak of *interval E-uncertainty* or *interval uncertainty of E-type.*

As a consequence, in order to correctly specify a "solution" for an interval problem, we need to connect either the universal quantifier "$\forall$" or the existential quantifier "$\exists$" with each interval parameter $\boldsymbol{u}$ of our problem. This is organized as the corresponding elementary quantifier prefix, either $(\forall\, u \in \boldsymbol{u})$ or $(\exists\, u \in \boldsymbol{u})$, put before $P(u)$, and then such prefixes unite with each other to form the general quantifier prefix for the entire problem. Overall, we get a formula of the predicate calculus that determines solutions to a given interval problem, and such formulas are called *selecting predicates* [31].

For interval systems of linear relations (1), the interval uncertainty of parameters can be specified by the interval matrix $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$, the quantifier matrix $\mathcal{A}$ of the same size as $\boldsymbol{A}$, the interval vector $\boldsymbol{b} \in \mathbb{IR}^m$ and the quantifier vector β of length m. All the elementary quantifier prefixes can be written down in an arbitrary order, and we denote the resulting prefix of the length $m(n+1)$ as $Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)$. The following definitions are useful.

Definition 1. For given interval matrix $\boldsymbol{A}$, interval vector $\boldsymbol{b}$, and quantifier prefix $Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)$, the *interval-quantifier linear system of relations*, or *interval-quantifier linear system* in short, will be called the predicate of the form $Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)(Ax\, \sigma\, b)$.

This definition is, in fact, a further refinement of the general ideas expressed in [31]. Also, the above introduced interval-quantifier linear systems are a natural generalization of interval linear systems, which have long been studied in interval analysis. *Interval linear system* of the form $\boldsymbol{A}x\, \sigma\, \boldsymbol{b}$ is a formal record, for which we specially stipulate what is considered a solution in each specific case. Usually, interval linear systems of only equations or of only inequalities of the same sign are considered.

Definition 2. A vector $\tilde{x} \in \mathbb{R}^n$ is referred to as *solution* to the interval-quantifier linear system if the predicate $Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)(Ax\, \sigma\, b)$ takes the value "true" in $\tilde{x}$.

In order to agree with the existing terminology in this field, the solutions to the interval-quantifier linear system $Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)(Ax\, \sigma\, b)$ will also be called the *quantifier solutions* of the interval linear system $\boldsymbol{A}x\, \sigma\, \boldsymbol{b}$. Thus, interval linear systems of relations are considered in this case as separate independent objects for which various sets of solutions (solution sets) are then defined. In recent decades, the objects of intensive study in interval analysis have been formal (algebraic) solutions, AE-solutions, strong solutions, weak solutions, tolerable solutions, controllable solutions (sometimes called simply "control solutions"), etc. (see [31, 33; 3, Chap. 2] and references in these publications) or our quantifier solutions. Let us recall some definitions.

Definition 3 (31, Definition 3.1). *AE-solutions* are quantifier solutions to interval equations (inequalities, etc.) for which the selecting predicate has AE-form, that is, for which, in the prefix $Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)$, all occurrences of the universal quantifier "$\forall$" precede the occurrences of the existential quantifier "$\exists$".

Definition 4. The united solution set of the interval linear system $\boldsymbol{A}x\,\sigma\,\boldsymbol{b}$ is the set

$$\Xi_{\text{uni}}(\boldsymbol{A}, \boldsymbol{b}) = \{x \in \mathbb{R}^n \mid (\exists\, A \in \boldsymbol{A})(\exists\, b \in \boldsymbol{b})(Ax\,\sigma\,b)\},$$

formed by solutions to all the point systems $Ax\,\sigma\,b$ with $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. Its members are also called *weak solutions* of the system of relations.

Definition 5. The tolerable solution set of the interval linear system $\boldsymbol{A}x\,\sigma\,\boldsymbol{b}$ is the set

$$\Xi_{\text{tol}}(\boldsymbol{A}, \boldsymbol{b}) = \{x \in \mathbb{R}^n \mid (\forall\, A \in \boldsymbol{A})(\exists\, b \in \boldsymbol{b})(Ax\,\sigma\,b)\},$$

formed by all such vectors $x \in \mathbb{R}^n$ that, for any matrix $A \in \boldsymbol{A}$, the product Ax satisfies the relation $Ax\,\sigma\,b$ for some $b \in \boldsymbol{b}$.

Definition 6. The controllable solution set of the interval linear system $\boldsymbol{A}x\,\sigma\,\boldsymbol{b}$ is the set

$$\Xi_{\text{ctr}}(\boldsymbol{A}, \boldsymbol{b}) = \{x \in \mathbb{R}^n \mid (\forall\, b \in \boldsymbol{b})(\exists\, A \in \boldsymbol{A})(Ax\,\sigma\,b)\},$$

formed by all such vectors $x \in \mathbb{R}^n$ that, for any vector $b \in \boldsymbol{b}$, we can satisfy the relation $Ax\,\sigma\,b$ for an appropriate choice of the matrix $A \in \boldsymbol{A}$.

Definition 7. A vector $\tilde{x} \in \mathbb{R}^n$ is called *strong solution* to the interval linear system $\boldsymbol{A}x\,\sigma\,\boldsymbol{b}$ if the predicate $(\forall\, A \in \boldsymbol{A})(\forall\, b \in \boldsymbol{b})(Ax\;\sigma\;b)$ takes the value "true" for $x = \tilde{x}$.

Note that AE-solutions, strong solutions, weak solutions, tolerable solutions, controllable solutions to the system $\boldsymbol{A}x\,\sigma\,\boldsymbol{b}$ are subsumed under the quantifier solutions.

The notation $Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)(Ax\,\sigma\,b)$ defines all possible interval-quantifier linear systems in parametric form. The parameters of the description are $\boldsymbol{A}$, $\boldsymbol{b}$, $\mathcal{A}$, β, σ and the order of elementary quantifier prefixes in Q (since the elementary prefixes with different quantifiers do not always commute). Imposing additional constraints on the parameters, we obtain different classes (subsets) of interval-quantifier linear systems. For example, if we require equality to be the value of each component of the relation vector σ, then we obtain a class of interval-quantifier systems of linear equations. For fuzzy systems of linear equations, the above formulated ideas and concepts were developed, in particular, in the work [23].

The paper presents a revised and expanded version of the work originally reported in [24, 25].

1.2. *Transition to analytical descriptions*

Interval-quantifier linear systems and their solutions were introduced in the previous section through a logical predicate of the first order. Predicative description is close to the formulation of practical problems in natural language, but it allows very limited means of theoretical investigation and is not at all suitable for calculations. As a consequence, the following problem arises.

Problem. *For the widest possible subset of interval-quantifier linear systems, find a convenient quantifier-free analytical description of their solutions and solution sets in algebraic systems* (*arithmetics*) *with sufficiently developed tools for equivalent transformations*, *study and computation.*

Usually, the solution sets to interval systems of equations and inequalities are described using real arithmetic in $\mathbb{R}$ [3, Chap. 2; 2, pp. 93–95; 5, 11, 12, 19, 20, 34], since it is simple, familiar, has good properties, and we can apply developed numerical methods in $\mathbb{R}$. For various subclasses of interval-quantifier systems of linear equations, a number of analytical descriptions have been obtained in interval arithmetics [1, 17, 31, 33]. Despite poor algebraic properties of the interval arithmetics (the absence of the distributivity, etc.), these descriptions turned out to be very useful. For instance, the description of the AE-solution sets of interval systems of linear equations made it possible to construct a general theory of these solutions and interval numerical methods for inner and outer estimation of the AE-solution sets [31, 33].

The features of the quantifier-free analytical descriptions proposed in this paper are as follows:

(1) They expand the class of the interval-quantifier linear systems for which a convenient description of solutions can be given in comparison with those known descriptions where non-negativity of x is not required. (The non-negativity requirement on the vector of unknowns can be formulated as an additional restriction on the parameters $\boldsymbol{A}$, $\boldsymbol{b}$, and σ. Vatolin in [34] obtained analytical descriptions for solutions of general interval-quantifier linear systems, but his result is only valid under non-negativity condition on the variables, which is quite severe limitation in practice. The class Q^{σ} of interval-quantifier linear systems we discuss in this paper has no restrictions on $\boldsymbol{A}$, $\boldsymbol{b}$, and σ, but it has a restriction on the order of the elementary quantifier prefixes.)

(2) Analytical descriptions of the solutions are obtained in ordinary real arithmetic $\mathbb{R}$, classical interval arithmetic $\mathbb{IR}$, and in Kaucher interval arithmetic $\mathbb{KR}$. This enables us to carry out investigation of the solution sets and computation with them by both real and interval methods.

In this paper, we derive analytic descriptions for solution sets in the form of inclusions and inequalities. It should be noted that there is an alternative way of describing the solution sets — with the help of so-called recognizing functionals, which is also very useful and popular for systems of equations. It was developed, for instance, in the works [26, 28].

2. Necessary Facts About Interval Arithmetics

In this section, we give the necessary information about various interval arithmetics, since it is scattered across various publications and has not become publicly available yet.

Arithmetic operations can be defined between real intervals, and the most popular way to do this is to introduce these operations "by representatives", that is, in accordance with the rule

$$\boldsymbol{x} \star \boldsymbol{y} = \{x \star y \,|\, x \in \boldsymbol{x},\, y \in \boldsymbol{y}\}, \quad \star \in \{+, -, \cdot, /\}.$$

Expanded constructive formulas for interval arithmetic operations are as follows (see e.g. [15, 16, 18, 33]):

$$\begin{aligned}
\boldsymbol{x} + \boldsymbol{y} &= [\underline{\boldsymbol{x}} + \underline{\boldsymbol{y}}, \overline{\boldsymbol{x}} + \overline{\boldsymbol{y}}],\\
\boldsymbol{x} - \boldsymbol{y} &= [\underline{\boldsymbol{x}} - \overline{\boldsymbol{y}}, \overline{\boldsymbol{x}} - \underline{\boldsymbol{y}}],\\
\boldsymbol{x} \cdot \boldsymbol{y} &= [\min\{\underline{\boldsymbol{x}}\,\underline{\boldsymbol{y}}, \underline{\boldsymbol{x}}\,\overline{\boldsymbol{y}}, \overline{\boldsymbol{x}}\,\underline{\boldsymbol{y}}, \overline{\boldsymbol{x}}\,\overline{\boldsymbol{y}}\}, \max\{\underline{\boldsymbol{x}}\,\underline{\boldsymbol{y}}, \underline{\boldsymbol{x}}\,\overline{\boldsymbol{y}}, \overline{\boldsymbol{x}}\,\underline{\boldsymbol{y}}, \overline{\boldsymbol{x}}\,\overline{\boldsymbol{y}}\}],\\
\boldsymbol{x}/\boldsymbol{y} &= \boldsymbol{x} \cdot [1/\overline{\boldsymbol{y}},\ 1/\underline{\boldsymbol{y}}] \quad \text{for } \boldsymbol{y} \not\ni 0.
\end{aligned}$$

The algebraic properties of the above operations are rather poor. In particular, for any intervals of nonzero width, there are no inverse elements with respect to these operations. The desire to improve the properties of the classical interval arithmetic $\mathbb{IR}$ led to the appearance of its various extensions. The most popular of them is *Kaucher interval arithmetic* $\mathbb{KR}$ developed by Kaucher [8]. Later, Gardeñes and Trepat [4] and then Markov [14] proposed another similar constructions. All these researchers constructed extensions of the classical interval arithmetic $\mathbb{IR}$ on the basis of different principles, which was reflected in the names of the corresponding algebraic structures: extended interval arithmetic [8], modal interval arithmetic [4], arithmetic of directed intervals [14]. However, despite the difference in their construction, all three algebraic systems coincide up to notation.

Interval on the real line $\mathbb{R}$ is its bounded closed and connected subset, that is, the set

$$[\eta, \theta] = \{u \in \mathbb{R} \,|\, \eta \le u \le \theta\}.$$

Formally, we can assume that the interval is a pair $[\eta, \theta]$ of real numbers, for which the condition $\eta \le \theta$ is satisfied. The set of all intervals over $\mathbb{R}$ is denoted as $\mathbb{IR}$. In $\mathbb{IR}$, the values η and θ should satisfy the requirement $\eta \le \theta$, but in Kaucher arithmetic this is not necessary. Intervals are also denoted by boldface letters, e.g. $\boldsymbol{u} \in \mathbb{KR}$. If

$\boldsymbol{u}$ and $[\eta, \theta]$ denote the same interval, then η is called left (lower) endpoint of the interval, which is written as $\underline{\boldsymbol{u}}$, and θ is called the right (upper) endpoint of the interval $\boldsymbol{u}$, which is written as $\overline{\boldsymbol{u}}$. Therefore, we take $\boldsymbol{u} \equiv [\underline{\boldsymbol{u}}, \overline{\boldsymbol{u}}]$.

In this paper, we will mainly use concepts and properties of Kaucher interval arithmetic, and they are presented below.

Two intervals are considered equal if both their left and right endpoints coincide:

$$\boldsymbol{u} = \boldsymbol{v} \quad \overset{\text{def}}{\Leftrightarrow} \quad \begin{cases} \underline{\boldsymbol{u}} = \underline{\boldsymbol{v}}, \\ \overline{\boldsymbol{u}} = \overline{\boldsymbol{v}}. \end{cases}$$

The inclusion relation "$\subseteq$" in $\mathbb{KR}$ continues the inclusion relation in $\mathbb{IR}$ that considers intervals as sets. So, we have

$$\boldsymbol{u} \subseteq \boldsymbol{v} \quad \overset{\text{def}}{\Leftrightarrow} \quad \begin{cases} \underline{\boldsymbol{u}} \geq \underline{\boldsymbol{v}}, \\ \overline{\boldsymbol{u}} \leq \overline{\boldsymbol{v}}. \end{cases} \tag{2}$$

In particular, $[6, 3] \subseteq [4, 5]$.

The operations of taking the least upper bound (supremum) and greatest lower bound (infimum) with respect to inclusion are introduced for families of intervals bounded from above and from below, respectively, using the infimum and supremum in $\mathbb{R}$:

$$\bigvee_{i \in I} \boldsymbol{u}_i := \sup_{i \in I} {}_{\subseteq} \boldsymbol{u}_i := \left[\inf_{i \in I} \underline{\boldsymbol{u}}_i, \sup_{i \in I} \overline{\boldsymbol{u}}_i \right],$$

$$\bigwedge_{i \in I} \boldsymbol{u}_i := \inf_{i \in I} {}_{\subseteq} \boldsymbol{u}_i := \left[\sup_{i \in I} \underline{\boldsymbol{u}}_i, \inf_{i \in I} \overline{\boldsymbol{u}}_i \right].$$

We need the following unary operations on intervals:

$$\operatorname{mid} \boldsymbol{u} := \check{\boldsymbol{u}} := (\underline{\boldsymbol{u}} + \overline{\boldsymbol{u}})/2 \quad \text{— the midpoint,}$$

$$\operatorname{rad} \boldsymbol{u} := \hat{\boldsymbol{u}} := (\overline{\boldsymbol{u}} - \underline{\boldsymbol{u}})/2 \quad \text{— the radius,}$$

$$\operatorname{dual} \boldsymbol{u} := [\overline{\boldsymbol{u}}, \underline{\boldsymbol{u}}] \quad \begin{array}{l} \text{— the dualization, i.e. swapping} \\ \text{the endpoints of the interval,} \end{array}$$

$$\operatorname{pro} \boldsymbol{u} := \begin{cases} \boldsymbol{u} & \text{if } \underline{\boldsymbol{u}} \leq \overline{\boldsymbol{u}}, \\ \operatorname{dual} \boldsymbol{u} & \text{if } \underline{\boldsymbol{u}} > \overline{\boldsymbol{u}} \end{cases} \quad \begin{array}{l} \text{— the proper projection} \\ \text{of the interval.} \end{array}$$

Note that the dualization makes sense only in $\mathbb{KR}$.

Arithmetic operations of addition, subtraction, multiplication, and division are determined through the corresponding real operations and taking exact lower and upper bounds by inclusion so that

$$\boldsymbol{u} * \boldsymbol{v} = \bigwedge\!\!\bigvee^{\boldsymbol{u}} \bigwedge\!\!\bigvee^{\boldsymbol{v}} (u * v), \quad \text{where } \bigwedge\!\!\bigvee^{\boldsymbol{u}} := \begin{cases} \bigvee\limits_{\boldsymbol{u}} & \text{if } \underline{\boldsymbol{u}} \leq \overline{\boldsymbol{u}}, \\ \bigwedge\limits_{\operatorname{pro} \boldsymbol{u}} & \text{if } \underline{\boldsymbol{u}} \geq \overline{\boldsymbol{u}}, \end{cases}$$

for each operation $* \in \{+, -, \cdot, /\}$. Naturally, division is determined only for such intervals $\boldsymbol{v}$ that $0 \notin \operatorname{pro} \boldsymbol{v}$. The addition and multiplication are commutative. The addition is defined "by endpoints":

$$\boldsymbol{u} + \boldsymbol{v} = [\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}, \overline{\boldsymbol{u}} + \overline{\boldsymbol{v}}]. \tag{3}$$

The real numbers $\eta \in \mathbb{R}$ are identified with intervals of zero radius $[\eta, \eta]$. Multiplication of an interval by the number $\eta \in \mathbb{R}$ satisfy the following properties:

$$\eta \boldsymbol{u} = \begin{cases} [\eta \underline{\boldsymbol{u}}, \eta \overline{\boldsymbol{u}}] & \text{for } \eta \geq 0, \\ [\eta \overline{\boldsymbol{u}}, \eta \underline{\boldsymbol{u}}] & \text{for } \eta \leq 0, \end{cases} \tag{4}$$

$$(\operatorname{dual} \boldsymbol{u})\eta \overset{(4)}{=} \operatorname{dual}(\boldsymbol{u}\eta) = [\overline{\boldsymbol{u}\eta}, \underline{\boldsymbol{u}\eta}]. \tag{5}$$

The symbol $-\boldsymbol{u}$ means the result of multiplication $(-1) \cdot \boldsymbol{u}$, not the opposite interval for $\boldsymbol{u}$ with respect to the addition.

The matrices and vectors whose elements are intervals are called interval matrices and interval vectors, respectively. We denote by $\boldsymbol{A}_{i:}$ the ith row of the matrix $\boldsymbol{A}$. For interval vectors and matrices, their endpoints, the relations "$=$", "$\geq$", "$\leq$" and "$\subseteq$", the operations "mid", "rad", "dual", "pro", as well as addition, subtraction, and multiplication by numbers are introduced componentwise and elementwise. For example,

$$(\operatorname{dual} \boldsymbol{A})_{ij} := \operatorname{dual}(\boldsymbol{A}_{ij}), \quad (\boldsymbol{A} - \boldsymbol{B})_{ij} := \boldsymbol{A}_{ij} - \boldsymbol{B}_{ij}, \quad (-\boldsymbol{A})_{ij} = -\boldsymbol{A}_{ij}.$$

The multiplication rules for interval vectors and matrices are interval extensions of analogous rules for the non-interval case:

$$(\boldsymbol{A}\boldsymbol{B})_{ij} := \sum_k \boldsymbol{A}_{ik} \boldsymbol{B}_{kj}. \tag{6}$$

Also, we need the property

$$(\operatorname{dual} \boldsymbol{A})x = \operatorname{dual}(\boldsymbol{A}x) \quad \text{for } \boldsymbol{A} \in \mathbb{KR}^{m \times n}, \;\; x \in \mathbb{R}^n, \tag{7}$$

which can be easily derived from the definition of interval matrix–vector product (a particular case of (6)) with the use of (3) and (5).

3. Main Results

3.1. *Analytical descriptions in interval arithmetics*

First of all, we are going to develop quantifier-free analytical descriptions, for interval-quantifier linear systems and their solutions, in interval arithmetics. We need the following notation:

$Q_{i:}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)$ will denote a quantifier prefix obtained from $Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)$ by deleting all those elementary prefixes that are not related to the ith row of the system;

$Q^{\forall\,\exists}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)$ will denote a quantifier prefix of the form $Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)$ satisfying the additional condition: for each $i \in \{1, \ldots, m\}$, the universal quantifiers in $Q_{i:}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)$ (if any) precede the existential quantifiers (if any);
$Q^{\mathrm{AE}}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)$ will denote a quantifier prefix of the form $Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)$ with the additional condition: all the universal quantifiers (if any) precede all the existential quantifiers (if there are such quantifiers);
$\boldsymbol{A}^{\forall}, \boldsymbol{A}^{\exists} \in \mathbb{IR}^{m\times n}$, $\boldsymbol{b}^{\forall}, \boldsymbol{b}^{\exists} \in \mathbb{IR}^{m}$, $\boldsymbol{A}^{\mathfrak{c}} \in \mathbb{KR}^{m\times n}$, $\boldsymbol{b}^{\mathfrak{c}} \in \mathbb{KR}^{m}$ will denote interval vectors and matrices defined by the rules

$$\boldsymbol{A}^{\forall}_{ij} := \begin{cases} \boldsymbol{A}_{ij} & \text{if } \mathcal{A}_{ij} = \forall, \\ 0 & \text{if } \mathcal{A}_{ij} = \exists, \end{cases} \qquad \boldsymbol{A}^{\exists}_{ij} := \begin{cases} \boldsymbol{A}_{ij} & \text{if } \mathcal{A}_{ij} = \exists, \\ 0 & \text{if } \mathcal{A}_{ij} = \forall, \end{cases}$$
$$\boldsymbol{b}^{\forall}_{i} := \begin{cases} \boldsymbol{b}_{i} & \text{if } \beta_{i} = \forall, \\ 0 & \text{if } \beta_{i} = \exists, \end{cases} \qquad \boldsymbol{b}^{\exists}_{i} := \begin{cases} \boldsymbol{b}_{i} & \text{if } \beta_{i} = \exists, \\ 0 & \text{if } \beta_{i} = \forall, \end{cases} \tag{8}$$

$$\boldsymbol{A}^{\mathfrak{c}}_{ij} := \begin{cases} \boldsymbol{A}_{ij} & \text{if } \mathcal{A}_{ij} = \forall, \\ \operatorname{dual} \boldsymbol{A}_{ij} & \text{if } \mathcal{A}_{ij} = \exists, \end{cases} \qquad \boldsymbol{b}^{\mathfrak{c}}_{i} := \begin{cases} \operatorname{dual} \boldsymbol{b}_{i} & \text{if } \beta_{i} = \forall, \\ \boldsymbol{b}_{i} & \text{if } \beta_{i} = \exists. \end{cases} \tag{9}$$

Definition 8. The matrix $\boldsymbol{A}^{\mathfrak{c}}$ and vector $\boldsymbol{b}^{\mathfrak{c}}$ will be called *characteristic matrix* and *characteristic vector* that correspond to the distribution of interval uncertainty types (A-type or E-type) described by the quantifier matrix $\mathcal{A}$ and vector β in the interval linear system under study.

The characteristic matrix and characteristic vector of the right-hand side were introduced in [31] for interval linear systems of equations (see also [33]) and proved to be extremely useful in the study and computation of solution sets to interval-quantifier systems of equations. We extend these concepts to interval linear inequalities and more general systems of relations. The Gothic letter "$\mathfrak{c}$" as the superscript of $\boldsymbol{A}$ and $\boldsymbol{b}$ in formula (9) means "characteristic".

We should write out the property of interval-quantifier linear systems, which we will repeatedly apply in the sequel: each elementary quantifier prefix from $Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)$ can be carried to the row of the system in which the parameter of this prefix is present. This means

$$Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (Ax\,\sigma\, b) \quad \Leftrightarrow \mathop{\&}_{i\in\{1,\ldots,m\}} Q_{i:}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (A_{i:}x\,\sigma_i\, b_i). \tag{10}$$

The substantiation for this property is that the system of relations $(Ax\,\sigma\, b)$ is, in terms of logic, the conjunction of the relations, that is,

$$\mathop{\&}_{i}\ (A_{i:}x\,\sigma_i\, b_i).$$

Additionally, for the conjunction, there hold equivalences

$$\forall\, t \in \mathcal{S}(P_1(t)\&P_2) \quad \Leftrightarrow \quad (\forall\, t \in \mathcal{S}\ P_1(t))\&P_2,$$
$$\exists\, t \in \mathcal{S}(P_1(t)\&P_2) \quad \Leftrightarrow \quad (\exists\, t \in \mathcal{S}\ P_1(t))\&P_2,$$

where $\mathcal{S}$ is the set of values of the variable t, P_1, P_2 are formulas, and P_2 does not depend on t.

In view of (10), it is obvious that

$$Q^{\forall\,\exists}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (Ax\,\sigma\,b) \quad \Leftrightarrow \quad Q^{\mathrm{AE}}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (Ax\,\sigma\,b), \tag{11}$$

i.e. the vector x is a solution to the system $Q^{\forall\,\exists}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (Ax\,\sigma\,b)$ if and only if it is a solution to the system $Q^{\mathrm{AE}}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (Ax\,\sigma\,b)$. Thus, although the class of systems of the form $Q^{\forall\,\exists}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\,(Ax\,\sigma\,b)$ is wider than the class of systems of the form $Q^{\mathrm{AE}}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (Ax\,\sigma\,b)$, the statements proved for the solutions to the system $Q^{\mathrm{AE}}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (Ax\,\sigma\,b)$ are trivially generalized into statements for the solutions to the system $Q^{\forall\,\exists}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (Ax\,\sigma\,b)$.

Now, let us turn to the interval-quantifier systems of linear equations. Analytical descriptions for the widest subset of such systems have been obtained by Shary. In [29, 30], he first proved that

$$Q^{\mathrm{AE}}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (Ax = b) \quad \Leftrightarrow \quad \boldsymbol{A}^{\forall}x - \boldsymbol{b}^{\forall} \subseteq \boldsymbol{b}^{\exists} - \boldsymbol{A}^{\exists}x \quad \Leftrightarrow \quad \boldsymbol{A}^{\mathfrak{c}}x \subseteq \boldsymbol{b}^{\mathfrak{c}} \tag{12}$$

(see also [31]). Equivalence (11) allows us to make the following generalization of (12).

Theorem 1. *The following equivalence holds*:

$$Q^{\forall\,\exists}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)(Ax = b) \quad \Leftrightarrow \quad \boldsymbol{A}^{\forall}x - \boldsymbol{b}^{\forall} \subseteq \boldsymbol{b}^{\exists} - \boldsymbol{A}^{\exists}x \quad \Leftrightarrow \quad \boldsymbol{A}^{\mathfrak{c}}x \subseteq \boldsymbol{b}^{\mathfrak{c}}. \tag{13}$$

Theorem 1 for the interval-quantifier system of equations $Q^{\forall\,\exists}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)(Ax = b)$ gives equivalent analytical inclusion systems, in $\mathbb{IR}$

$$\boldsymbol{A}^{\forall}x - \boldsymbol{b}^{\forall} \subseteq \boldsymbol{b}^{\exists} - \boldsymbol{A}^{\exists}x$$

and in $\mathbb{KR}$

$$\boldsymbol{A}^{\mathfrak{c}}x \subseteq \boldsymbol{b}^{\mathfrak{c}}.$$

Definition 9. The interval-quantifier systems of relations, in which the vector of relations σ consists of the same components, will be called *relationally homogeneous systems.*

The result of Theorem 1 refers to systems of equations, and our immediate goal is to obtain similar results for relationally homogeneous systems of inequalities.

Theorem 2. *The following equivalences hold*:

$$Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (Ax \geq b) \quad \Leftrightarrow \quad \underline{\boldsymbol{A}^{\forall}x} + \overline{\boldsymbol{A}^{\exists}x} \geq \overline{\boldsymbol{b}}^{\forall} + \underline{\boldsymbol{b}}^{\exists} \quad \Leftrightarrow \quad \underline{\boldsymbol{A}^{\mathfrak{c}}x} \geq \underline{\boldsymbol{b}^{\mathfrak{c}}}, \tag{14}$$

$$Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (Ax \leq b) \quad \Leftrightarrow \quad \overline{\boldsymbol{A}^{\forall}x} + \underline{\boldsymbol{A}^{\exists}x} \leq \underline{\boldsymbol{b}}^{\forall} + \overline{\boldsymbol{b}}^{\exists} \quad \Leftrightarrow \quad \overline{\boldsymbol{A}^{\mathfrak{c}}x} \leq \overline{\boldsymbol{b}^{\mathfrak{c}}}. \tag{15}$$

Proof. We carry out the detailed proof only for the chain of equivalences (14). For (15), the substantiation is completely similar.

(1) From (10), it follows that

$$Q(\boldsymbol{A},\boldsymbol{b},\mathcal{A},\beta)\ (Ax\ge b)\quad\Leftrightarrow\quad \mathop{\&}_{i\in\{1,\ldots,m\}} Q_{i:}(\boldsymbol{A},\boldsymbol{b},\mathcal{A},\beta)\ (A_{i:}x\ge b_i). \tag{16}$$

(2) Using the fact that

$$A_{i:}x\ge b_i\quad\Leftrightarrow\quad \sum_{j=1}^{n} A_{ij}x_j+(-b_i)\ge 0$$

and that, for any continuous functions $h:\mathbb{R}^2\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$ and an interval $\boldsymbol{u}\in\mathbb{IR}$, there holds

$$(\forall\, u\in\boldsymbol{u})(h(u,x)+g(x)\ge 0)\quad\Leftrightarrow\quad \min_{u\in\boldsymbol{u}} h(u,x)+g(x)\ge 0,$$

$$(\exists\, u\in\boldsymbol{u})\ (h(u,x)+g(x)\ge 0)\quad\Leftrightarrow\quad \max_{u\in\boldsymbol{u}} h(u,x)+g(x)\ge 0,$$

enables us to get an analytical description for $Q_{i:}(\boldsymbol{A},\boldsymbol{b},\mathcal{A},\beta)(A_{i:}x\ge b_i)$:

$$Q_{i:}(\boldsymbol{A},\boldsymbol{b},\mathcal{A},\beta)\ (A_{i:}x\ge b_i)\quad\Leftrightarrow\quad \sum_{j=1}^{n}\operatorname*{extr}^{\mathcal{A}_{ij}}_{A_{ij}\in\boldsymbol{A}_{ij}}(A_{ij}x_j)+\operatorname*{extr}^{\beta_i}_{b_i\in\boldsymbol{b}_i}(-b_i)\ge 0, \tag{17}$$

where "extr" means conditional extremum, such that

$$\operatorname{extr}^{\forall}=\min,\quad \operatorname{extr}^{\exists}=\max\,.$$

(3) Thanks to the equalities

$$\min_{u\in\boldsymbol{u}}(ux)=\underline{\boldsymbol{u}x},\quad \max_{u\in\boldsymbol{u}}(ux)=\overline{\boldsymbol{u}x},\quad \min_{u\in\boldsymbol{u}}(u)=\underline{u},\quad \max_{u\in\boldsymbol{u}}(u)=\overline{u},$$

which are valid for any interval $\boldsymbol{u}\in\mathbb{IR}$, and taking into account (3), the sum of the extrema in (17) can be expressed in terms of the matrices $\boldsymbol{A}^{\forall}$, $\boldsymbol{A}^{\exists}$ and the vectors $\boldsymbol{b}^{\forall}$, $\boldsymbol{b}^{\exists}$ from (8):

$$\sum_{j=1}^{n}\operatorname*{extr}^{\mathcal{A}_{ij}}_{A_{ij}\in\boldsymbol{A}_{ij}}(A_{ij}x_j)+\operatorname*{extr}^{\beta_i}_{b_i\in\boldsymbol{b}_i}(-b_i)\ge 0\quad\Leftrightarrow\quad \underline{\boldsymbol{A}^{\forall}_{i:}x}+\overline{\boldsymbol{A}^{\exists}_{i:}x}\ge\overline{\boldsymbol{b}}^{\forall}_i+\underline{\boldsymbol{b}}^{\exists}_i. \tag{18}$$

(4) From (16)–(18), it follows that

$$Q(\boldsymbol{A},\boldsymbol{b},\mathcal{A},\beta)\ (Ax\ge b)\quad\Leftrightarrow\quad \underline{\boldsymbol{A}^{\forall}x}+\overline{\boldsymbol{A}^{\exists}x}\ge\overline{\boldsymbol{b}}^{\forall}+\underline{\boldsymbol{b}}^{\exists}.$$

(5) Let us prove the second equivalence in the chain (14). For the matrix $\boldsymbol{A}^{\mathfrak{c}}$ in (9), we have

$$[\underline{\boldsymbol{A}^{\mathfrak{c}}x},\overline{\boldsymbol{A}^{\mathfrak{c}}x}]=\boldsymbol{A}^{\mathfrak{c}}x\ \overset{\substack{\text{definitions}\\ \text{of } \boldsymbol{A}^{\mathfrak{c}},\boldsymbol{A}^{\forall},\boldsymbol{A}^{\exists}}}{=}\ \boldsymbol{A}^{\forall}x+(\operatorname{dual}\boldsymbol{A}^{\exists})x$$

$$\overset{\substack{\text{properties}\\ \text{(7) and (3)}}}{=}\ [\underline{\boldsymbol{A}^{\forall}x}+\overline{\boldsymbol{A}^{\exists}x},\ \overline{\boldsymbol{A}^{\forall}x}+\underline{\boldsymbol{A}^{\exists}x}], \tag{19}$$

and therefore $\underline{\boldsymbol{A}^{\mathfrak{c}}x}=\underline{\boldsymbol{A}^{\forall}x}+\overline{\boldsymbol{A}^{\exists}x}$. The definitions of the vectors $\boldsymbol{b}^{\mathfrak{c}}$, $\boldsymbol{b}^{\forall}$, and $\boldsymbol{b}^{\exists}$ give

$$[\underline{\boldsymbol{b}}^{\mathfrak{c}},\overline{\boldsymbol{b}}^{\mathfrak{c}}]=\boldsymbol{b}^{\mathfrak{c}}=\operatorname{dual}(\boldsymbol{b}^{\forall})+\boldsymbol{b}^{\exists}=[\overline{\boldsymbol{b}}^{\forall}+\underline{\boldsymbol{b}}^{\exists},\underline{\boldsymbol{b}}^{\forall}+\overline{\boldsymbol{b}}^{\exists}], \tag{20}$$

hence $\underline{\boldsymbol{b}^{\mathsf{c}}} = \overline{\boldsymbol{b}}^{\forall} + \underline{\boldsymbol{b}}^{\exists}$. Overall, we get

$$\underline{\boldsymbol{A}^{\forall}x} + \overline{\boldsymbol{A}^{\exists}x} \geq \overline{\boldsymbol{b}}^{\forall} + \underline{\boldsymbol{b}}^{\exists} \quad \Leftrightarrow \quad \underline{\boldsymbol{A}^{\mathsf{c}}x} \geq \underline{\boldsymbol{b}^{\mathsf{c}}}.$$

The proof of Theorem 2 is complete. □

In the interval arithmetics $\mathbb{IR}$ and $\mathbb{KR}$, the relations "$\geq$" and "$\leq$" are applicable, and they are continuations of the same relations over $\mathbb{R}$. For vectors, "$\geq$" and "$\leq$" are introduced componentwise. This allows us to formally refer to the records with $\boldsymbol{A}^{\forall}$, $\boldsymbol{A}^{\exists}$, $\boldsymbol{b}^{\forall}$, $\boldsymbol{b}^{\exists}$ in (14) and (15) as inequalities in classical interval arithmetic, while the records with $\boldsymbol{A}^{\mathsf{c}}$ and $\boldsymbol{b}^{\mathsf{c}}$ will be called inequalities in the Kaucher arithmetic. Still, in practice it is more convenient to understand all inequalities from (14) and (15) as componentwise inequalities in $\mathbb{R}^m$.

From (10) and Theorem 2, the following remarkable fact follows.

Corollary 1. *For any relation vector $\sigma \in \{\geq, \leq\}^m$, the solution sets of interval-quantifier systems of linear inequalities do not depend on the order of the elementary quantifier prefixes in their selecting predicates. In other words, all interval-quantifier systems of linear inequalities with identical $\boldsymbol{A}$, $\boldsymbol{b}$, $\mathcal{A}$, β, and σ have the same solution sets.*

This property essentially distinguishes interval systems of inequalities from interval systems of equations.

We give a corollary of Theorems 1 and 2, which establishes the relation between AE-solution sets of interval systems of linear equations and quantifier solution sets of interval relationally homogeneous systems of linear inequalities.

Corollary 2. *The following equivalence holds*:

$$\begin{aligned} Q^{\mathrm{AE}}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\,(Ax = b) \quad &\Leftrightarrow \quad Q^{\forall\,\exists}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\,(Ax = b) \\ &\Leftrightarrow \quad \begin{cases} Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)(Ax \geq b), \\ Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)(Ax \leq b). \end{cases} \end{aligned}$$

The proof is given by the following chain of equivalences:

$$\begin{cases} Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (Ax \geq b) \\ Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (Ax \leq b) \end{cases} \overset{\text{Theorem 2}}{\Leftrightarrow} \begin{cases} \underline{\boldsymbol{A}^{\mathsf{c}}x} \geq \underline{\boldsymbol{b}^{\mathsf{c}}} \\ \overline{\boldsymbol{A}^{\mathsf{c}}x} \leq \overline{\boldsymbol{b}^{\mathsf{c}}} \end{cases} \overset{\text{definition of } \subseteq}{\Leftrightarrow} \boldsymbol{A}^{\mathsf{c}}x \subseteq \boldsymbol{b}^{\mathsf{c}}$$

$$\overset{\text{Theorem 1}}{\Leftrightarrow} \quad Q^{\forall\,\exists}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (Ax = b) \quad \overset{(11)}{\Leftrightarrow} \quad Q^{\mathrm{AE}}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)(Ax = b).$$

Theorems 1 and 2 give analytical descriptions for relationally homogeneous systems. Next, we turn to the consideration of systems with an arbitrary relationship vector σ.

Definition 10. We denote by $Q^{\sigma}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)$ a quantifier prefix of the form $Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)$ satisfying the following condition: if σ_i is "$=$", then the universal quantifiers (if any) precede the existential quantifiers (if any) in $Q^{\sigma}_{i:}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)$.

Definition 11. The *class* Q^σ within the set of all interval-quantifier systems of linear relations is a subset consisting of all systems of the form $Q^\sigma(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)(Ax \,\sigma\, b)$.

The following theorem gives a quantifier-free analytical description of the class Q^σ in the interval arithmetics $\mathbb{KR}$ and $\mathbb{IR}$, with the use of componentwise inequalities from $\overline{\mathbb{R}}^m$, where $\overline{\mathbb{R}}$ denotes the extended real line, i.e. $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$.

Theorem 3. *The following equivalence holds*:

$$Q^\sigma(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (Ax \,\sigma\, b) \quad\Leftrightarrow\quad \begin{cases} \underline{\boldsymbol{A}^{\mathrm{c}} x} \geq \underline{\boldsymbol{b}^{\mathrm{c}}} + u, \\ \overline{\boldsymbol{A}^{\mathrm{c}} x} \leq \overline{\boldsymbol{b}^{\mathrm{c}}} + v \end{cases}$$

$$\Leftrightarrow\quad \begin{cases} \underline{\boldsymbol{A}^{\forall}\, x} + \overline{\boldsymbol{A}^{\exists}\, x} \geq \overline{\boldsymbol{b}}^{\forall} + \underline{\boldsymbol{b}}^{\exists} + u, \\ \overline{\boldsymbol{A}^{\forall}}\, x + \underline{\boldsymbol{A}^{\exists}\, x} \leq \underline{\boldsymbol{b}}^{\forall} + \overline{\boldsymbol{b}}^{\exists} + v, \end{cases} \tag{21}$$

where $\boldsymbol{A}^{\mathrm{c}}$ *and* $\boldsymbol{b}^{\mathrm{c}}$ *are from* (9), $\boldsymbol{A}^{\forall}$, $\boldsymbol{A}^{\exists}$, $\boldsymbol{b}^{\forall}$, $\boldsymbol{b}^{\exists}$ *are from* (8), *while the vectors* $u, v \in \overline{\mathbb{R}}^m$ *are defined as*

$$u_i := \begin{cases} 0 & \text{if } \sigma_i \text{ is "$=$" or "$\geq$"}, \\ -\infty & \text{if "σ_i" is "$\leq$"}, \end{cases} \qquad v_i := \begin{cases} 0 & \text{if } \sigma_i \text{ is "$=$" or "$\leq$"}, \\ \infty & \text{if } \sigma_i \text{ is "$\geq$"}. \end{cases}$$

Proof. [Proof step by step] (1) Due to the fact that each interval parameter (either an element of the matrix $\boldsymbol{A}$ or a component of the vector $\boldsymbol{b}$) enters only one row of the system $\boldsymbol{A}x \,\sigma\, \boldsymbol{b}$, we have (10) and, in particular,

$$Q^\sigma(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (Ax \,\sigma\, b) \quad\Leftrightarrow \mathop{\&}_{i \in \{1,\dots,m\}} Q^\sigma_{i:}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)(A_{i:}x \,\sigma_i\, b_i). \tag{22}$$

(2) We eliminate quantifier prefixes in the predicate $Q^\sigma_{i:}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (A_{i:}x \,\sigma_i\, b_i)$ using Theorems 1 and 2, based on the specific values of σ_i:

$$Q^\sigma_{i:}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (A_{i:}x = b_i) \overset{(13)}{\Leftrightarrow} (\boldsymbol{A}^{\mathrm{c}}x)_i \subseteq \boldsymbol{b}^{\mathrm{c}}_i \Leftrightarrow ((\underline{\boldsymbol{A}^{\mathrm{c}}x})_i \geq \underline{\boldsymbol{b}^{\mathrm{c}}_i}) \& \ ((\overline{\boldsymbol{A}^{\mathrm{c}}x})_i \leq \overline{\boldsymbol{b}^{\mathrm{c}}_i}),$$

$$Q^\sigma_{i:}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (A_{i:}x \geq b_i) \overset{(14)}{\Leftrightarrow} (\underline{\boldsymbol{A}^{\mathrm{c}}x})_i \geq \underline{\boldsymbol{b}^{\mathrm{c}}_i} \Leftrightarrow ((\underline{\boldsymbol{A}^{\mathrm{c}}x})_i \geq \underline{\boldsymbol{b}^{\mathrm{c}}_i}) \& ((\overline{\boldsymbol{A}^{\mathrm{c}}x})_i \leq \infty),$$

$$Q^\sigma_{i:}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (A_{i:}x \leq b_i) \overset{(15)}{\Leftrightarrow} (\overline{\boldsymbol{A}^{\mathrm{c}}x})_i \leq \overline{\boldsymbol{b}^{\mathrm{c}}_i} \Leftrightarrow ((\underline{\boldsymbol{A}^{\mathrm{c}}x})_i \geq -\infty) \& ((\overline{\boldsymbol{A}^{\mathrm{c}}x})_i \leq \overline{\boldsymbol{b}^{\mathrm{c}}_i}).$$

(3) Introducing the vectors u and v, we pass to the matrix–vector inequalities

$$Q^\sigma(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (Ax \,\sigma\, b) \quad\Leftrightarrow\quad \begin{cases} \underline{\boldsymbol{A}^{\mathrm{c}} x} \geq \underline{\boldsymbol{b}^{\mathrm{c}}} + u, \\ \overline{\boldsymbol{A}^{\mathrm{c}} x} \leq \overline{\boldsymbol{b}^{\mathrm{c}}} + v. \end{cases}$$

(4) The equivalence

$$\begin{cases} \underline{\boldsymbol{A}^{\mathrm{c}} x} \geq \underline{\boldsymbol{b}^{\mathrm{c}}} + u, \\ \overline{\boldsymbol{A}^{\mathrm{c}} x} \leq \overline{\boldsymbol{b}^{\mathrm{c}}} + v \end{cases} \quad\Leftrightarrow\quad \begin{cases} \underline{\boldsymbol{A}^{\forall}\, x} + \overline{\boldsymbol{A}^{\exists}\, x} \geq \overline{\boldsymbol{b}}^{\forall} + \underline{\boldsymbol{b}}^{\exists} + u, \\ \overline{\boldsymbol{A}^{\forall}}\, x + \underline{\boldsymbol{A}^{\exists}\, x} \leq \underline{\boldsymbol{b}}^{\forall} + \overline{\boldsymbol{b}}^{\exists} + v \end{cases}$$

is obvious in view of (19) and (20). The proof of Theorem 3 is complete. □

Convenient analytical representations for the class Q^σ can be obtained from Theorem 3, if we introduce the sets of intervals $\mathbb{K}\overline{\mathbb{R}} = \{[\underline{z}, \overline{z}] \mid \underline{z}, \overline{z} \in \overline{\mathbb{R}}\}$ and $\mathbb{I}\overline{\mathbb{R}} = \{[\underline{z}, \overline{z}] \mid \underline{z}, \overline{z} \in \overline{\mathbb{R}}, \underline{z} \le \overline{z}\}$, and continue relation "$\subseteq$" according to rule (2). Then

$$Q^\sigma(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (Ax\,\sigma\,b) \Leftrightarrow \boldsymbol{A}^{\mathfrak{c}} x \subseteq \boldsymbol{b}^{\mathfrak{c}} + \boldsymbol{w} \Leftrightarrow \boldsymbol{A}^{\forall} x - \boldsymbol{b}^{\forall} \subseteq \boldsymbol{b}^{\exists} - \boldsymbol{A}^{\exists} x + \boldsymbol{w}, \tag{23}$$

where $\boldsymbol{A}^{\mathfrak{c}}$ and $\boldsymbol{b}^{\mathfrak{c}}$ from (9), $\boldsymbol{A}^{\forall}$, $\boldsymbol{A}^{\exists}$, $\boldsymbol{b}^{\forall}$, $\boldsymbol{b}^{\exists}$ from (8), and the interval vector $\boldsymbol{w} \in \mathbb{I}\overline{\mathbb{R}}^m$ is such that

$$\boldsymbol{w}_i := \begin{cases} 0 & \text{if } \sigma_i \text{ is "="}, \\ [0, \infty] & \text{if } \sigma_i \text{ is "}\ge\text{"}, \\ [-\infty, 0] & \text{if } \sigma_i \text{ is "}\le\text{"}. \end{cases}$$

The inclusion

$$\boldsymbol{A}^{\mathfrak{c}} x \subseteq \boldsymbol{b}^{\mathfrak{c}} + \boldsymbol{w}$$

provides an analytical description of the solution set to the quantifier interval linear system $Q^\sigma(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)\ (Ax\,\sigma\,b)$ in any interval arithmetic that extends the Kaucher arithmetic to the set $\mathbb{K}\overline{\mathbb{R}}$. An example of such an extension is given in [7]. We agree to denote the arithmetic extension, as well as its basic set, through $\mathbb{K}\overline{\mathbb{R}}$.

Similarly, the inclusion

$$\boldsymbol{A}^{\forall} x - \boldsymbol{b}^{\forall} \subseteq \boldsymbol{b}^{\exists} - \boldsymbol{A}^{\exists} x + \boldsymbol{w}$$

provides an analytical description of the solution set to the system $Q^\sigma(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)(Ax\,\sigma\,b)$ in interval arithmetic that extends $\mathbb{IR}$ to the set $\mathbb{I}\overline{\mathbb{R}}$. Examples of the extension of the classical interval arithmetic to a set of intervals with infinite endpoints are described in [13]. Let us agree to refer to any such extension as arithmetic $\mathbb{I}\overline{\mathbb{R}}$. Thus, the relation (23) gives analytical descriptions of the solution sets to quantifier interval linear systems of class Q^σ in the interval arithmetics $\mathbb{K}\overline{\mathbb{R}}$ and $\mathbb{I}\overline{\mathbb{R}}$.

Comparing the analytical descriptions obtained for the solution sets to quantifier interval linear systems, we can see that,

> on the one hand, the analytical description in $\mathbb{KR}$ ($\mathbb{K}\overline{\mathbb{R}}$) is much more remote from the initial data $\boldsymbol{A}$, $\boldsymbol{b}$, $\mathcal{A}$ and β due to multilevel notation, and,
>
> on the other hand, the description in $\mathbb{KR}$ ($\mathbb{K}\overline{\mathbb{R}}$) is more concise and convenient for analysis than a similar description in $\mathbb{IR}$ ($\mathbb{I}\overline{\mathbb{R}}$).

3.2. *Analytical descriptions in real arithmetic*

In this section, we derive quantifier-free analytical descriptions of the quantifier solution sets to interval linear systems in the real arithmetic $\mathbb{R}$. To do that, we will need *Hadamard product* of matrices (entrywise product), denoted by the symbol "$\circ$" (see e.g. [6]). Hadamard product is defined for two matrices of the same dimensions and produces another matrix in which the ijth element is the product of the ijth elements of the original matrices:

$$(A \circ B)_{ij} := A_{ij} B_{ij}.$$

Also, note that the operation of taking the modulus of a vector is understood componentwise. If, for instance, $x \in \mathbb{R}^n$, then $|x|$ is a non-negative vector with the components $|x|_i = |x_i|$.

Theorem 4. *The following equivalences hold*:

$$Q^{\forall\exists}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)(Ax = b) \quad \Leftrightarrow \quad |\check{\boldsymbol{A}}x - \check{\boldsymbol{b}}| \le (\mathcal{A}^s \circ \hat{\boldsymbol{A}})|x| + \beta^s \circ \hat{\boldsymbol{b}}, \tag{24}$$

$$Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)(Ax \ge b) \quad \Leftrightarrow \quad \check{\boldsymbol{b}} - \check{\boldsymbol{A}}x \le (\mathcal{A}^s \circ \hat{\boldsymbol{A}})|x| + \beta^s \circ \hat{\boldsymbol{b}}, \tag{25}$$

$$Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)(Ax \le b) \quad \Leftrightarrow \quad \check{\boldsymbol{A}}x - \check{\boldsymbol{b}} \le (\mathcal{A}^s \circ \hat{\boldsymbol{A}})|x| + \beta^s \circ \hat{\boldsymbol{b}}, \tag{26}$$

$$Q^{\sigma}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)(Ax \,\sigma\, b) \quad \Leftrightarrow \quad \operatorname{abs}^{\sigma}(\check{\boldsymbol{A}}x - \check{\boldsymbol{b}}) \le (\mathcal{A}^s \circ \hat{\boldsymbol{A}})|x| + \beta^s \circ \hat{\boldsymbol{b}}, \tag{27}$$

where

$$\mathcal{A}^s_{ij} = \begin{cases} 1 & \text{if } \mathcal{A}_{ij} = \exists, \\ -1 & \text{if } \mathcal{A}_{ij} = \forall, \end{cases} \qquad \beta^s_i = \begin{cases} 1 & \text{if } \beta_i = \exists, \\ -1 & \text{if } \beta_i = \forall, \end{cases} \tag{28}$$

$$\operatorname{abs}_i{}^{\sigma}(y) = \begin{cases} |y_i| & \text{if } \sigma_i \text{ is } \text{“}=\text{”}, \\ -y_i & \text{if } \sigma_i \text{ is } \text{“}\ge\text{”}, \\ y_i & \text{if } \sigma_i \text{ is } \text{“}\le\text{”}. \end{cases}$$

Proof. (1) Equivalence (24) was proposed and proved by Jiri Rohn at the international conference Interval'96 (September–October of 1996, Würzburg, Germany), in a private talk with Sergey Shary and Anatoly Lakeyev. Later, its reformulation with the use of Hadamard product was proposed by Lakeyev in the work [12], and a similar proof was given by Rohn in [22]. Below, we present our own proof.

In view of Theorem 1,

$$Q^{\forall\exists}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)(Ax = b) \quad \Leftrightarrow \quad \boldsymbol{A}^{\mathrm{c}} x \subseteq \boldsymbol{b}^{\mathrm{c}}.$$

Then, using the properties of Kaucher arithmetic

$$(\forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{KR}^m)(\boldsymbol{u} \subseteq \boldsymbol{v} \Leftrightarrow |\check{\boldsymbol{u}} - \check{\boldsymbol{v}}| \le \hat{\boldsymbol{v}} - \hat{\boldsymbol{u}}),$$

$$\operatorname{mid}(\boldsymbol{A}^{\mathrm{c}} x) = \check{\boldsymbol{A}}^{\mathrm{c}} x, \quad \operatorname{rad}(\boldsymbol{A}^{\mathrm{c}} x) = \hat{\boldsymbol{A}}^{\mathrm{c}}\,|x|, \tag{29}$$

we get

$$\boldsymbol{A}^{\mathrm{c}} x \subseteq \boldsymbol{b}^{\mathrm{c}} \quad \Leftrightarrow \quad |\check{\boldsymbol{A}}^{\mathrm{c}} x - \check{\boldsymbol{b}}^{\mathrm{c}}| \le \hat{\boldsymbol{b}}^{\mathrm{c}} - \hat{\boldsymbol{A}}^{\mathrm{c}}\,|x|.$$

From the definitions of (9) and (28) for $\boldsymbol{A}^{\mathrm{c}}$, $\boldsymbol{b}^{\mathrm{c}}$, $\mathcal{A}^s$, and β^s, we have

$$\check{\boldsymbol{A}}^{\mathrm{c}} = \check{\boldsymbol{A}}, \quad \hat{\boldsymbol{A}}^{\mathrm{c}} = -\mathcal{A}^s \circ \hat{\boldsymbol{A}}, \quad \check{\boldsymbol{b}}^{\mathrm{c}} = \check{\boldsymbol{b}}, \quad \hat{\boldsymbol{b}}^{\mathrm{c}} = \beta^s \circ \hat{\boldsymbol{b}}. \tag{30}$$

(2) Let us prove equivalence (25). According to Theorem 2

$$Q(\boldsymbol{A},\boldsymbol{b},\mathcal{A},\beta)(Ax \geq b) \quad \Leftrightarrow \quad \underline{\boldsymbol{A}^{\mathrm{c}}x} \geq \underline{\boldsymbol{b}^{\mathrm{c}}}.$$

Drawing on the obvious property of the Kaucher arithmetic

$$(\forall\, \boldsymbol{u},\boldsymbol{v} \in \mathbb{KR}^m)(\underline{u} \geq \underline{v} \Leftrightarrow \check{\boldsymbol{v}} - \check{\boldsymbol{u}} \leq \hat{\boldsymbol{v}} - \hat{\boldsymbol{u}}),$$

which allows us to replace the inequality between the endpoints by the inequality between centers and radii, and then involving (29), we get

$$\underline{\boldsymbol{A}^{\mathrm{c}}x} \geq \underline{\boldsymbol{b}^{\mathrm{c}}} \quad \Leftrightarrow \quad \check{\boldsymbol{b}^{\mathrm{c}}} - \check{\boldsymbol{A}^{\mathrm{c}}}x \leq \hat{\boldsymbol{b}^{\mathrm{c}}} - \hat{\boldsymbol{A}^{\mathrm{c}}}\,|x|.$$

Finally, we use (30).

(3) Equivalence (26) is proved similarly to (25).

(4) It remains to substantiate equivalence (27). Just as in the item (1) of the proof of Theorem 3, we have (22), i.e. the problem splits in rows. We apply, to each row, one of the equivalences (24), (25), or (26), depending on the corresponding binary relation, and convolve the resulting system of inequalities using the operation abs^{σ}.

The proof of Theorem 4 is complete. □

From equivalences (24)–(26), one more proof of Corollary 2 becomes obvious. In addition, it is not difficult to identify the following connection between relationally homogeneous systems of inequalities of the opposite signs.

Corollary 3. *The following equivalences hold*:

$$Q(\boldsymbol{A},\boldsymbol{b},\mathcal{A},\beta)(Ax \geq b) \quad \Leftrightarrow \quad Q(-\boldsymbol{A},-\boldsymbol{b},\mathcal{A},\beta)(Ax \leq b), \tag{31}$$

$$Q(-\boldsymbol{A},-\boldsymbol{b},\mathcal{A},\beta)(Ax \geq b) \quad \Leftrightarrow \quad Q(\boldsymbol{A},\boldsymbol{b},\mathcal{A},\beta)\ (Ax \leq b). \tag{32}$$

Proof. Based on the properties of intervals

$$\mathrm{mid}(-\boldsymbol{u}) = -\check{\boldsymbol{u}} \quad \text{and} \quad \mathrm{rad}(-\boldsymbol{u}) = \hat{\boldsymbol{u}}, \tag{33}$$

we can show the validity of relation (32):

$$\begin{aligned} Q(-\boldsymbol{A},-\boldsymbol{b},\mathcal{A},\beta)\ (Ax \geq b) &\overset{(25)}{\Leftrightarrow} \mathrm{mid}(-\boldsymbol{b}) - \mathrm{mid}(-\boldsymbol{A})x \\ &\quad\ \leq (\mathcal{A}^s \circ\, \mathrm{rad}(-\boldsymbol{A}))|x| + \beta^s \circ\, \mathrm{rad}(-\boldsymbol{b}) \\ &\overset{(33)}{\Leftrightarrow} -\check{\boldsymbol{b}} + \check{\boldsymbol{A}}x\ \leq (\mathcal{A}^s \circ \hat{\boldsymbol{A}})|x| + \beta^s \circ\ \hat{\boldsymbol{b}} \\ &\overset{(26)}{\Leftrightarrow} Q(\boldsymbol{A},\boldsymbol{b},\mathcal{A},\beta)\ (Ax \leq b). \end{aligned}$$

Relation (31) is proved similarly. □

Corollary 3 means that, if the sign of the inequality and the signs of all intervals of the parameter values are reversed to the opposite, then the set of quantifier solutions to the interval system of linear inequalities does not change. For example, the solution sets to the systems $(\forall\, A \in \boldsymbol{A})\,(\exists\, b \in \boldsymbol{b})\,(Ax \geq b)$ and $(\forall\, A \in -\boldsymbol{A})\,(\exists\, b \in -\boldsymbol{b})\,(Ax \leq b)$ coincide.

3.3. *Analytical descriptions in* $\mathbb{KR}$*,* $\mathbb{IR}$*, and* $\mathbb{R}$ *for systems of basic types*

So far, when considering the interval-quantifier linear systems $Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)$ $(Ax\,\sigma\,b)$, we tried to obtain results in which there were no constraints on the parameters $\boldsymbol{A}$, $\boldsymbol{b}$, $\mathcal{A}$, β, and σ, while the restrictions on the order of the elementary quantifier prefixes in Q were minimal. In this sense, the most general descriptions have been found for the class Q^σ. In this section, we consider subsets of interval-quantifier linear systems of class Q^σ, which are distinguished by the requirement of homogeneity of the quantifier matrix $\mathcal{A}$ and the homogeneity of the quantifier vector β. Elements of all these subsets will be called systems of *basic types*, and their solutions will be referred to as *quantifier solutions of basic types* for interval linear systems of the form $\boldsymbol{A}x\,\sigma\,\boldsymbol{b}$.

Depending on which quantifiers fill the matrix $\mathcal{A}$ and the vector β, all the interval-quantifier linear systems of the basic types are divided into four subsets, or four types. This subdivision is presented in the last column of Table 1. For each of the main types of solutions to the systems, we give a proper name that continues the one used in [3, 33] for solutions of relationally homogeneous systems of this type. The names of the solutions are listed in the first column of Table 1, the values of the elements of the matrix $\mathcal{A}$ and components of the vector β are listed in the second and third columns, and the fourth column gives the general form for the interval-quantifier systems of the corresponding basic type.

Quantifier-free analytical descriptions in $\mathbb{KR}$, $\mathbb{IR}$, and $\mathbb{R}$ for systems of the basic types can be obtained as corollaries of the corresponding descriptions for systems of class Q^σ. Let us explain that for relationally homogeneous systems using Table 2.

In Table 2, columns 4–7, corresponding to the basic types of quantifier solutions, are obtained, in row-wise manner, from column 3 corresponding to quantifier solutions with the prefix Q^σ. It is necessary to use definition (9) of the matrix $\boldsymbol{A}^{\mathrm{c}}$ and vector $\boldsymbol{b}^{\mathrm{c}}$ in the rows corresponding to the Kaucher arithmetic. In the rows that correspond to the classical interval arithmetic, we have to use definition (8) of the matrices $\boldsymbol{A}^\forall$, $\boldsymbol{A}^\exists$ and the vectors $\boldsymbol{b}^\forall$, $\boldsymbol{b}^\exists$. Finally, the rows corresponding to real non-interval arithmetic, definition (28) of the matrix $\mathcal{A}^s$, vector β^s and the definition of the product "$\circ$" should be used.

Approximately half of the descriptions of the basic types of quantifier solutions for interval linear systems, presented in columns 4–7 of Table 2, have been obtained

Table 1. Basic types of quantifier solutions to the interval system $\boldsymbol{A}x\,\sigma\,\boldsymbol{b}$.

Name of solution	Values of elements for matrix $\mathcal{A}$	Values of elements for vector β	Interval-quantifier system of basic type
Weak	$\exists$	$\exists$	$(\exists\, A \in \boldsymbol{A})(\exists\, b \in \boldsymbol{b})\ (Ax\,\sigma\,b)$
Tolerable	$\forall$	$\exists$	$(\forall\, A \in \boldsymbol{A})(\exists\, b \in \boldsymbol{b})\ (Ax\,\sigma\,b)$
Controllable	$\exists$	$\forall$	$(\forall\, b \in \boldsymbol{b})(\exists\, A \in \boldsymbol{A})\ (Ax\,\sigma\,b)$
Strong	$\forall$	$\forall$	$(\forall\, A \in \boldsymbol{A})(\forall\, b \in \boldsymbol{b})\ (Ax\,\sigma\,b)$

Table 2. Characterization of relationally homogeneous interval-quantifier linear systems and their main solution types.

		Type of solution and corresponding quantifier prefix $Q(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)$				
			Basic types of solutions			
$Ax\,\sigma\,b$	Space of description	Quantifier $Q^{\sigma}(\boldsymbol{A}, \boldsymbol{b}, \mathcal{A}, \beta)$	Weak $(\exists\, A \in \boldsymbol{A})\ (\exists\, b \in \boldsymbol{b})$	Tolerable $(\forall\, A \in \boldsymbol{A})\ (\exists\, b \in \boldsymbol{b})$	Controllable $(\forall\, b \in \boldsymbol{b})\ (\exists\, A \in \boldsymbol{A})$	Strong $(\forall\, A \in \boldsymbol{A})\ (\forall\, b \in \boldsymbol{b})$
$Ax = b$	$\mathbb{KR}$	$\boldsymbol{A}^{\mathfrak{c}} x \subseteq \boldsymbol{b}^{\mathfrak{c}}$	$(\mathrm{dual}\,\boldsymbol{A})x \subseteq \boldsymbol{b}$	$\boldsymbol{A}x \subseteq \boldsymbol{b}$	$(\mathrm{dual}\,\boldsymbol{A})x \subseteq \mathrm{dual}\,\boldsymbol{b}$	$\boldsymbol{A}x \subseteq \mathrm{dual}\,\boldsymbol{b}$
	$\mathbb{IR}$	$\boldsymbol{A}^{\forall} x - \boldsymbol{b}^{\forall} \subseteq \boldsymbol{b}^{\exists} - \boldsymbol{A}^{\exists} x$	$0 \in \boldsymbol{b} - \boldsymbol{A}x$	$\boldsymbol{A}x \subseteq \boldsymbol{b}$	$\boldsymbol{b} \subseteq \boldsymbol{A}x$	$\boldsymbol{A}x - \boldsymbol{b} \subseteq 0$
	$\mathbb{R}$	$\lvert\check{\boldsymbol{A}}x - \check{\boldsymbol{b}}\rvert \le (\mathcal{A}^s \circ \hat{\boldsymbol{A}})\lvert x\rvert + \beta^s \circ \hat{\boldsymbol{b}}$	$\lvert\check{\boldsymbol{A}}x - \check{\boldsymbol{b}}\rvert \le \hat{\boldsymbol{A}}\lvert x\rvert + \hat{\boldsymbol{b}}$	$\lvert\check{\boldsymbol{A}}x - \check{\boldsymbol{b}}\rvert \le -\hat{\boldsymbol{A}}\lvert x\rvert + \hat{\boldsymbol{b}}$	$\lvert\check{\boldsymbol{A}}x - \check{\boldsymbol{b}}\rvert \le \hat{\boldsymbol{A}}\lvert x\rvert - \hat{\boldsymbol{b}}$	$\lvert\check{\boldsymbol{A}}x - \check{\boldsymbol{b}}\rvert \le -\hat{\boldsymbol{A}}\lvert x\rvert - \hat{\boldsymbol{b}}$
$Ax \ge b$	$\mathbb{KR}$	$\underline{\boldsymbol{A}^{\mathfrak{c}} x} \ge \underline{\boldsymbol{b}^{\mathfrak{c}}}$	$\underline{(\mathrm{dual}\,\boldsymbol{A})x} \ge \underline{\boldsymbol{b}}$	$\underline{\boldsymbol{A}x} \ge \underline{\boldsymbol{b}}$	$\underline{(\mathrm{dual}\,\boldsymbol{A})x} \ge \overline{\boldsymbol{b}}$	$\underline{\boldsymbol{A}x} \ge \overline{\boldsymbol{b}}$
	$\mathbb{IR}$	$\overline{\boldsymbol{A}^{\exists} x} + \underline{\boldsymbol{A}^{\forall} x} \ge \underline{\boldsymbol{b}}^{\exists} + \overline{\boldsymbol{b}}^{\forall}$	$\overline{\boldsymbol{A}x} \ge \underline{\boldsymbol{b}}$	$\underline{\boldsymbol{A}x} \ge \underline{\boldsymbol{b}}$	$\overline{\boldsymbol{A}x} \ge \overline{\boldsymbol{b}}$	$\underline{\boldsymbol{A}x} \ge \overline{\boldsymbol{b}}$
	$\mathbb{R}$	$\check{\boldsymbol{b}} - \check{\boldsymbol{A}}x \le (\mathcal{A}^s \circ \hat{\boldsymbol{A}})\lvert x\rvert + \beta^s \circ \hat{\boldsymbol{b}}$	$\check{\boldsymbol{b}} - \check{\boldsymbol{A}}x \le \hat{\boldsymbol{A}}\lvert x\rvert + \hat{\boldsymbol{b}}$	$\check{\boldsymbol{b}} - \check{\boldsymbol{A}}x \le -\hat{\boldsymbol{A}}\lvert x\rvert + \hat{\boldsymbol{b}}$	$\check{\boldsymbol{b}} - \check{\boldsymbol{A}}x \le \hat{\boldsymbol{A}}\lvert x\rvert - \hat{\boldsymbol{b}}$	$\check{\boldsymbol{b}} - \check{\boldsymbol{A}}x \le -\hat{\boldsymbol{A}}\lvert x\rvert - \hat{\boldsymbol{b}}$
$Ax \le b$	$\mathbb{KR}$	$\overline{\boldsymbol{A}^{\mathfrak{c}} x} \le \overline{\boldsymbol{b}^{\mathfrak{c}}}$	$\overline{(\mathrm{dual}\,\boldsymbol{A})x} \le \overline{\boldsymbol{b}}$	$\overline{\boldsymbol{A}x} \le \overline{\boldsymbol{b}}$	$\overline{(\mathrm{dual}\,\boldsymbol{A})x} \le \underline{\boldsymbol{b}}$	$\overline{\boldsymbol{A}x} \le \underline{\boldsymbol{b}}$
	$\mathbb{IR}$	$\underline{\boldsymbol{A}^{\exists} x} + \overline{\boldsymbol{A}^{\forall} x} \le \overline{\boldsymbol{b}}^{\exists} + \underline{\boldsymbol{b}}^{\forall}$	$\underline{\boldsymbol{A}x} \le \overline{\boldsymbol{b}}$	$\overline{\boldsymbol{A}x} \le \overline{\boldsymbol{b}}$	$\underline{\boldsymbol{A}x} \le \underline{\boldsymbol{b}}$	$\overline{\boldsymbol{A}x} \le \underline{\boldsymbol{b}}$
	$\mathbb{R}$	$\check{\boldsymbol{A}}x - \check{\boldsymbol{b}} \le (\mathcal{A}^s \circ \hat{\boldsymbol{A}})\lvert x\rvert + \beta^s \circ \hat{\boldsymbol{b}}$	$\check{\boldsymbol{A}}x - \check{\boldsymbol{b}} \le \hat{\boldsymbol{A}}\lvert x\rvert + \hat{\boldsymbol{b}}$	$\check{\boldsymbol{A}}x - \check{\boldsymbol{b}} \le -\hat{\boldsymbol{A}}\lvert x\rvert + \hat{\boldsymbol{b}}$	$\check{\boldsymbol{A}}x - \check{\boldsymbol{b}} \le \hat{\boldsymbol{A}}\lvert x\rvert - \hat{\boldsymbol{b}}$	$\check{\boldsymbol{A}}x - \check{\boldsymbol{b}} \le -\hat{\boldsymbol{A}}\lvert x\rvert - \hat{\boldsymbol{b}}$

earlier. The descriptions that were found first, obtained their own proper names. These are

> the Oettli–Prager characterization in $\mathbb{R}$ [19]and the Beeck characterization in $\mathbb{IR}$ [1] for weak solutions of the equation $\boldsymbol{A}x = \boldsymbol{b}$;
> the Gerlach description in $\mathbb{R}$ for weak solutions of the inequality $\boldsymbol{A}x \le \boldsymbol{b}$ [5].

The analytical descriptions of the set of tolerable solutions to the equation $\boldsymbol{A}x = \boldsymbol{b}$ was obtained in $\mathbb{R}$ by Rohn [20] and in $\mathbb{IR}$ by Neumaier [17]. The description in $\mathbb{R}$ was further investigated by Lakeyev and Noskov in [11], and they also presented, as an evident one, a description in $\mathbb{IR}$ for the set of controllable solutions to the equation $\boldsymbol{A}x = \boldsymbol{b}$ (see also [27]). The remaining descriptions for the basic types of quantifier solutions to the equation $\boldsymbol{A}x = \boldsymbol{b}$ in the interval arithmetics $\mathbb{IR}$ and $\mathbb{KR}$ are also known, for example, as obvious corollaries of statement (12), proved by Shary in [29, 30]. In [3, Theorem 2.25], an analytical description in $\mathbb{R}$ for strong solutions to the interval inequality $\boldsymbol{A}x \le \boldsymbol{b}$ was presented. Finally, in [32], alternative descriptions were given for the set of weak solutions (united solution set) for interval linear systems of equations in the field of real numbers $\mathbb{R}$

For interval-quantifier systems of basic types in which the relationship vector σ is not homogeneous, analytical descriptions in $\mathbb{K}\overline{\mathbb{R}}$ can be obtained from (23) and (9). The descriptions in $\mathbb{I}\overline{\mathbb{R}}$ can be derived from (23) and (8), and the descriptions in $\mathbb{R}$ follows from (27) and (28). Below, we give these descriptions only in $\mathbb{I}\overline{\mathbb{R}}$ and $\mathbb{R}$ (in $\mathbb{K}\overline{\mathbb{R}}$, they are less expressive and differ from the descriptions in $\mathbb{I}\overline{\mathbb{R}}$ by obvious arithmetic transformations, in the same way as the descriptions in $\mathbb{KR}$ and $\mathbb{IR}$ differ from each other in Table 2):

$$
\begin{aligned}
&(\exists\, A \in \boldsymbol{A})(\exists\, b \in \boldsymbol{b})(Ax\,\sigma\, b) \quad\Leftrightarrow\quad 0 \in \boldsymbol{b} - \boldsymbol{A}x + \boldsymbol{w} \quad\Leftrightarrow\quad \operatorname{abs}^{\sigma}(\check{\boldsymbol{A}}x - \check{\boldsymbol{b}}) \le \hat{\boldsymbol{A}}|x| + \hat{\boldsymbol{b}};\\
&(\forall\, A \in \boldsymbol{A})(\exists\, b \in \boldsymbol{b})(Ax\,\sigma\, b) \quad\Leftrightarrow\quad \boldsymbol{A}x \subseteq \boldsymbol{b} + \boldsymbol{w} \quad\Leftrightarrow\quad \operatorname{abs}^{\sigma}(\check{\boldsymbol{A}}x - \check{\boldsymbol{b}}) \le -\hat{\boldsymbol{A}}|x| + \hat{\boldsymbol{b}};\\
&(\forall\, b \in \boldsymbol{b})(\exists\, A \in \boldsymbol{A})(Ax\,\sigma\, b) \quad\Leftrightarrow\quad \boldsymbol{b} \subseteq \boldsymbol{A}x + \boldsymbol{w} \quad\Leftrightarrow\quad \operatorname{abs}^{\sigma}(\check{\boldsymbol{A}}x - \check{\boldsymbol{b}}) \le \hat{\boldsymbol{A}}|x| - \hat{\boldsymbol{b}};\\
&(\forall\, A \in \boldsymbol{A})(\forall\, b \in \boldsymbol{b})(Ax\,\sigma\, b) \quad\Leftrightarrow\quad \boldsymbol{A}x - \boldsymbol{b} \subseteq \boldsymbol{w} \quad\Leftrightarrow\quad \operatorname{abs}^{\sigma}(\check{\boldsymbol{A}}x - \check{\boldsymbol{b}}) \le -\hat{\boldsymbol{A}}|x| - \hat{\boldsymbol{b}}.
\end{aligned}
$$

4. Conclusion

The main results of the paper are presented in Theorems 2–4 (equivalence (24) was previously known) and in Corollaries 1–2.

Among the statements that have no restrictions on the parameters $\boldsymbol{A}$, $\boldsymbol{b}$, $\mathcal{A}$, β, and σ, those that give analytical descriptions of the solutions to the interval-quantifier linear systems of class Q^{σ} have the greatest generality. These are relation (21), which enables transition to $\mathbb{KR}$ and $\mathbb{IR}$, relation (23) for transition to $\mathbb{K}\overline{\mathbb{R}}$ and $\mathbb{I}\overline{\mathbb{R}}$, and equivalence (27) that allows us to go into $\mathbb{R}$.

The usefulness of the quantifier-free analytical descriptions from (21), (23), and (27) is that they give us the opportunity

- to study all interval-quantifier linear systems of class Q^σ simultaneously and in a uniform way, and to derive results for their subclasses (in particular, for interval-quantifier systems of basic types) as consequences of the general result;
- to design such solution methods for problems related to interval-quantifier linear systems that are suitable for all systems of class Q^σ (an example is the author's software packages `IntLinInc2D` and `IntLinInc3D` for visualization of quantifier solution sets to interval linear systems, freely available at http://www.nsc.ru/interval/sharaya/).

Analytical descriptions, in interval arithmetic, for various classes of interval-quantifier linear systems and for their solutions, both previously known (for example, relation (12)) and those obtained in this work in the form of relations (13)–(15), (21), (23), allow us

- to investigate interval-quantifier linear systems by interval methods, i.e. to study the properties of their solution sets, the relationships between systems with various conditions on the parameters $\boldsymbol{A}$, $\boldsymbol{b}$, $\mathcal{A}$, β, σ and the order of the quantifier prefixes (an example is the proof of Corollary 2);
- to construct interval numerical methods (that is, essentially using interval arithmetic) for the solution of problems in which the formulation involves interval-quantifier linear systems (examples of such methods for systems of equations can be found in [31, 33], while for inequalities and systems of class Q^σ constructing such methods is a matter of the future research).

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