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Linear Static Systems Under Interval Uncertainty: Algorithms To Solve Control And Stabilization Problems

Sergey P. Shary

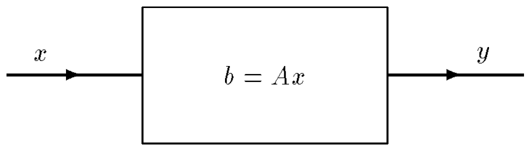
***Abstract*— This paper analyzes mathematical and computational aspects linear static systems under interval uncertainty.**

By an interval uncertainty, we mean that only intervals of possible values are known for input data and for the system's parameters. In other words, for each of this parameters p, we know an estimate $\tilde{p}$, and we know the upper bound for possible error $\tilde{p} - p$ of this estimate.

I. Description of a Linear (Static) System

A. Schematic Description

Let us denote the input and the output of the linear system by by vectors $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ respectively. Usually, the following structural scheme is used as a model for linear static systems:



(1)

How is this system described in mathematical terms?

B. Mathematical Description: Idealized Case

In the ideal case, when we can neglect the measurement inaccuracies, we can assume that we know (or can determine) precisely:

- all n components x_j of the input x;
- all m components b_i of the output b;
- all the components a_{ij} of the $m \times n$ matrix A.

In this case, relation between input and output is describe by a linear equation

$$b = Ax \tag{2}$$

C. Mathematical Description: A More Realistic Case (Of Interval Uncertainty)

The assumption that we know the parameters precisely is an idealization. In reality, we can only determine the *intervals* that contain the desired values:

- for each of n components of the input x, we know the *interval* $\mathbf{x}_j$ that contains the (unknown) actual value x_j;
- all for each of m components of the output b, we know the interval $\mathbf{b}_i$ that contains the (unknown) actual value b_i;
- for each of $m \times n$ components of the matrix A, we know the *interval* $\mathbf{a}_{ij}$ that contains the (unknown) actual value of a_{ij}.

In this case, the relation between input and output is describe by an interval analog of the system (2):

$$\mathbf{A}x = \mathbf{b}, \tag{3}$$

where $\mathbf{A} = (\,\mathbf{a}_{ij})$ and $\mathbf{b} = (\mathbf{b}_i)$.

II. Problems

For every linear system, usually, two kinds of problems are solved:

- *direct problems*, when we know the input, and we want to predict the output;
- *inverse problems*, when we know the output, and we want to estimate the input.

Both in the idealized case, and in the case of interval uncertainty, direct problems are easy to solve. Therefore, in this paper, we will analyze inverse problems.

In ideal (non-interval) case, an inverse problem consists of solving a system (2) with known A and b. In case of interval uncertainty, we also have to somehow "solve" the interval linear algebraic system (ILAS) (3). In this case, however, it is not automatically clear what a "solution" means. Actually, as we will see, several solutions make sense here.

III. Possible Interpretations Of Interval Uncertainty, and Corresponding Formulations Of The Inverse Problem For Linear Systems

One and the same interval can describe two different types of uncertainty. Let us illustrate these two types of uncertainty on a simplified example, in

The author is with Computer Center, Krasnoyarsk, Russia, email shary@cckr.krasnoyarsk.su.

which only one coefficient a_{11} of the matrix A is not known precisely. Then, the relation (3) can means two different things:

- First, it can mean that we have one one linear system with fixed (but unknown) parameters. For this system, we do not know the exact value of a_{11}, but we know that it belongs to the interval $\mathbf{a}_{11}$. For this particular system, and for the input x, the output belongs to the vector interval $\mathbf{b}$. In mathematical terms, this means that for *some* $A \in \mathbf{A}$, we have $Ax \in \mathbf{b}$. Another situation that corresponds to this interpretation is when the changes in the value of the parameter are possible, but we have no control over these changes. In this case, still, we can only guarantee that there is a value of this parameter for which the desired inclusion $Ax \in \mathbf{b}$ holds.
- Second, it can mean that we have several similar systems; the values of a_{11} for these systems are different, but they all lie in the interval $\mathbf{a}_{11}$, and, potentially, all numbers from this interval can be represented as the values of a_{11} for some system from this class. In this case, a natural interpretation of the relation (3) is that for *all* possible values of $A \in \mathbf{A}$, we have $Ax \in \mathbf{b}$. Another situation when a similar interpretation is possible is when we have one system, but we can *control* the values of this parameter.

In the second case, when the condition describes what happens $\forall A \in \mathbf{A}$, the equation (3) contains more information about the system. Therefore, this prevailing type of information is called *uncertainty of the I-st type*. Correspondingly, the case when the condition is $\exists A \in \mathbf{A}$ is called *uncertainty of the II-nd type*.

Different components of the coefficients matrix A and of the output vector b can correspond to different types of uncertainty. So, in order to describe the most general case, we must do the following:

- Let the entire set of the index pairs (i, j) (that describe components a_{ij} of the matrix A) be divided into two non-intersecting parts:
  - $\Omega' = \{\omega'_1, \dots, \omega'_p\}$ and
  - $\Omega'' = \{\omega''_1, \dots, \omega''_r\}$.

  Here, $p+r = m \cdot n$. These sets have the following interpretation:
  - If $(i, j) \in \Omega'$, then the parameter a_{ij} is of the first type of uncertainty (uncontrolled *perturbation*).
  - If $(i, j) \in \Omega''$, the the parameter a_{ij} is of the second type of uncertainty (*control*).
- Similarly, we introduce two non-intersecting sets of integer indices:
  - $\Theta' = \{\vartheta'_1, \dots, \vartheta'_s\}$ and
  - $\Theta'' = \{\vartheta''_1, \dots, \vartheta''_t\}$,

  such that $\Theta' \cup \Theta'' = \{1, 2, \dots, m\}$. These sets have the following interpretation:
  - If $i \in \Theta'$, then the parameter b_i is of the first type of uncertainty.
  - If $i \in \Theta''$, the the parameter b_{ij} is of the second type of uncertainty.

Some of the sets Ω', Ω'', Θ', Θ'' may be empty. The above-mentioned interpretation enables us to formulate what we mean by a solution of the ILAS.

Definition. *We define the set of $\mathfrak{Q}\mathfrak{q}$-solutions to the interval linear system (3) as follows:*

$$\Sigma_{\mathfrak{Q}\mathfrak{q}}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid$$
$$(\forall a_{\omega'_1} \in \mathbf{a}_{\omega'_1}) \dots (\forall a_{\omega'_p} \in \mathbf{a}_{\omega'_p})$$
$$(\forall b_{\vartheta'_1} \in \mathbf{b}_{\vartheta'_1}) \dots (\forall b_{\vartheta'_s} \in \mathbf{b}_{\vartheta'_s})$$
$$(\exists a_{\omega''_1} \in \mathbf{a}_{\omega''_1}) \dots (\exists a_{\omega''_r} \in \mathbf{a}_{\omega''_r})$$
$$(\exists b_{\vartheta''_1} \in \mathbf{b}_{\vartheta''_1}) \dots (\exists b_{\vartheta''_t} \in \mathbf{b}_{\vartheta''_t})(Ax = b)\} \qquad (4)$$

where the quantifier $m \times n$-matrix $\mathfrak{Q} = (\mathfrak{Q}_{ij})$ and m-vector $\mathfrak{q} = (\mathfrak{q}_i)$ are such that

$$\mathfrak{Q}_{ij} = \begin{cases} \forall, & \text{if } (i,j) \in \Omega', \\ \exists, & \text{if } (i,j) \in \Omega'', \end{cases} \qquad \mathfrak{q}_i = \begin{cases} \forall, & \text{if } i \in \Theta', \\ \exists, & \text{if } i \in \Theta''. \end{cases}$$

We will also call the set (4) the solution set of the type $\mathfrak{Q}\mathfrak{q}$.

This general definition contains all known solutions of interval analysis as particular cases:

- *the united solution set*

  $$\Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b}) =$$
  $$\{x \in \mathbb{R}^n \mid (\exists A \in \mathbf{A})(\exists b \in \mathbf{b})(Ax = b)\},$$

  formed by the solutions of all systems $Ax = b$ with $A \in \mathbf{A}$ and $b \in \mathbf{b}$ (historically first and, undoubtedly, the most popular of the solution sets to ILAS),
- *the tolerable solution set*

  $$\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) =$$
  $$\{x \in \mathbb{R}^n \mid (\forall A \in \mathbf{A})(\exists b \in \mathbf{b})(Ax = b)\},$$

  formed by all vectors $x \in \mathbb{R}^n$, such that the product Ax falls into $\mathbf{b}$ for any $A \in \mathbf{A}$,

- *the controlled solution set*

$$\Sigma_{\exists\forall}(\mathbf{A},\mathbf{b}) =$$

$$\{x \in \mathbb{R}^n \mid (\forall\, b \in \mathbf{b})(\exists\, A \in \mathbf{A})(Ax = b)\},$$

  formed by vectors $x \in \mathbb{R}^n$ such that for any desired $b \in \mathbf{b}$ we can find an appropriate $A \in \mathbf{A}$ satisfying $Ax = b$,

These particular cases are extreme points of the class of all possible solution sets to (3).

IV. An Even More General Definition Is Possible, e.g., In Game Theory And Decision Making

The above definition (4) is not the most general interpretation of (3). Indeed, in *game theory*, and in *multi-stage decision making*, we must not only describe which parameters are controllable, but also what parameters are controllable by whom, and in what order. It is possible, e.g., that players move in turn, and a parameter a_{1i} is chosen by the first player on his $i-$th move, and the parameter a_{2i} is chosen by the second player on his $i-$th move. Then, if the condition $Ax \in \mathbf{b}$ means that the first player has won, then the equation (3) can be interpreted as follows: There exists such a first move of the first player that, no matter how the second one moves, the first can move again, etc, and get Ax to be in $\mathbf{b}$, i.e., as:

$$\exists a_{11}\forall a_{21}\exists a_{12}\forall a_{22}...(Ax \in \mathbf{b}).$$

In general, we can have an arbitrary number of quantifier combinations.

In the present paper, we will only consider the set (4), in which *all occurrences of the universal quantifier* $\forall$ *precede to occurrences of the existential quantifier* $\exists$. In logical terms, we can rephrase this condition by saying that the corresponding formula must have an *AE-form.*

V. Main Result

To formulate our results, we must introduce the following denotations.

For the interval linear system $\mathbf{A}x = \mathbf{b}$, we define interval matrices $\mathbf{A}^\forall = (\mathbf{a}_{ij}^\forall)$ and $\mathbf{A}^\exists = (\mathbf{a}_{ij}^\exists)$ and interval vectors $\mathbf{b}^\forall = (\mathbf{b}_i^\forall)$ and $\mathbf{b}^\exists = (\mathbf{b}_i^\exists)$ of the same size as $\mathbf{A}$ and $\mathbf{b}$ as follows:

$$\mathbf{a}_{ij}^\forall = \begin{cases} \mathbf{a}_{ij}, & \text{if } \mathfrak{Q}_{ij} = \forall, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathbf{a}_{ij}^\exists = \begin{cases} \mathbf{a}_{ij}, & \text{if } \mathfrak{Q}_{ij} = \exists, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathbf{b}_i^\forall = \begin{cases} \mathbf{b}_i, & \text{if } \mathfrak{q}_i = \forall, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathbf{b}_i^\exists = \begin{cases} \mathbf{b}_i, & \text{if } \mathfrak{q}_i = \exists, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $\mathbf{A} = \mathbf{A}^\forall + \mathbf{A}^\exists$ and $\mathbf{b} = \mathbf{b}^\forall + \mathbf{b}^\exists$. The fundamental result of our research is

PROPOSITION 1. *The point* x *belongs to the solution set* $\Sigma_{\mathfrak{Q}\mathfrak{q}}(\mathbf{A},\mathbf{b})$ *if and only if*

$$\mathbf{A}^\forall x - \mathbf{b}^\forall \subseteq \mathbf{b}^\exists - \mathbf{A}^\exists x. \tag{5}$$

VI. It Is Very Difficult To Describe The Set Of All Solutions, So, Instead, Let Us Find Some Solutions

It is not hard to prove that for all $\mathfrak{Q}$ and $\mathfrak{q}$, the intersection of the solution set $\Sigma_{\mathfrak{Q}\mathfrak{q}}(\mathbf{A},\mathbf{b})$ with each orthant of the space $\mathbb{R}^n$ is a convex polyhedron. So, in principle, the set of all x that satisfy the condition (5) is a union of 2^n convex polyhedra.

Each polyhedron can be easily described, but since there are 2^n of them (as many as orthants), the resulting length of the direct description of $\Sigma_{\mathfrak{Q}\mathfrak{q}}(\mathbf{A},\mathbf{b})$ grows exponentially with n. As a result, even from moderate values of n, this description becomes practically useless.

This is not because the method is bad: even the problems of deciding whether the united solution set and the controlled solution set are empty or not are known to be NP-hard.

Because of this, instead of trying to describe the solution set itself, to try to find an element from this set. In other words, we arrive at the following problem:

Find an interval vector

that is contained in the solution set

$\Sigma_{\mathfrak{Q}\mathfrak{q}}(\mathbf{A},\mathbf{b})$ *(if it is nonempty)*

of the interval linear system. (6)

Several particular cases of this problem have direct practical interpretation:

- For the tolerable solution set $\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b})$, the problem (6) is the classical *linear tolerance problem* with numerous and fruitful practical applications.
- For $\Sigma_{\mathfrak{Q}\mathfrak{q}}(\mathbf{A},\mathbf{b}) = \Sigma_{\exists\forall}(\mathbf{A},\mathbf{b})$ or $\Sigma_{\mathfrak{Q}\mathfrak{q}}(\mathbf{A},\mathbf{b}) = \Sigma_{\exists\exists}(\mathbf{A},\mathbf{b})$ the problem (6) is the *control problem* or the *identification problem* respectively for the interval linear system [2].

- The linear tolerance problem can be also interpreted as a problem of stabilization within the required output state corridor $\mathbf{b}$ for the system of which *all* parameters a_{ij} are subjected to bounded perturbations.
- If some a_{ij} are disturbing parameters while some are controlled ones, and all $\mathfrak{q}_i = \exists$, $i = 1, 2, \ldots, m$, then we come to the stabilization problem with a control possibility, or, in other words, to the problem of insuring survival of the system.
- Alternately, if a part of a_{ij}'s are disturbing parameters and a part of them are controlled while all $\mathfrak{q}_i = \forall$, $i = 1, 2, \ldots, m$, then we have the control problem under bounded perturbations.

VII. A New Method Of Solving The Problem (6)

In this paper, we present a new, algorithmically efficient approach to the analysis of the linear static systems under interval uncertainty, namely, to the solution of the problem (6).

The underlying idea of our result is unusual for interval computations: it uses the concept of *interval algebraic solution* to the interval equation, that is, an interval vector $\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_n)$, for which $\mathbf{Ax}$ (interpreted as a normal interval product) coincides with $\mathbf{b}$.

To be more precise, we reduce the problem (6) to the problem of finding algebraic interval solution to a special systems of equations in the *extended Kaucher interval arithmetic* $\mathbb{IR}$ [1], thus reducing the original problem to a purely algebraic problem of the numerical analysis.

This reduction does not always work: there exist cases in which the set of solutions (4) is not empty, but the corresponding algebraic problem has no solutions. However, if the algebraic problem does have a solution, then we can solve the problem (6) as well. And, in many reasonable cases, the algebraic problem does have a solution.

To formulate our result, let us briefly recall what Kaucher arithmetic is. In this formalism, the basic elements are the pairs $[\underline{x}, \overline{x}]$ of real numbers, for which (unlike standard interval arithmetic) the condition $\underline{x} \leq \overline{x}$ is not required. Thus, $\mathbb{IR}$ is obtained from the set of standard intervals by adding *improper* intervals $[\underline{x}, \overline{x}]$, $\underline{x} > \overline{x}$, to the set $\mathrm{I}\mathbb{R} = \{[\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in \mathbb{R}, \underline{x} \leq \overline{x}\}$ of the *proper* intervals and the real numbers. Proper and improper intervals (the two major parts of $\mathbb{IR}$, can change places as the result of the duality mapping

$$\mathrm{dual} : \mathbb{IR} \to \mathbb{IR},$$

defined as dual $[\underline{x}, \overline{x}] = [\overline{x}, \underline{x}]$.

PROPOSITION 2. *If the proper interval vector $\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_n)$ is an algebraic interval solution of the equation*

$$(\mathbf{A}^{\forall} + \mathrm{dual}\ \mathbf{A}^{\exists})\mathbf{x} = \mathrm{dual}\ \mathbf{b}^{\forall} + \mathbf{b}^{\exists}, \tag{7}$$

then $\mathbf{x} \subseteq \Sigma_{\mathfrak{Q}\mathfrak{q}}(\mathbf{A}, \mathbf{b})$, i.e., the interval vector $\mathbf{x}$ is a solution to the problem (6).

This algebraic approach is remarkable for its property to almost always give solutions to the problem (6) which are *maximal by inclusion*:

PROPOSITION 3. *If the proper interval vector $\mathbf{x}$ is an inclusion-maximal algebraic interval solution to the system (7), then it is also an inclusion-maximal interval vector contained in $\Sigma_{\mathfrak{Q}\mathfrak{q}}(\mathbf{A}, \mathbf{b})$, i.e., it presents an inclusion-maximal solution to the problem (6).*

VIII. Other Related Results

We have also:

- investigated existence and uniqueness of the algebraic interval solutions,
- proposed a number of practical numerical algorithms to compute algebraic solutions (in particular, the subdifferential Newton method) and
- proved convergence of these algorithms.

References

[1] E. Kaucher, "Interval analysis in the extended interval space $\mathbb{IR}$", *Computing*, Suppl. 2, 1980, pp. 33–49.

[2] S. P. Shary, "Algebraic approach to the interval linear static identification, tolerance and control problems, or One more application of Kaucher arithmetic", *Interval Computations*, 1995, to appear.