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On Full-Rank Interval Matrices

S. P. Shary*

Institute of Computational Technologies, Siberian Branch, Russian Academy of Sciences, pr. Akad. Lavrent'eva 6, Novosibirsk, 630090 Russia

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Abstract—For interval matrices, the paper considers the problem of determining whether a matrix has full rank. We propose a full-rank criterion that relies on the search for diagonal dominance as well as criteria based on pseudoinversion of the midpoint matrix and comparison of the midpoint and the radius matrices for the interval matrix under study.

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To the 60th anniversary of Anatoly Valentinovich Lakeyev

1. INTRODUCTION AND FORMULATION OF THE PROBLEM

In this study, intervals are understood as closed, bounded, and connected subsets of the real axis $\mathbb{R}$, i.e., sets of the form $[\eta, \theta] = \{\, x \in \mathbb{R} \mid \eta \le x \le \theta \,\}$ for real $\eta, \theta \in \mathbb{R}$ and $\eta \le \theta$. The intervals and interval quantities are denoted by bold symbols, whereas noninterval (point) quantities are not specially marked. Underlining and overlining ($\underline{\boldsymbol{a}}$ and $\overline{\boldsymbol{a}}$) denote the left and right endpoints of the interval $\boldsymbol{a} \subset \mathbb{R}$, so that $\boldsymbol{a} = [\underline{\boldsymbol{a}}, \overline{\boldsymbol{a}}] = \{\, x \in \mathbb{R} \mid \underline{\boldsymbol{a}} \le x \le \overline{\boldsymbol{a}} \,\}$ as a whole. In addition, we denote the midpoint of the interval by $\operatorname{mid} \boldsymbol{a} = \frac{1}{2}(\overline{\boldsymbol{a}} + \underline{\boldsymbol{a}})$, the radius of the interval by $\operatorname{rad} \boldsymbol{a} = \frac{1}{2}(\overline{\boldsymbol{a}} - \underline{\boldsymbol{a}})$, the absolute value of the interval by $|\boldsymbol{a}| = \max\{\, |\overline{\boldsymbol{a}}|, |\underline{\boldsymbol{a}}| \,\}$, and the magnitude of the interval (the smallest distance from its points to zero) by $\langle \boldsymbol{a} \rangle$:

$$\langle \boldsymbol{a} \rangle = \begin{cases} \min\{\, |\overline{\boldsymbol{a}}|, |\underline{\boldsymbol{a}}| \,\} & \text{if } 0 \notin \boldsymbol{a}, \\ 0 & \text{otherwise.} \end{cases}$$

The interval matrix is a rectangular table consisting of intervals, which is indicated by $\boldsymbol{A} = (\boldsymbol{a}_{ij})$, meaning that the intersection of the ith row and the jth column contains the element $\boldsymbol{a}_{ij}$. The above-described operations mid, rad, and $|\cdot|$ are applied to interval vectors and matrices componentwise and elementwise. Similarly, the inequalities between point vectors and set-theoretical inclusions and memberships are understood in the componentwise sense. In particular, for matrices $A = (a_{ij})$ and $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ of identical dimensions, the relation $A \in \boldsymbol{A}$ means that $a_{ij} \in \boldsymbol{a}_{ij}$ for all matrix elements.

This paper deals with methods of finding some properties of interval matrices. It is known that a square matrix is called a nonsingular (nondegenerate or regular) matrix if its determinant is not equal to zero [1–3, 10, 13]. This property is equivalent to the absence of a linear dependence between the rows (columns) of such a matrix. Otherwise, the matrix is called a singular (degenerate) matrix.

An interval square matrix $\boldsymbol{A}$ is called a *nonsingular* matrix if all point matrices $A \in \boldsymbol{A}$ are nonsingular [14, 19, 23]. An interval square matrix $\boldsymbol{A}$ is called a *singular* matrix if it is not nonsingular, which is equivalent to the fact that the matrix $\boldsymbol{A}$ contains at least one singular point matrix. The nearest generalization of nonsingular matrices (both point and interval matrices) is full-rank matrices.

The rank of the matrix is the maximum number of its linearly independent rows or columns. As is shown in matrix analysis, these numbers coincide and are equal to the maximum order of nonzero minors

*E-mail: shary@ict.nsc.ru

of the considered matrix [1–3, 10, 13]. A real $m \times n$ matrix is called a *full-rank matrix* if its rank is equal to the minimum number among m and n (it cannot be greater). An interval matrix is called a *full-rank matrix* if it contains only full-rank point matrices. Otherwise, we say that this matrix has *incomplete rank*.

We are mainly interested in interval matrices in connection with interval systems of linear algebraic equations of the form

$$\left\{\begin{array}{ccccccccc} \boldsymbol{a}_{11}x_1 & + & \boldsymbol{a}_{12}x_2 & + & \ldots & + & \boldsymbol{a}_{1n}x_n & = & \boldsymbol{b}_1, \\ \boldsymbol{a}_{21}x_1 & + & \boldsymbol{a}_{22}x_2 & + & \ldots & + & \boldsymbol{a}_{2n}x_n & = & \boldsymbol{b}_2, \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ \boldsymbol{a}_{m1}x_1 & + & \boldsymbol{a}_{m2}x_2 & + & \ldots & + & \boldsymbol{a}_{mn}x_n & = & \boldsymbol{b}_m, \end{array}\right. \tag{1}$$

where $\boldsymbol{a}_{ij}$ are the interval coefficients and $\boldsymbol{b}_i$ are the interval free terms, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. If $\boldsymbol{A} = (\, \boldsymbol{a}_{ij})$ is an interval $m \times n$ matrix, $x = (x_1, x_2, \ldots, x_n)^\top$, and $\boldsymbol{b} = (\, \boldsymbol{b}_i)$ is an interval m-vector, then (1) can be rewritten in a short form as

$$\boldsymbol{A}x = \boldsymbol{b}. \tag{2}$$

Systems (1) and (2) are understood as sets of point systems of linear algebraic equations $Ax = b$ with $m \times n$ matrices A belonging to $\boldsymbol{A}$, and m-vectors b from $\boldsymbol{b}$. The *solution set* of an interval linear system is the set

$$\varXi(\boldsymbol{A}, \boldsymbol{b}) \;=\; \bigl\{\, x \in \mathbb{R}^n \mid (\exists\, A \in \boldsymbol{A})(\exists\, b \in \boldsymbol{b})(\, Ax = b\,) \,\bigr\},$$

formed by all possible solutions of point systems $Ax = b$ with $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$ (see, e.g., [9, 14, 19, 23]).

For systems (1) and (2), the fact that the matrix has full rank is important for solving the data fitting problem for inexact data with interval uncertainty, where interval systems of linear algebraic equations arise in which the number of variables usually differs from the number of equations (see, in particular, [15] and references therein). These systems are often overdetermined, though sometimes they are underdetermined (e.g., in spectroscopy). If the interval matrix constructed on the basis of observations has incomplete rank, this is a consequence of improper organization of experiments because the data of some experiments turned out to be results (linear combinations) of others. Vice versa, a full-rank matrix provides the maximum information both at $m \le n$ and at $m \ge n$.

For overdetermined interval systems of linear equations, in particular, the following proposition is valid.

Proposition 1. *If an interval $m \times n$ matrix $\boldsymbol{A}$, $m \ge n$, has full rank, then the solution set $\varXi(\boldsymbol{A}, \boldsymbol{b})$ of the interval system of linear algebraic equations $\boldsymbol{A}x = \boldsymbol{b}$ is bounded.*

Proof. For the interval linear system $\boldsymbol{A}x = \boldsymbol{b}$, in addition to the above-determined solution set $\varXi(\boldsymbol{A}, \boldsymbol{b})$, we also introduce the set

$$\widetilde{\varXi}(\boldsymbol{A}, \boldsymbol{b}) = \bigl\{\, x \in \mathbb{R}^n \mid (\exists\, A \in \boldsymbol{A})(\exists\, b \in \boldsymbol{b})\bigl(\, A^\top A x = A^\top b\,\bigr) \,\bigr\},$$

formed by solutions of all systems of linear equations $A^\top A x = A^\top b$ for $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. It is known that the compatible system of equations $Ax = b$ is equivalent to the system $A^\top A x = A^\top b$ [1, 2]; therefore, $\varXi(\boldsymbol{A}, \boldsymbol{b}) \subseteq \widetilde{\varXi}(\boldsymbol{A}, \boldsymbol{b})$. The proof of Proposition 1 is based on demonstrating that this set $\widetilde{\varXi}(\boldsymbol{A}, \boldsymbol{b})$ is bounded, though it is larger in the general case.

If A is an $m \times n$ full-rank matrix and $m \ge n$, then $A^\top A$ is a nonsingular $n \times n$ matrix. Therefore, the solution $\check{x}$ of the system $A^\top A x = A^\top b$ is defined by the formula

$$\check{x} = \bigl(A^\top A\bigr)^{-1} A^\top b.$$

The result of this formula continuously depends on A and b for full-rank matrices A. Let us consider the mapping $\mathbb{R}^{m\times n} \times \mathbb{R}^m \to \mathbb{R}^n$, which is defined by the rule $(A, b) \mapsto \bigl(A^\top A\bigr)^{-1} A^\top b$ and is determined for all $m \times n$ full-rank matrices A, $m \geq n$, and all n-vectors b. It is continuous, and the set $\widetilde{\Xi}(\boldsymbol{A}, \boldsymbol{b})$ is the image of the compact $\boldsymbol{A} \times \boldsymbol{b} \subset \mathbb{R}^{m \times n} \times \mathbb{R}^m$ with this continuous mapping. As a consequence, $\widetilde{\Xi}(\boldsymbol{A}, \boldsymbol{b})$ is also compact (see, e.g., [6]) and, therefore, bounded. For this reason, the original solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ of the interval linear system of equations $\boldsymbol{A}x = \boldsymbol{b}$ contained in this set is also bounded. □

The main idea of the proof performed above coincides with the idea described in [1, p. 92], but Neumaier [19] for some reason does not stipulate the condition $m \geq n$, and his considerations concerning the topological properties of sets and mappings and the relationship between solutions of different systems of equations are careless.

The inverse statement to Proposition 1 is caused by additional properties of the matrix of the system and the solution set; in the general case, it may fail to be satisfied (contrary to the result announced in [11]). As an example, let us consider the system of linear equations

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix} x = \begin{pmatrix} [10, 11] \\ [40, 42] \\ [90, 93] \end{pmatrix}, \tag{3}$$

which contains the intervals only in the right-hand side. All rows in the matrix of the system are proportional, so the matrix has incomplete rank 1. Moreover, system (3) is, obviously, incompatible, because its three equations define three parallel bands in $\mathbb{R}^2$, which do not intersect each other. Thus, the solution set of the system is empty and, therefore, bounded.

Below we give some pieces of advice for studying whether interval matrices have full rank. In the general case, verification of the fact whether the interval matrix has full rank is an NP-hard problem [23]. It follows from the fact that its particular case, i.e., the problem of recognition of the nonsingular character of the interval matrix, is also NP-hard [18, 20].

2. CRITERIA ON THE BASIS OF DIAGONAL DOMINANCE

Let us recall that a point square $n \times n$ matrix $A = (a_{ij})$ is called a *diagonally dominant matrix* if the following relation is valid for all $i = 1, 2, \ldots, n$:

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|. \tag{4}$$

It is known that diagonally dominant matrices are nonsingular, and this result is the essence of the Hadamard criterion [3] (it is also often called a Levy–Desplanque theorem, see [13]). The Hadamard criterion can be easily extended to interval matrices.

We say that an interval matrix is *diagonally dominant* if all point matrices contained in this interval matrix are diagonally dominant. It is easy to see that this definition is equivalent to the following: an interval $n \times n$ matrix $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ is diagonally dominant if the following inequalities are satisfied for all $i = 1, 2, \ldots, n$:

$$\langle \boldsymbol{a}_{ii} \rangle > \sum_{j \neq i} |\boldsymbol{a}_{ij}|. \tag{5}$$

Theorem 1 (interval Hadamard criterion). *If an interval matrix is diagonally dominant, i.e., it satisfies* (5)*, then it is nonsingular.*

Apparently, this simple result was first noticed and used in [17].

Proof. The proof obviously follows from the usual Hadamard criterion and from the definition of diagonal dominance of interval matrices. □

Considering the problem of recognition whether the matrix has full rank, we can say that an interval $m \times n$ matrix has full rank if it contains a diagonally dominant square submatrix of size $\min\{m, n\}$. However, the property of diagonal dominance is rather flaky and is not preserved after permutation of rows, whereas the property of having full rank remains unchanged. For this reason, it makes sense to search in this matrix for submatrices where diagonal dominance can be reached by using an appropriate permutation of rows rather than diagonally dominant submatrices.

Finding this permutation is a combinatorial problem; however, owing to its properties, its solution can be found by comparatively simple methods. The table below shows a simple algorithm based on "greedy" heuristics, which tries to construct a diagonally dominant square matrix of size $n \times n$ from the rows of the considered interval $m \times n$ matrix $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ with $m \geq n$. In the case of success, the original matrix is assumed to have full rank; in the case of failure, an additional study is needed. Our algorithm solves the problem of finding a diagonally dominant submatrix up to the end, because each row of the original matrix can have no more than one element satisfying (5).

```
DO FOR i = 1 TO n
    from rows with the numbers k = i, i+1, ..., m, choose
    the row having the greatest difference δ_k = ⟨a_kk⟩ − Σ_{j≠k} |a_kj|,
    denote its number by l;
    IF δ_l ≤ 0 THEN
        issue the message "Additional study of the matrix is needed";
        STOP
    END IF
    permute the ith row with the lth row;
END DO
issue the message "The considered matrix has full rank"
```

A finer criterion of the full rank can be also constructed on the basis of the notion of nonstrict (weak) diagonal dominance. If, instead of (4), the nonstrict inequalities

$$|\, a_{ii}| \geq \sum_{j \neq i} |\, a_{ij}|, \qquad i = 1, 2, \ldots, n, \tag{6}$$

are valid for the elements of the matrix $A = (a_{ij})$, we will speak about *nonstrict diagonal dominance* in A. Further, an $n \times n$ matrix $A = (\, a_{ij})$ is called a *reducible* matrix if the set $\{\, 1, 2, \ldots, n\,\}$ of the first n natural numbers can be decomposed into two nonintersecting subsets I and J, such that $a_{ij} = 0$ for $i \in I$ and $j \in J$. An equivalent definition reads as follows: an $n \times n$ matrix A is reducible if it can be transformed by means of permutations of rows and columns to a block-triangular form $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ with square blocks A_{11} and A_{22}. Matrices that are not reducible are called *irreducible* matrices. The most important example of irreducible matrices are matrices with all elements not equal to zero, in particular, with all element being nonnegative. According to the known Taussky theorem (see [3]) if conditions (6) are satisfied for an irreducible matrix A; if at least one of these conditions is strictly satisfied, the matrix A is nonsingular.

If the following nonstrict inequalities are satisfied for the interval matrix $\boldsymbol{A} = (\boldsymbol{a}_{ij})$:

$$\langle\, \boldsymbol{a}_{ii} \rangle \geq \sum_{j \neq i} |\, \boldsymbol{a}_{ij}| \qquad \text{for all } i = 1, 2, \ldots, n, \tag{7}$$

we will speak about *nonstrict diagonal dominance* in $\boldsymbol{A}$. Further, the interval matrix $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ is called *irreducible* if all point matrices $A \in \boldsymbol{A}$ are irreducible. Obviously, the following theorem is valid.

Theorem 2 (Taussky interval theorem). *Let an irreducible interval square matrix $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ possess nonstrict diagonal dominance* (7)*, and the inequality in* (7) *is rigorously satisfied at least for one row. Then the matrix $\boldsymbol{A}$ is nonsingular.*

In view of this result, it is possible to construct a modification of the algorithm, which is designed to identify a square matrix with nonstrict diagonal dominance. We will not dwell on the details of this construction in the present paper.

3. NECESSARY AND SUFFICIENT CONDITIONS OF THE FULL RANK

The following result is a generalization of the nonsingularity criterion of square interval matrices and yields the necessary and sufficient conditions of the full rank of interval matrices.

Theorem 3 (Rohn [23]). *An interval $m \times n$ matrix $\boldsymbol{A}$, $m \geq n$, has full rank if and only if the system of inequalities*

$$|\,(\operatorname{mid}\boldsymbol{A})\,x\,| \leq (\operatorname{rad}\boldsymbol{A})\,|x|, \qquad x \in \mathbb{R}^n, \tag{8}$$

has a unique zero solution.

Neither Rohn [23] nor other authors proved this result, apparently, because it can be easily derived from the proof for square matrices published in [21]. For our paper to be self-sufficient, we write out the full proof of the theorem.

Proof. Sufficiency. If the $m \times n$ matrix $\boldsymbol{A}$, $m \geq n$, contains a matrix with incomplete rank $\tilde{A}$, then $\tilde{A}\tilde{x} = 0$ for a certain nonzero vector $\tilde{x} \in \mathbb{R}^n$. Therefore, we have

$$|\,(\operatorname{mid}\boldsymbol{A})\,\tilde{x}\,| \;=\; |\,(\operatorname{mid}\boldsymbol{A} - \tilde{A})\,\tilde{x}\,| \;\leq\; |\tilde{A} - \operatorname{mid}\boldsymbol{A}|\,|\tilde{x}| \;\leq\; (\operatorname{rad}\boldsymbol{A})\,|\tilde{x}|,$$

because $|\tilde{A} - \operatorname{mid}\boldsymbol{A}| \leq \operatorname{rad}\boldsymbol{A}$. Thus, the vector $\tilde{x}$ is a nontrivial solution of the system of inequalities (8).

Necessity. If inequality (8) indeed has the solution $\tilde{x} \neq 0$, we form vectors $y = (\,y_i) \in \mathbb{R}^m$ and $z = (\,z_j) \in \mathbb{R}^n$ such that

$$y_i = \begin{cases} \dfrac{(\operatorname{mid}\boldsymbol{A}\cdot\tilde{x})_i}{(\operatorname{rad}\boldsymbol{A}\cdot|\tilde{x}|)_i} & \text{if } (\operatorname{rad}\boldsymbol{A}\cdot|\tilde{x}|)_i \neq 0, \\[2ex] 1 & \text{if } (\operatorname{rad}\boldsymbol{A}\cdot|\tilde{x}|)_i = 0, \end{cases} \qquad i = 1, 2, \ldots, m,$$

and

$$z_j = \begin{cases} 1 & \text{if } \tilde{x}_j \geq 0, \\ -1 & \text{if } \tilde{x}_j < 0, \end{cases} \qquad j = 1, 2, \ldots, n,$$

and, using these vectors, we construct a matrix $\tilde{A}$ with the element $(\operatorname{mid}\boldsymbol{A})_{ij} - y_i z_j (\operatorname{rad}\boldsymbol{A})_{ij}$ at the ijth place. In the matrix form, it is represented as

$$\tilde{A} = \operatorname{mid}\boldsymbol{A} - \operatorname{diag}\{y\}\cdot\operatorname{rad}\boldsymbol{A}\cdot\operatorname{diag}\{z\}.$$

As all $|\,y_i z_j| \leq 1$, then, obviously, $\tilde{A}$ belongs to $\boldsymbol{A}$. At the same time, it has incomplete rank because its product by the nonzero vector $\tilde{x}$ turns to zero. Indeed, we have

$$\begin{aligned} \tilde{A}\tilde{x} \;&=\; (\operatorname{mid}\boldsymbol{A})\,\tilde{x} - \operatorname{diag}\{y\}\,(\operatorname{rad}\boldsymbol{A})\operatorname{diag}\{z\}\,\tilde{x} \\ &=\; (\operatorname{mid}\boldsymbol{A})\,\tilde{x} - \operatorname{diag}\{y\}\,(\operatorname{rad}\boldsymbol{A})\,|\tilde{x}|, \end{aligned}$$

moreover, if $(\operatorname{rad}\boldsymbol{A}\cdot|\tilde{x}|)_i \neq 0$, then the ith component of this vector should be equal to the difference

$$\big(\,(\operatorname{mid}\boldsymbol{A})\,\tilde{x}\,\big)_i - \frac{(\operatorname{mid}\boldsymbol{A}\cdot\tilde{x})_i}{(\operatorname{rad}\boldsymbol{A}\cdot|\tilde{x}|)_i}\,\big(\operatorname{rad}\boldsymbol{A}\cdot|\tilde{x}|\,\big)_i \;=\; \big(\operatorname{mid}\boldsymbol{A}\cdot\tilde{x}\,\big)_i - \big(\operatorname{mid}\boldsymbol{A}\cdot\tilde{x}\,\big)_i;$$

if $(\operatorname{rad}\boldsymbol{A}\cdot|\tilde{x}|)_i = 0$, it should be equal to the difference of two zeros by virtue of (8). This fact proves the sufficiency of the conditions of this theorem. ∎

It is not easy to find nontrivial solutions of (8) in the general case, and the main purpose of Theorem 3 is to form the basis for designing more practical criteria of the full rank of interval matrices.

Corollary 1. *An interval $m \times n$ matrix $\boldsymbol{A}$, $m \leq n$, has full rank if and only if the system of inequalities*

$$\left| x^\top (\mathrm{mid}\, \boldsymbol{A}) \right| \;\leq\; |x|^\top (\mathrm{rad}\, \boldsymbol{A}), \quad x \in \mathbb{R}^n,$$

has a unique zero solution.

To justify this statement, we use the fact that the rank of the matrix remains unchanged after transposition and apply Theorem 3 to the matrix $A^\top$.

4. CRITERION BASED ON PSEUDOINVERSION AND SPECTRAL RADIUS

Let us recall (see [1–3, 13]) that a pseudoinverse matrix for a real $m \times n$ matrix A is a real $n \times m$ matrix A^+ such that AA^+ and A^+A are symmetric matrices and

$$AA^+A = A, \qquad A^+AA^+ = A^+.$$

If A is a full-rank matrix and $m \geq n$, then $A^+ = (A^\top A)^{-1} A^\top$ [1, 2]. In this case, A^+A is a unit $n \times n$ matrix. On the other hand, if A is a full-rank matrix and $m \leq n$, then $A^+ = A^\top (AA^\top)^{-1}$ [1, 2]. Then AA^+ is a unit $m \times m$ matrix. The pseudoinverse matrix was actually involved in proving Proposition 1 in the expression for the solution $\check{x}$.

The formulation of the next result can also be found in the handbook [23], but the reader is referred to [22] for the proof, where the considerations are not transparent, and the required result is not even explicitly formulated. Another proof is given below.

Theorem 4. *Let an interval $m \times n$ matrix $\boldsymbol{A}$ be such that $m \geq n$, the mid-point matrix* $\mathrm{mid}\, \boldsymbol{A}$ *have full rank, and*

$$\rho\bigl(\, |(\mathrm{mid}\, \boldsymbol{A})^+| \cdot \mathrm{rad}\, \boldsymbol{A} \,\bigr) < 1,$$

where $\rho(\cdot)$ means taking the spectral radius of the square matrix. Then $\boldsymbol{A}$ has full rank.

Proof. Let us assume, on the opposite, that the interval matrix $\boldsymbol{A}$ has incomplete rank. Then, according to Theorem 3, there exists a nonzero n-vector $\tilde{x}$ such that

$$|(\mathrm{mid}\, \boldsymbol{A})\, \tilde{x}\,| \leq (\mathrm{rad}\, \boldsymbol{A})\, |\tilde{x}|.$$

Thus, we have

$$|\tilde{x}| \;=\; |(\mathrm{mid}\, \boldsymbol{A})^+ (\mathrm{mid}\, \boldsymbol{A})\, \tilde{x}| \;\leq\; |(\mathrm{mid}\, \boldsymbol{A})^+| \cdot |(\mathrm{mid}\, \boldsymbol{A})\, \tilde{x}| \;\leq\; |(\mathrm{mid}\, \boldsymbol{A})^+| \cdot (\mathrm{rad}\, \boldsymbol{A}) \cdot |\tilde{x}|,$$

i.e., for the nonzero nonnegative vector $v = |x|$, we have

$$|(\mathrm{mid}\, \boldsymbol{A})^+| (\mathrm{rad}\, \boldsymbol{A}) v \geq v,$$

and the matrix $|(\mathrm{mid}\, \boldsymbol{A})^+|\; (\mathrm{rad}\, \boldsymbol{A})$ is nonnegative.

Let us now recall the following fact from the theory of nonnegative matrices. If G is a nonnegative $n \times n$ matrix, $\rho(G)$ is its spectral radius, and α is a positive real number, then

$$\rho(G) \;\geq\; \alpha \Leftrightarrow \bigl(\exists v \in \mathbb{R}^n\bigr) \bigl(v \geq 0,\, v \neq 0 \,\&\, Gv \geq \alpha v\bigr). \tag{9}$$

The proof of this equivalence can be found, e.g., in the monograph of Horn and Johnson [13, Thm. 8.3.1] or in [16, 19]. On the other hand, this result is implicitly justified in the Wielandt's proof of the Perron–Frobenius theorem on nonnegative matrices, which can be found in many handbooks on the matrix theory, for instance, in the classical book of Gantmacher [3].

Thus, it follows from (9) and from the inequality $|(\mathrm{mid}\, \boldsymbol{A})^+| \cdot (\mathrm{rad}\, \boldsymbol{A}) \cdot |\tilde{x}| \;\geq\; |\tilde{x}|$ derived by us that the spectral radius of the nonnegative matrix $|(\mathrm{mid}\, \boldsymbol{A})^+|(\mathrm{rad}\, \boldsymbol{A})$ is equal to or greater than unity. We come to a contradiction! ☐

Theorem 4 is a direct generalization of the Ris–Beeck nonsingularity criterion for interval matrices, which is well known in interval analysis (see [14, 21, 23]).

Corollary 2. *Let an interval $m \times n$ matrix $\boldsymbol{A}$ be such that $m \le n$, the midpoint matrix* $\operatorname{mid} \boldsymbol{A}$ *have full rank, and*

$$\rho\big(\operatorname{rad} \boldsymbol{A} \cdot |(\operatorname{mid} \boldsymbol{A})^{+}|\big) < 1,$$

where $\rho(\cdot)$ means taking of the spectral radius. Then $\boldsymbol{A}$ has full rank.

For justification, we use Corollary 1 from Theorem 3 and the fact that the spectrum of the square matrix remains unchanged after transposition.

As an example of using the results obtained, let us consider the matrix

$$\boldsymbol{B} \;=\; \begin{pmatrix} [1,2] & [3,4] \\ [5,6] & [7,8] \\ [9,10] & [11,12] \end{pmatrix},$$

which has full rank equal to 2. To verify this fact, we consider its submatrix

$$\begin{pmatrix} [1,2] & [3,4] \\ [9,10] & [11,12] \end{pmatrix} \tag{10}$$

and find the interval estimate of the range of the values of its determinant:

$$[1,2] \cdot [11,12] - [9,10] \cdot [3,4] = [11,24] - [27,40] = [-29,-3].$$

In the case of 2×2 matrices, all variables (matrix elements) in the first power are included into the expression for the determinant only once; therefore, by virtue of the main theorem of interval arithmetics [9, 14, 19], the result of interval estimation coincides with the exact domain of the determinant values. As $0 \notin [-29,-3]$, the matrix (10) is nonsingular.

At the same time,

$$\operatorname{rad} \boldsymbol{B} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}, \qquad \operatorname{mid} \boldsymbol{B} = \begin{pmatrix} 1.5 & 3.5 \\ 5.5 & 7.5 \\ 9.5 & 11.5 \end{pmatrix},$$

the matrix $\operatorname{mid} \boldsymbol{B}$ has full rank and $\rho\big(\,|(\operatorname{mid} \boldsymbol{B})^{+}| \cdot \operatorname{rad} \boldsymbol{B}\,\big) = 0.979167 < 1$. We see that the criterion derived in Theorem 4 also shows that the matrix $\boldsymbol{B}$ has full rank.

5. CRITERION BASED ON COMPARISON OF SINGULAR VALUES OF THE MID-POINT AND RADIUS MATRICES

Let us recall that a singular value σ of a real $m \times n$ matrix A is a nonnegative solution to the system

$$\begin{pmatrix} 0 & A^{\top} \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \;=\; \sigma \begin{pmatrix} x \\ y \end{pmatrix},$$

corresponding to nonzero vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. The singular values of the matrix A can also be defined as arithmetic square roots of the common eigenvalues of the matrices $A^{\top}A$ and $AA^{\top}$ (see [13]). Thus, the singular values of the $m \times n$ matrix are a set of $\min\{m,n\}$ nonnegative values. We use $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$ to denote the smallest and greatest singular values of the matrix A.

Theorem 5. *If the inequality*

$$\sigma_{\max}(\operatorname{rad} \boldsymbol{A}) \ < \ \sigma_{\min}(\operatorname{mid} \boldsymbol{A}) \tag{11}$$

is satisfied for the interval $m \times n$ matrix $\boldsymbol{A}$, then it has full rank.

Proof. Let $m \geq n$, and let us assume the opposite to the theorem to be proved: the matrix $\boldsymbol{A}$ has incomplete rank. Then, according to Theorem 3, inequality (8) is valid for a certain vector $\tilde{x} \neq 0$, i.e.,

$$|\,(\operatorname{mid} \boldsymbol{A})\,\tilde{x}\,| \leq (\operatorname{rad} \boldsymbol{A})\,|\tilde{x}|, \tag{12}$$

and the equivalent componentwise inequality for vector-columns is also valid:

$$|(\operatorname{mid} \boldsymbol{A})\,\tilde{x}|^\top \leq \big((\operatorname{rad} \boldsymbol{A})|\tilde{x}|\big)^\top. \tag{13}$$

The components of the vectors in the left and right sides of inequalities (12) and (13) are nonnegative; therefore, multiplying the vectors from the like parts, we obtain an inequality of the same meaning:

$$|(\operatorname{mid} \boldsymbol{A})\,\tilde{x}|^\top|(\operatorname{mid} \boldsymbol{A})\,\tilde{x}| \ \leq \ \big(\,(\operatorname{rad} \boldsymbol{A})|\tilde{x}|\,\big)^\top(\operatorname{rad} \boldsymbol{A})|\tilde{x}|. \tag{14}$$

Without loss of generality, we can assume that $\tilde{x}^\top\tilde{x} = 1$, i.e., the vector $\tilde{x}$ is normalized in the Euclidean norm, which is traditionally denoted by $\|\cdot\|_2$. Let us recall the variational description of eigenvalues of a symmetric matrix, which is based on the Rayleigh quotient (see, e.g., [2, 3, 5, 10, 13]): if $H \in \mathbb{R}^{n\times n}$, $H^\top = H$, and $\lambda_{\min}(H)$ and $\lambda_{\max}(H)$ are the smallest and greatest eigenvalues of the matrix H, respectively, then

$$\lambda_{\min}(H) \ = \ \min_{y\neq 0} \frac{y^\top H\,y}{y^\top y} \ = \ \min_{\|x\|_2=1} x^\top H\,x, \qquad \lambda_{\max}(H) \ = \ \max_{y\neq 0} \frac{y^\top H\,y}{y^\top y} \ = \ \max_{\|x\|_2=1} x^\top H\,x.$$

As a consequence, we have

$$\begin{aligned}
\sigma^2_{\min}(\operatorname{mid} \boldsymbol{A}) \ &= \ \lambda_{\min}\big(\,(\operatorname{mid} \boldsymbol{A})^\top(\operatorname{mid} \boldsymbol{A})\,\big) \\
&= \ \min_{\|x\|_2=1} \ \big(\,x^\top(\operatorname{mid} \boldsymbol{A})^\top(\operatorname{mid} \boldsymbol{A})\,x\,\big) \quad \text{by virtue of the Rayleigh characterization} \\
&\leq \ \big(\,(\operatorname{mid} \boldsymbol{A})\,\tilde{x}\,\big)^\top\big(\,(\operatorname{mid} \boldsymbol{A})\,\tilde{x}\,\big) \ \leq \ |(\operatorname{mid} \boldsymbol{A})\,\tilde{x}|^\top|(\operatorname{mid} \boldsymbol{A})\,\tilde{x}| \\
&\leq \ \big(\,(\operatorname{rad} \boldsymbol{A})|\tilde{x}|\,\big)^\top(\operatorname{rad} \boldsymbol{A})|\tilde{x}| \quad \text{by virtue of (14)} \\
&= \ |\tilde{x}|^\top(\operatorname{rad} \boldsymbol{A})^\top\,(\operatorname{rad} \boldsymbol{A})\,|\tilde{x}| \leq \ \max_{\|x\|_2=1} \Big(x^\top\big(\,(\operatorname{rad} \boldsymbol{A})^\top(\operatorname{rad} \boldsymbol{A})\,\big)\,x\Big) \\
&= \ \lambda_{\max}\big(\,(\operatorname{rad} \boldsymbol{A})^\top(\operatorname{rad} \boldsymbol{A})\,\big) \quad \text{by virtue of the Rayleigh characterization} \\
&= \ \sigma^2_{\max}(\operatorname{rad} \boldsymbol{A}).
\end{aligned} \tag{15}$$

Comparing the beginning and end of (15), as a whole we obtain $\sigma_{\min}(\operatorname{mid} \boldsymbol{A}) \leq \sigma_{\max}(\operatorname{rad} \boldsymbol{A})$, which contradicts condition (11).

In proving the case with $m \leq n$, our considerations are based on Corollary 1. □

If $\boldsymbol{A} = A$ is a point matrix, then $\operatorname{mid} \boldsymbol{A} = A$, $\operatorname{rad} \boldsymbol{A} = 0$, and all singular values of the matrix $\operatorname{rad} \boldsymbol{A}$ are also equal to zero. Then the result of Theorem 5 expresses the well-known (from matrix analysis) full rank criterion: condition $\sigma_{\min}(A){>}0$.

For the case of essentially interval matrices, the result of Theorem 5 is a generalization of the known Rump criterion [24] (see also [14, 21]): if $\sigma_{\max}(\operatorname{rad} \boldsymbol{A}) < \sigma_{\min}(\operatorname{mid} \boldsymbol{A})$ for an interval square matrix $\boldsymbol{A}$, then this matrix is nonsingular.

As the first example of using the criterion obtained, we consider the matrix proposed by Irene Sharaya:

$$\boldsymbol{A} = \begin{pmatrix} 1 & [0,1] \\ -1 & [0,1] \\ [-1,1] & 1 \end{pmatrix}.$$

It demonstrates that traditional methods of linear algebra and matrix analysis and the intuition based on them may fail to work for interval matrices. The midpoint and radius matrices for $\boldsymbol{A}$ have the form

$$\operatorname{mid} \boldsymbol{A} = \begin{pmatrix} 1 & 0.5 \\ -1 & 0.5 \\ 0 & 1 \end{pmatrix}, \qquad \operatorname{rad} \boldsymbol{A} = \begin{pmatrix} 0 & 0.5 \\ 0 & 0.5 \\ 1 & 0 \end{pmatrix}.$$

Correspondingly, $\sigma_{\max}(\operatorname{rad} \boldsymbol{A}) = 1$, $\sigma_{\min}(\operatorname{mid} \boldsymbol{A}) = 1.22474$, and the considered matrix has full rank 2 in accordance with Theorem 5. The same result can be achieved by using Theorem 4. At the same time, the matrix $\boldsymbol{A}$ does not contain nonsingular interval 2×2 submatrices, which can easily be determined by complete search for all submatrices of this kind:

$$\begin{pmatrix} 1 & [0,1] \\ -1 & [0,1] \end{pmatrix}, \qquad \begin{pmatrix} 1 & [0,1] \\ [-1,1] & 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & [0,1] \\ [-1,1] & 1 \end{pmatrix}.$$

These matrices contain singular matrices

$$\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix},$$

respectively. Let us recall that the full-rank matrix in a usual noninterval case by definition has a square nonsingular matrix whose order is equal to the matrix rank.

As the next example, we consider the interval matrix

$$\boldsymbol{B} = \begin{pmatrix} [1,2] & [3,4] \\ [5,6] & [7,8] \\ [9,10] & [11,12] \end{pmatrix}.$$

In the previous section, we found that this matrix has full rank by using Theorem 4. At the same time, $\sigma_{\max}(\operatorname{rad} \boldsymbol{B}) = 1.22474$ and $\sigma_{\min}(\operatorname{mid} \boldsymbol{B}) = 1.09151$ for this matrix; therefore, Theorem 5 does not allow us to be sure whether the matrix $\boldsymbol{B}$ has full rank.

Let us consider an example of another property. We consider the interval matrix

$$\boldsymbol{C} = \begin{pmatrix} 3.3 & [0,2] & [0,2] \\ [0,2] & 3.3 & [0,2] \\ [0,2] & [0,2] & 3.3 \\ [0,1] & [0,1] & [0,1] \end{pmatrix}.$$

For this matrix, we have $\rho\big(\,|(\operatorname{mid} \boldsymbol{C})^+| \cdot \operatorname{rad} \boldsymbol{C}\,\big) = 1.06291 > 1$. In this case, however, we also have $\sigma_{\max}(\operatorname{rad} \boldsymbol{C}) = 2.17945$ and $\sigma_{\min}(\operatorname{mid} \boldsymbol{C}) = 2.3$. It turns out that the criterion of Theorem 5 says that the considered matrix $\boldsymbol{C}$ has full rank, whereas Theorem 4 does not allow us to make any definite conclusions.

6. CRITERION BASED ON COMPARISON OF THE NORMS OF THE MIDPOINT AND RADIUS MATRICES

Let us recall that the *subordinate matrix norm*, for a given vector norm $\|\cdot\|$, is defined as

$$\|A\|' = \max_{x\neq 0} \frac{\|Ax\|}{\|x\|}. \tag{16}$$

The *absolute norm* is the norm depending only on the absolute values of the elements (of the vector or matrix). Among popular norms, the absolute norms are

$$\|A\|_1 \;=\; \max_{1\leq j\leq n} \left(\sum_{i=1}^{m} |a_{ij}| \right) \quad \text{and} \quad \|A\|_\infty \;=\; \max_{1\leq i\leq m} \left(\sum_{j=1}^{n} |a_{ij}| \right), \quad A \in \mathbb{R}^{m\times n},$$

which are subordinate to the vector norms

$$\|x\|_1 \;=\; \sum_{i=1}^{n} |x_i| \quad \text{and} \quad \|x\|_\infty \;=\; \max_{1\leq i\leq n} |x_i|, \quad x \in \mathbb{R}^n.$$

The basic result of this part of our work is the following theorem.

Theorem 6. *Let $\|\cdot\|$ be an absolute subordinate matrix norm. If, for an interval $m\times n$ matrix $\boldsymbol{A}$, $m \geq n$, the midpoint matrix* $\operatorname{mid}\boldsymbol{A}$ *has full rank and the condition*

$$\|\operatorname{rad}\boldsymbol{A}\| \;<\; \left\|(\operatorname{mid}\boldsymbol{A})^+\right\|^{-1}$$

is satisfied, then $\boldsymbol{A}$ also has full rank.

Our proof is essentially based on the concept of the *lower bound of the matrix* [10]. For a fixed vector norm $\|\cdot\|$, the lower bound of the matrix A with respect to this norm will be referred to as $\operatorname{lob}(A)$ (from <u>lo</u>wer <u>b</u>ound) defined as

$$\operatorname{lob}(A) \;=\; \min_{x\neq 0} \frac{\|Ax\|}{\|x\|} \;=\; \min_{\|y\|=1} \|Ay\|. \tag{17}$$

The lower bound of the matrix is, in a certain sense, an antipode of the subordinate matrix norm because expression (16) determining the norm is absolutely similar to (17) with the only difference that it has max instead of min. In this terminology, the subordinate matrix norm could be called an upper bound of the matrix.

It is clear that the lower bound of the matrix is a nonnegative number such that the following expression is valid for all vectors $x \in \mathbb{R}^n$:

$$\|Ax\| \;\geq\; \operatorname{lob}(A)\,\|x\|. \tag{18}$$

The lower bound of square matrices, i.e., in the case of $m = n$, is equal to zero if and only if the matrix is singular (degenerate). For rectangular $m\times n$ matrices with $m \geq n$, the lower bound is equal to zero if and only if the matrix has incomplete rank. For rectangular $m\times n$ matrices with $m < n$, the lower bound is always equal to zero because x in (17) can be chosen as a nonzero n-vector orthogonal to all m rows of the matrix A.

The following proposition is valid in the general case.

Proposition 2. *Let a full-rank $m\times n$ matrix A, $m \geq n$, and a vector norm be given. Then the lower bound of A with respect to this norm has the estimate from below* $\operatorname{lob}(A) \geq \|A^+\|^{-1}$*, where $\|\cdot\|$ is a subordinate matrix norm.*

Proof. Let A be a full-rank $m \times n$ matrix and $m \geq n$. Then $Ax \neq 0$ for $x \neq 0$. As a consequence, we have

$$\operatorname{lob}(A) \;=\; \min_{x\neq 0} \frac{\|Ax\|}{\|x\|} = \left(\max_{x\neq 0} \left(\frac{\|Ax\|}{\|x\|}\right)^{-1}\right)^{-1} = \left(\max_{x\neq 0} \frac{\|x\|}{\|Ax\|} \right)^{-1}.$$

Let us replace $Ax = y$. Then for an arbitrary vector $y \in \mathbb{R}^m$ that belongs to the image $\operatorname{Im} A$ of the linear mapping $x \mapsto Ax$, we obtain $x = A^+y$. We can continue the considerations:

$$\operatorname{lob}(A) = \left(\max_{x\neq 0} \frac{\|x\|}{\|Ax\|} \right)^{-1} = \left(\max_{\substack{y\in \operatorname{Im} A\\ y\neq 0}} \frac{\|A^+y\|}{\|y\|} \right)^{-1} \geq \left(\max_{y\neq 0} \frac{\|A^+y\|}{\|y\|} \right)^{-1} = \|A^+\|^{-1},$$

which had to be proved. $\square$

Proof of Theorem 6. For any $x \in \mathbb{R}^n$, $x \neq 0$, and for any $\tilde{A} \in \boldsymbol{A}$, we have

$$\tilde{A}x = \big(\operatorname{mid} \boldsymbol{A} + (\tilde{A} - \operatorname{mid} \boldsymbol{A})\big)x = (\operatorname{mid} \boldsymbol{A})x + (\tilde{A} - \operatorname{mid} \boldsymbol{A})x,$$

where, obviously,

$$|\tilde{A} - \operatorname{mid} \boldsymbol{A}| \leq \operatorname{rad} \boldsymbol{A}. \tag{19}$$

Therefore, we have

$$\begin{aligned}
\big\|(\operatorname{mid} \boldsymbol{A})x\big\| \;&\geq\; \operatorname{lob}(\operatorname{mid} \boldsymbol{A})\,\|x\| \quad \text{by virtue of (18)}\\
&\geq\; \big\|(\operatorname{mid} \boldsymbol{A})^+\big\|^{-1}\|x\| \quad \text{by virtue of Proposition 2}\\
&>\; \|\operatorname{rad} \boldsymbol{A}\|\,\|x\| \quad \text{based on the condition of this theorem} \qquad (20)\\
&\geq\; \big\|\tilde{A} - \operatorname{mid} \boldsymbol{A}\big\|\,\|x\| \quad \text{by virtue of (19) and the absolute matrix norm}\\
&\geq\; \big\|(\tilde{A} - \operatorname{mid} \boldsymbol{A})\,x\big\| \quad \text{by virtue of submultiplicativity of the matrix norm.}
\end{aligned}$$

Comparing the beginning and end of this chain, we can conclude that the sum $(\operatorname{mid} \boldsymbol{A})x + (\tilde{A} - \operatorname{mid} \boldsymbol{A})x$ should not vanish. Otherwise, we would have $(\operatorname{mid} \boldsymbol{A})x = -(\tilde{A} - \operatorname{mid} \boldsymbol{A})x$ and, hence, $\|(\operatorname{mid} \boldsymbol{A})x\| = \|(\tilde{A} - \operatorname{mid} \boldsymbol{A})x\|$ despite the strict inequality (20). Thus, we have $\tilde{A}x \neq 0$, and the matrix $\tilde{A}$ has full rank. $\square$

For the case of square interval matrices, a similar result on nonsingularity was formulated in [7] as a consequence of research performed in [18]. The proof given above covers a more general case of rectangular matrices and, moreover, is shorter and more transparent.

As an example of using Theorem 6, let us consider the matrix

$$\boldsymbol{A} = \begin{pmatrix} [-1,1] & [3,5] \\ [7,9] & [11,13] \\ [13,15] & [19,21] \end{pmatrix},$$

for which the midpoint and radius matrices are

$$\operatorname{mid} \boldsymbol{A} = \begin{pmatrix} 0 & 4 \\ 8 & 12 \\ 14 & 20 \end{pmatrix} \quad \text{and} \quad \operatorname{rad} \boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then $\sigma_{\min}(\operatorname{mid}\boldsymbol{A}) = 2.27684$ and $\sigma_{\max}(\operatorname{rad}\boldsymbol{A}) = 2.44949$, and the criterion based on singular values (Theorem 5) does not work.

However, as the smallest singular value of the midpoint matrix $\operatorname{mid}\boldsymbol{A}$ is noticeably different from zero, it has full rank. Moreover, the pseudoinverse matrix for $\operatorname{mid}\boldsymbol{A}$ has the form

$$\begin{pmatrix} -0.356061 & -0.0075758 & 0.0757576 \\ 0.246212 & 0.0265152 & -0.0151515 \end{pmatrix}$$

and $\|(\operatorname{mid}\boldsymbol{A})^{+}\|_{\infty}^{-1} = 2.27586$, whereas $\|\operatorname{rad}\boldsymbol{A}\|_{\infty} = 2$. Based on Theorem 6, from here we can conclude that the interval matrix $\boldsymbol{A}$ also has full rank. The same conclusion can be obtained by using the criterion of Theorem 4.

It is known that the maximum singular value of the matrix is a matrix norm (the so-called "spectral norm"); therefore, the result of Theorem 6 is similar to Theorem 5. However, formally it does not include the latter because the spectral norm of the matrices is not the absolute matrix norm. Moreover, Theorem 5 does not require that the midpoint matrix $\operatorname{mid}\boldsymbol{A}$ should have full rank. Nevertheless, it is useful to provide substantiation of Theorem 5 on the basis of the ideas of this section.

Let us first note that the following expression is valid for $m \times n$ matrices with $m \geq n$ by virtue of the Rayleigh relation:

$$\begin{aligned} \min_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} &= \min_{x\neq 0} \frac{\sqrt{(Ax)^\top Ax}}{\sqrt{x^\top x}} = \sqrt{\min_{x\neq 0} \frac{(Ax)^\top Ax}{x^\top x}} = \sqrt{\min_{x\neq 0} \frac{x^\top (A^\top A)\, x}{x^\top x}} \\ &= \sqrt{\lambda_{\min}(A^\top A)} = \sigma_{\min}(A). \end{aligned}$$

In other words, the lower bound of the $m \times n$ matrix A, $m \geq n$, with respect to the Euclidean norm is $\operatorname{lob}_2(A) = \sigma_{\min}(A)$.

Passing to the proof of the theorem, let us take some matrix $\tilde{A} \in \boldsymbol{A}$. Obviously, $|\tilde{A} - \operatorname{mid}\boldsymbol{A}| \leq \operatorname{rad}\boldsymbol{A}$; therefore, we have

$$\begin{pmatrix} 0 & (\operatorname{rad}\boldsymbol{A})^\top \\ \operatorname{rad}\boldsymbol{A} & 0 \end{pmatrix} \geq \begin{pmatrix} 0 & |\tilde{A} - \operatorname{mid}\boldsymbol{A}|^\top \\ |\tilde{A} - \operatorname{mid}\boldsymbol{A}| & 0 \end{pmatrix}. \tag{21}$$

Let us use the following fact of the Perron–Frobenius theory for nonnegative matrixes: if $G \geq |H|$, then $\rho(G) \geq \rho(H)$ (see [13, Thm. 8.1.18]). Then, on the basis of (18), we can conclude that

$$\rho \begin{pmatrix} 0 & (\operatorname{rad}\boldsymbol{A})^\top \\ \operatorname{rad}\boldsymbol{A} & 0 \end{pmatrix} \geq \rho \begin{pmatrix} 0 & (\tilde{A} - \operatorname{mid}\boldsymbol{A})^\top \\ (\tilde{A} - \operatorname{mid}\boldsymbol{A}) & 0 \end{pmatrix},$$

i.e.,

$$\sigma_{\max}(\operatorname{rad}\boldsymbol{A}) \geq \sigma_{\max}(\tilde{A} - \operatorname{mid}\boldsymbol{A}). \tag{22}$$

Taking into account the inequalities derived above, we have

$$\begin{aligned} \|(\operatorname{mid}\boldsymbol{A})x\|_2 &\geq \sigma_{\min}(\operatorname{mid}\boldsymbol{A})\, \|x\|_2 \quad \text{from the expression obtained for } \operatorname{lob}_2(A) \\ &> \sigma_{\max}(\operatorname{rad}\boldsymbol{A})\, \|x\|_2 \quad \text{based on the condition of Proposition 2} \\ &\geq \sigma_{\max}(\tilde{A} - \operatorname{mid}\boldsymbol{A})\, \|x\|_2 \quad \text{by virtue of (22)} \\ &= \left\|\tilde{A} - \operatorname{mid}\boldsymbol{A}\right\|_2 \|x\|_2 \quad \text{by definition of the spectral norm} \\ &\geq \left\|(\tilde{A} - \operatorname{mid}\boldsymbol{A})\, x\right\|_2 \quad \text{by virtue of submultiplicativity of the norm.} \end{aligned}$$

We have to note that the product $\tilde{A}x = (\mathrm{mid}\,\boldsymbol{A})x + (\tilde{A} - \mathrm{mid}\,\boldsymbol{A})x$ cannot vanish because it would lead to $(\mathrm{mid}\,\boldsymbol{A})x = -(\tilde{A} - \mathrm{mid}\,\boldsymbol{A})x$. Then it should be $\|(\mathrm{mid}\,\boldsymbol{A})x\|_2 = \|(\tilde{A} - \mathrm{mid}\,\boldsymbol{A})x\|_2$, which contradicts the results of the considerations discussed above. As a consequence, $\tilde{A}x \neq 0$, and the matrix $\tilde{A}$ has full rank.

To finalize the proof of Theorem 5, it is necessary to consider the case of a rectangular matrix with $m < n$. For this purpose, we can transpose the matrix and our considerations can be based on the proof given above because neither the matrix rank nor its singular values change after transposition. The same consideration refers to application of Theorem 6 in practice.

To conclude, we should note that the computation of the spectral radius, singular values of matrices, and pseudoinverse matrices is well developed in modern numerical analysis (see, e.g., [4, 5]). Reliable algorithms have been elaborated for computing these objects, and thoroughly tested subroutines implementing these algorithms are included into standard software libraries for numerical linear algebra (all numerical data reported in this paper were obtained by Scilab, an open computer mathematics system [25]). For rough and rapid estimation of the spectral radius, which is used in Theorem 4, it is possible to use its upper bound based on a certain matrix norm.

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