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On Unbounded Tolerable Solution Sets

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Abstract. We prove that the nonempty tolerable solution set of the interval linear system $\boldsymbol{A}x = \boldsymbol{b}$ is unbounded if and only if the interval matrix of the system has linearly dependent degenerate columns. Also, we prove that the tolerable solution set may be represented as a sum of the linear subspace $\{c \in \mathbb{R}^n \mid \boldsymbol{A}c = 0\}$ and a bounded convex polyhedron and propose a way for suitable estimation of unbounded tolerable solution sets.

1. Introduction

We use definitions and properties of interval arithmetic (see, for example, [2]) and notation from [1].

For an interval linear system $\boldsymbol{A}x = \boldsymbol{b}$ with an interval matrix $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$, an interval vector $\boldsymbol{b} \in \mathbb{IR}^m$ and a real vector $x \in \mathbb{R}^n$ (m, n are positive integer numbers), the set

$$\Xi = \{x \mid (\forall A \in \boldsymbol{A})\,(\exists b \in \boldsymbol{b})\,(Ax = b)\} \tag{1.1}$$

is called *tolerable solution set* [7].

The definition (1.1) is equivalent to $\Xi = \{x \mid (\forall A \in \boldsymbol{A})\,(Ax \in \boldsymbol{b})\}$. Since $\{Ax \mid A \in \boldsymbol{A}\} = \boldsymbol{A}x$, the following *characterization of the tolerable solutions* is valid:

$$x \in \Xi \quad \Longleftrightarrow \quad \boldsymbol{A}x \subseteq \boldsymbol{b}. \tag{1.2}$$

Many geometrical properties of the tolerable solution set may be easily proved with the use of this characterization. For example:

1. Intersection of Ξ with each orthant of the space $\mathbb{R}^n$ is intersection of no more than $2m + n$ closed half-spaces. (Each orthant is intersection of n half-spaces. In each orthant, the inclusion $\boldsymbol{A}x \subseteq \boldsymbol{b}$ means a system of $2m$ nonstrict linear inequalities on x. The solution set of every nonstrict linear inequality is a half-space or empty set or the whole space $\mathbb{R}^n$. The empty set may be represented as intersection of two half-spaces.)
2. Ξ is convex. (Let $\boldsymbol{A}x, \boldsymbol{A}y \subseteq \boldsymbol{b}$ and $\lambda \in [0, 1]$. Applying subdistributivity, special cases of distributivity and monotonicity of interval operations we obtain $\boldsymbol{A}(\lambda x{+}(1{-}\lambda)y) \subseteq \boldsymbol{A}(\lambda x){+}\boldsymbol{A}((1{-}\lambda)y) = \lambda\boldsymbol{A}x{+}(1{-}\lambda)\boldsymbol{A}y \subseteq \lambda\boldsymbol{b}{+}(1{-}\lambda)\boldsymbol{b} = \boldsymbol{b}$.)

3. The tolerable solution set is an intersection of finite number of closed half-spaces or (for $\boldsymbol{A} = 0$, $\boldsymbol{b} = 0$) the whole space $\mathbb{R}^n$, i.e. a convex polyhedron (see Section 4 for definition). (This fact follows from the two previous properties.)

The characterization of the tolerable solutions and the fact that Ξ is convex polyhedron are well known, they have been proved in different ways by several authors (for example, [5], [6], [8]). Above, we repeated proofs from [6], because this publication is in Russian and difficult to get .

The tolerable solution set arises in control problems. In practice, its interval (inner or outer) estimates are usually used. For example, solution of popular interval linear tolerance problem is an inner interval estimate of the tolerable solution set. Naturally, the least inclusive (also called *optimal*) interval estimate is preferable among outer estimates, while a maximal one is preferable among inner estimates. For the nonempty bounded tolerable solution set, the optimal outer and a maximal inner estimates exist (Figure 1a) and may be found. Methods for the optimal outer interval estimation are described in [5], [7], for the maximal inner interval estimation—in [3], [7], [8]. Since we consider only classic (bounded) intervals, for unbounded convex polyhedron

1) outer estimate does not exist (Figure 1b, c, d),
2) each inner interval estimate is significantly smaller than the estimated set (Figure 1b, c, d),
3) sometimes maximal inner estimate does not exist (Figure 1b, c).

This paper is devoted to unbounded tolerable solution sets, and the following questions are considered here:

- When is the tolerable solution set unbounded?
- What structure does it have?
- How to estimate the unbounded tolerable solution set properly?

The paper is organized in the following way. Section 2 is auxiliary. In Section 3, we prove that the nonempty tolerable solution set is unbounded if and only if the matrix $\boldsymbol{A}$ has linearly dependent degenerate columns. In Section 4, we prove that the tolerable solution set is the sum of a linear subspace and a convex polytope (i.e. bounded convex polyhedron), and propose a way to estimate the unbounded tolerable solution set properly.

2. Interval Matrices with Linearly Dependent Degenerate Columns

We say that an interval matrix $\boldsymbol{A}$ *has a degenerate column* $A_{:j}$ if lower and upper bounds of the interval column $\boldsymbol{A}_{:j}$ coincide. Let us denote by J the set of all indexes j of the degenerate columns of $\boldsymbol{A}$. In linear algebra, a set of real vectors $\{A_{:j} \mid j \in J\}$ is called to be *linearly dependent* if such numbers $c_j \in \mathbb{R}$ exist that $\sum_{j \in J} A_{:j} c_j = 0$ and $\sum_{j \in J} |c_j| > 0$.

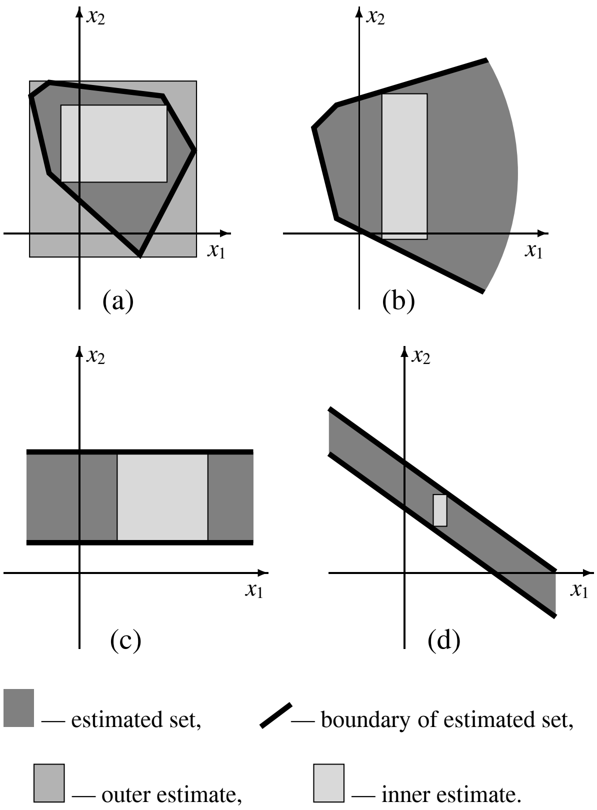

Figure 1. Interval estimation of convex polyhedrons.

LEMMA 2.1. *For $c \in \mathbb{R}^n$, the following equivalence takes place*

$$\boldsymbol{A}c = 0 \quad \Longleftrightarrow \quad \begin{cases} \sum_{j \in J} A_{:j} c_j = 0, \\ c_k = 0 \quad \text{for } \ k \notin J. \end{cases} \tag{2.1}$$

Proof.
$\Leftarrow$) is obvious.
$\Rightarrow$) The equation $\boldsymbol{A}c = 0$ yields $\operatorname{rad}(\boldsymbol{A}c) = 0$. As is known,

$$\operatorname{rad}(\boldsymbol{A}c) = \sum_{l=1}^{n} |c_l| \operatorname{rad} \boldsymbol{A}_{:l}.$$

In this sum, all terms are nonnegative. For $k \notin J$, $\operatorname{rad} \boldsymbol{A}_{:k} \neq 0$, therefore $c_k = 0$. Elimination of zero terms $\boldsymbol{A}_{:k} c_k$, $k \notin J$, from $\boldsymbol{A}c = 0$ gives $\sum_{j \in J} A_{:j} c_j = 0$. □

LEMMA 2.2. *An interval matrix $\boldsymbol{A}$ has linearly dependent degenerate columns if and only if such nonzero vector $c \in \mathbb{R}^n$ exists, that $\boldsymbol{A}c = 0$.*

Proof follows from the definition of linearly dependent vectors and Lemma 2.1. □

3. Unboundedness Criterion for the Tolerable Solution Set

LEMMA 3.1. *Let the tolerable solution set Ξ be nonempty. If the interval matrix $\boldsymbol{A}$ has linearly dependent degenerate columns, then Ξ is unbounded.*

Proof. According to the definition of linearly dependent vectors, there exist such $c_j \in \mathbb{R}$, $j \in J$, that $\sum_{j \in J} |c_j| > 0$ and

$$\sum_{j \in J} A_{:j} c_j = 0. \tag{3.1}$$

If we multiply (3.1) by any real number t and add the zero term $\sum_{\substack{k=1 \\ k \notin J}}^{n} \boldsymbol{A}_{:k}(t \cdot 0)$ to the left-hand side, then the equality is not violated, and therefore

$$\forall\, t \in \mathbb{R} \qquad \sum_{j \in J} A_{:j}(tc_j) + \sum_{\substack{k=1 \\ k \notin J}}^{n} \boldsymbol{A}_{:k}(t \cdot 0) = 0. \tag{3.2}$$

Let $\tilde{x}$ be an element of Ξ. By the characterization (1.2), we have

$$\sum_{j \in J} A_{:j} \tilde{x}_j + \sum_{\substack{k=1 \\ k \notin J}}^{n} \boldsymbol{A}_{:k} \tilde{x}_k \subseteq \boldsymbol{b}. \tag{3.3}$$

Adding (3.2) to (3.3) and applying particular cases of distributivity yield

$$\forall\, t \in \mathbb{R} \qquad \sum_{j \in J} A_{:j}(\tilde{x}_j + tc_j) + \sum_{\substack{k=1 \\ k \notin J}}^{n} \boldsymbol{A}_{:k}(\tilde{x}_k + t \cdot 0) \;\subseteq\; \boldsymbol{b}. \tag{3.4}$$

If we introduce a vector $c \in \mathbb{R}^n$ with the components c_j for $j \in J$ and $c_k = 0$ for $k \notin J$, then (3.4) may be rewritten in the form

$$\forall\, t \in \mathbb{R} \qquad \boldsymbol{A}(\tilde{x} + tc) \subseteq \boldsymbol{b}$$

or, briefly,

$$\boldsymbol{A}(\tilde{x} + \mathbb{R}c) \in \boldsymbol{b}. \tag{3.5}$$

In conformity with the characterization (1.2), the inclusion (3.5) means that the line $(\tilde{x} + \mathbb{R}c)$ is contained in Ξ. Hence, the tolerable solution set Ξ is unbounded. □

LEMMA 3.2. *If the tolerable solution set Ξ is unbounded, then the matrix $\boldsymbol{A}$ has linearly dependent degenerate columns.*

Proof. In the introduction, we showed that intersection of Ξ with each orthant is a convex polyhedron. As far as Ξ is unbounded, there exists such an orthant Ort that intersection $\Xi \bigcap \text{Ort}$ is unbounded. Therefore we can take a ray $\tilde{x}+tc$, $t \in \mathbb{R}^+$ (where $\tilde{x}$ is a starting point, c is a nonzero direction vector, t is the location parameter), from the convex polyhedron $\Xi \bigcap \text{Ort}$.

The ray $\tilde{x} + tc$, $t \in \mathbb{R}^+$, lies in Ξ, so by the characterization (1.2)

$$\forall\, t \in \mathbb{R}^+ \qquad \boldsymbol{A}(\tilde{x} + tc) \subseteq \boldsymbol{b}. \tag{3.6}$$

The ray $\tilde{x} + tc$, $t \in \mathbb{R}^+$, lies in Ort, so $(\forall j \in \{1, \ldots, n\})$ $(x_j c_j \geq 0)$, and in (3.6) we can open the parenthesis as in the distributive law:

$$\forall\, t \in \mathbb{R}^+ \qquad \boldsymbol{A}\tilde{x} + \boldsymbol{A}(tc) \subseteq \boldsymbol{b}.$$

Factorization of the real number t yields

$$\forall\, t \in \mathbb{R}^+ \qquad \boldsymbol{A}\tilde{x} + t(\boldsymbol{A}c) \subseteq \boldsymbol{b}.$$

Since t may be arbitrary large, it follows that

$$\boldsymbol{A}c \;=\; 0. \tag{3.7}$$

By Lemma 2.2, (3.7) means that the interval matrix $\boldsymbol{A}$ has linearly dependent degenerate columns. □

The following theorem is *unboundedness criterion* for the tolerable solution set.

THEOREM 3.1. *Let a tolerable solution set Ξ be nonempty. It is unbounded if and only if the interval matrix $\boldsymbol{A}$ has linearly dependent degenerate columns.*

Proof follows from Lemmas 3.1 and 3.2. □

How to apply the unboundedness criterion? Sometimes there is no need even to calculate, for example:

the interval matrix $\boldsymbol{A}$	the tolerable solution set Ξ
All elements are nondegenerate intervals.	bounded
A nondegenerate element exists in each column.[a]	bounded
It is evident that square matrix $\boldsymbol{A}$ is non-singular.	bounded
It is evident that $\boldsymbol{A}$ is real (degenerate) matrix with zero null space.	bounded
Zero column exists.	unbounded or empty
Proportional degenerate columns exist.	unbounded or empty
Number of degenerate columns is greater than m.	unbounded or empty

[a] This sufficient condition of boundedness is proved by another way in [5].

In general case, we should investigate the real matrix $A^J \in \mathbb{R}^{|J|\times m}$ that consists of the degenerate columns $A_{:j}$ from $\boldsymbol{A}$, by methods of linear algebra, on whether its null space has nonzero vectors or not. If null space of A^J contains only zero vector, then the corresponding tolerable solution set is bounded. Otherwise this set is unbounded or empty.

4. Structure of the Tolerable Solution Set

Let us agree about definitions and remind some facts from convex analysis. Intersection of $\mathbb{R}^n$ and finite number of closed half-spaces is called *convex polyhedron*. If number of half-spaces is zero, then convex polyhedron coincides with the whole space $\mathbb{R}^n$. Convex hull of a finite number of points is called *convex polytope*. If the number of points is zero, the convex polytope is empty. A convex polytope is a bounded convex polyhedron. A vector c is called *direction* of a convex set, if some ray with the direction c lies in this set. All the directions of a convex polyhedron form a convex cone. Each convex polyhedron may be represented as a sum of the convex cone of its directions and a convex polytope.

Now we return to our problems.

LEMMA 4.1. *Let a tolerable solution set Ξ be nonempty. For each nonzero vector $c \in \mathbb{R}^n$, the following statements (i)–(iii) are equivalent:*

(i) $(\forall x \in \Xi)\ (x + \mathbb{R}c \subseteq \Xi)$ *(for all points x from Ξ, the line passing through x and parallel to c lies in Ξ);*

(ii) $(\exists y \in \Xi)\ (y + \mathbb{R}^+c \subseteq \Xi)$ *(there exists a ray that has direction c and lies in Ξ);*

(iii) $\boldsymbol{A}c = 0$.

Proof.

(*i*) $\Rightarrow$ (*ii*) is obvious for nonempty Ξ.

(*ii*) $\Rightarrow$ (*iii*). Let Ort be the orthant containing c. Intersection of the ray $y + \mathbb{R}^+c$ with Ort is a ray that the reasoning from Lemma 3.2 may be applied for.

(*iii*) $\Rightarrow$ (*i*). By Lemma 2.1, $c_k = 0$ for $k \notin J$, so, in the formula $\boldsymbol{A}(x + \mathbb{R}c)$, we can open parenthesis according to the distributive law. We have $\boldsymbol{A}(x + \mathbb{R}c) = \boldsymbol{A}x + \boldsymbol{A}(\mathbb{R}c) = \boldsymbol{A}x + \mathbb{R}(\boldsymbol{A}c) = \boldsymbol{A}x$. The equality $\boldsymbol{A}(x + \mathbb{R}c) = \boldsymbol{A}x$ just obtained and the characterization (1.2) yield the statement (*i*).

THEOREM 4.1. *The tolerable solution set may be represented as a sum $\Xi = L + V$ of the subspace $L = \{c \in \mathbb{R}^n \mid \boldsymbol{A}c = 0\}$ and a convex polytope V.*

Proof. In the introduction, we proved that the tolerable solution set is a convex polyhedron. As is known from convex analysis, such a set may be represented as a sum of the convex cone of its directions and a convex polytope. The equivalence of the statements (*i*) and (*ii*) from Lemma 4.1 implies that the convex cone of directions is a linear subspace. Let us denote this subspace by L and the mentioned polytope by V. The equivalence of the statements (*ii*) and (*iii*) from Lemma 4.1 gives $L = \{c \in \mathbb{R}^n \mid \boldsymbol{A}c = 0\}$. □

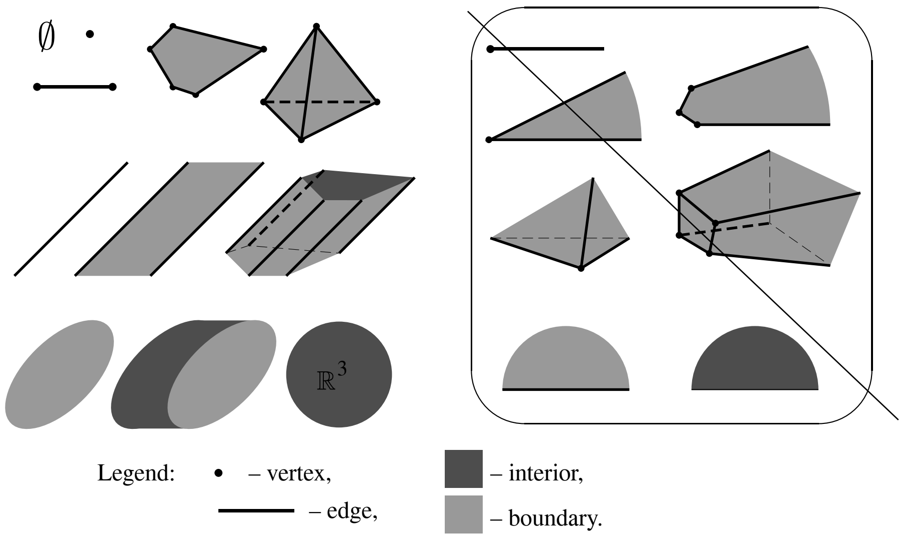

Figure 2. Examples of convex polyhedrons in $\mathbb{R}^3$. Framed sets cannot be represented as a sum of a linear subspace and a convex polytope. By Theorem 4.1, the tolerable solution set cannot have shapes as the sets within the frame.

On the one hand, Theorem 4.1 describes the shape of the tolerable solution set more exactly than the fact, that this set is a convex polyhedron (see Figure 2).

On the other hand, Theorem 4.1 gives a way for suitable estimation of the unbounded tolerable solution set. For such sets, a proper estimate may be found as a sum of the subspace L with an estimate of the convex polytope V.

By Lemma 2.1,

$$L = \left\{ c \in \mathbb{R}^n \mid \sum_{j \in J} A_{:j} c_j = 0,\ c_k = 0,\ k \notin J \right\},$$

therefore L may be found by methods of linear algebra. (The subspace L lies in the subspace $\mathbb{R}^J$ of variables x_j, $j \in J$, and coincides in $\mathbb{R}^J$ with the kernel of the real matrix $A^J \in \mathbb{R}^{|J| \times m}$, which consists of degenerate columns from $\boldsymbol{A}$.)

Let us denote by L' a subspace that is complementary to L in $\mathbb{R}^n$ and by V' the projection of the polytope V onto L' parallel to L. It is obvious that

1) $\Xi = L + V'$, i.e. in the statement of Theorem 4.1 the polytope may be chosen from the subspace L';
2) $V' = \Xi \cap L'$, i.e. the sought-for polytope V' in the subspace L' is unique and its description may be obtained from (1.1) by substitution of unknowns corresponding to L'.

Since V' is bounded and described as a tolerable solution set, its estimate P may be found by methods designed for the bounded tolerable solution set which we mentioned in the introduction. The proper estimate of Ξ may have the form $L + P$.

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