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Outer Estimation of Generalized Solution Sets to Interval Linear Systems

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Abstract. The work advances a numerical technique for computing enclosures of *generalized AE-solution sets* to interval linear systems of equations. We develop an approach (called *algebraic*) in which the outer estimation problem reduces to a problem of computing algebraic solutions of an auxiliary interval equation in Kaucher complete interval arithmetic.

1. Introduction

In our work, we will consider *generalized solution sets* for interval algebraic systems that naturally arise when interval parameters of a system express different kinds of uncertainty (ambiguity). We would like to remind that, basically, the interval data uncertainty and/or ambiguity can be understood in two ways, in accordance with the two-fold interpretation of the intervals.

In real life problems, one is hardly interested in intervals on their own, as integral and undivided objects, with no further internal structure. In most cases, we only use an interval $\mathbf{v}$ in connection with a property (let us denote it by P) that can be fulfilled or not for its point members. Under the circumstances, the following different situations may occur:

either the property $P(v)$ considered (that may be a point equation, inequality, etc.) holds for *all* members v from the given interval $\mathbf{v}$,

or the property $P(v)$ holds only for *some* members v from the interval $\mathbf{v}$, not necessarily all (maybe, only for one value).

In formal writing, this distinction is manifested in using the logical quantifiers—either the universal quantifier $\forall$ or the existential quantifier $\exists$:

- in the first case, we write "$(\forall v \in \mathbf{v})\ P(v)$" and shall speak of *$\forall$-type (A-type) of uncertainty*,
- in the second case, we write "$(\exists v \in \mathbf{v})\ P(v)$" and are going to speak of *$\exists$-type (E-type) of uncertainty*

(see also [13], [15], [17], [20]).

The above difference between the two uncertainty (ambiguity) types should be taken into account when strictly defining solutions and solution sets to interval equations, inequalities, etc. For instance, the most general definition of the solution set to the interval system of linear equations

$$\mathbf{A}x = \mathbf{b}, \tag{1.1}$$

with an interval $m \times n$-matrix $\mathbf{A} = (\mathbf{a}_{ij})$ and an interval right-hand side m-vector $\mathbf{b} = (\mathbf{b}_i)$, has the form

$$\{x \in \mathbb{R}^n \mid (Q_1 v_{\pi_1} \in \mathbf{v}_{\pi_1})(Q_2 v_{\pi_2} \in \mathbf{v}_{\pi_2}) \cdots (Q_{mn+m} v_{\pi_{mn+m}} \in \mathbf{v}_{\pi_{mn+m}})\ (Ax = b)\}, \tag{1.2}$$

where

$Q_1, Q_2, \ldots, Q_{mn+m}$
are the logical quantifiers $\forall$ or $\exists$,

$(v_1, v_2, \ldots, v_{mn+m}) := (a_{11}, \ldots, a_{mn}, b_1, \ldots, b_m) \in \mathbb{R}^{mn+m}$
is the aggregated (compound) parameter vector of the system of equations considered,

$(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{mn+m}) := (\mathbf{a}_{11}, \ldots, \mathbf{a}_{mn}, \mathbf{b}_1, \ldots, \mathbf{b}_m) \in \mathbb{IR}^{mn+m}$
is the aggregated vector of the intervals of the possible values of these parameters,

$(\pi_1, \pi_2, \ldots, \pi_{mn+m})$
is a permutation of the integers $1, 2, \ldots, mn + m$.

DEFINITION 1.1. The sets of the form (1.2) will be referred to as *generalized solution sets* to the interval system of equations $\mathbf{A}x = \mathbf{b}$.

DEFINITION 1.2. The logical formula written out after the vertical line in the definition of the set (1.2), which determines a characteristic property of the points of this set, will be called *selecting predicate* of the corresponding solution set (1.2) to the interval system of equations.

Definition 1.1 is very general. One can easily calculate, for example, that the number of the solution sets it comprehends far much exceeds even 2^{mn+m}. Such a great variety is, in particular, due to the fact that in logical formulas (the selecting predicates of the solution sets among them) the occurrences of the different quantifiers cannot be permuted with each other [8].

The generalized solution sets to interval equations and inequalities naturally come into being in operations research and decision making, they have interesting and significant applications. In our work, we shall not treat the solution sets of the most general form (1.2), with arbitrarily combined quantifiers at the interval parameters, but confine ourselves only to such solution sets of the interval equations for which the selecting predicate has *all the occurrences of the universal quantifier* $\forall$ *prior to the occurrences of the existential quantifier* $\exists$. To put it differently, we consider only the solution sets whose selecting predicate has *AE-form*. When interpreting in terms of systems analysis they simulate one-stage "perturbation-control" action on a system [13], [15].

DEFINITION 1.3. For the interval systems of equations, the generalized solution sets for which the selecting predicate has AE-form will be termed *AE-solution sets* (or *sets of AE-solutions*).

Such is, for example, *tolerable solution set**

$$\begin{aligned}\Sigma_{tol}(\mathbf{A},\mathbf{b}) = \{x \in \mathbb{R}^n \mid (\forall a_{11} \in \mathbf{a}_{11})(\forall a_{12} \in \mathbf{a}_{12}) \cdots (\forall a_{mn} \in \mathbf{a}_{mn})\\ (\exists b_1 \in \mathbf{b}_1)(\exists b_2 \in \mathbf{b}_2) \cdots (\exists b_m \in \mathbf{b}_m)(Ax = b)\},\end{aligned}$$

which corresponds to the case when all the entries of the matrix $\mathbf{A}$ have A-uncertainty and all the elements of the vector $\mathbf{b}$ have E-uncertainty. Usually, it is written in the following form

$$\begin{aligned}\Sigma_{tol}(\mathbf{A},\mathbf{b}) &= \{x \in \mathbb{R}^n \mid (\forall A \in \mathbf{A})(\exists b \in \mathbf{b})(Ax = b)\},\\ &= \{x \in \mathbb{R}^n \mid (\forall A \in \mathbf{A})(Ax \in \mathbf{b})\}\\ &= \{x \in \mathbb{R}^n \mid (\mathbf{A}x \subseteq \mathbf{b})\}.\end{aligned}$$

Another example, which is also subsumed under Definition 1.3, is *united solution set* of interval systems of equation, i.e., the set of solutions to all point systems with the coefficients from given intervals. For the interval linear system (1.1), it is strictly defined as

$$\begin{aligned}\Sigma_{uni}(\mathbf{A},\mathbf{b}) &= \{x \in \mathbb{R}^n \mid (\exists a_{11} \in \mathbf{a}_{11})(\exists a_{12} \in \mathbf{a}_{12}) \cdots (\exists a_{mn} \in \mathbf{a}_{mn})\\ &\qquad (\exists b_1 \in \mathbf{b}_1)(\exists b_2 \in \mathbf{b}_2) \cdots (\exists b_m \in \mathbf{b}_m)(Ax = b)\}\\ &= \{x \in \mathbb{R}^n \mid (\exists A \in \mathbf{A})(\exists b \in \mathbf{b})(Ax = b)\}\\ &= \{x \in \mathbb{R}^n \mid (\mathbf{A}x \cap \mathbf{b} \neq \emptyset)\}.\end{aligned}$$

2. Quantifier Formalization

In this section we consider, for the AE-solution sets, various possible ways of describing the uncertainty types distribution with respect to the interval parameters of the system.

* One can find surveys of the related results in [7], [10], [19]. For dynamic systems, an analog of this solution set is the *set of viable trajectories*, while the mathematical problem statement that gives rise to it is nothing but the *viability problem*.

1. As far as the order of the quantifiers is fixed, the simplest of such ways is to directly point out which quantifier is applied to this or that element of the interval system. Namely, let us introduce an $m \times n$-matrix $\alpha = (\alpha_{ij})$ and an m-vector $\beta = (\beta_i)$ made up of the logical quantifiers and such that

$$\alpha_{ij} := \begin{cases} \forall, & \text{if } \mathbf{a}_{ij} \text{ has A-uncertainty}, \\ \exists, & \text{if } \mathbf{a}_{ij} \text{ has E-uncertainty}, \end{cases}$$

$$\beta_i := \begin{cases} \forall, & \text{if } \mathbf{b}_i \text{ has A-uncertainty}, \\ \exists, & \text{if } \mathbf{b}_i \text{ has E-uncertainty}. \end{cases}$$

Specifying α and β, along with the interval system itself, completely determines the corresponding AE-solution set.

2. Another way to represent the uncertainty types corresponding to the elements of the interval linear system (1.1) is to trace out partitions of the index sets of both the entries of the matrix $\mathbf{A}$ and components of the right-hand side $\mathbf{b}$. More precisely, let the entire set of the index pairs (i,j) of the entries a_{ij}, that is, the set

$$\{(1,1),(1,2),\dots,(1,n),(2,1),(2,2),\dots,(2,n),\\ \dots,(m,1),(m,2),\dots,(m,n)\},$$

be divided into two nonintersecting parts $\hat{\Omega} := \{\hat{\omega}_1,\dots,\hat{\omega}_p\}$ and $\check{\Omega} := \{\check{\omega}_1,\dots,\check{\omega}_q\}$, $p+q=mn$, such that

a_{ij} is of the interval A-uncertainty for $(i,j) \in \hat{\Omega}$,

a_{ij} is of the interval E-uncertainty for $(i,j) \in \check{\Omega}$.

Similarly, we introduce nonintersecting sets of the integer indices $\hat{\Theta} := \{\hat{\vartheta}_1,\dots,\hat{\vartheta}_s\}$ and $\check{\Theta} := \{\check{\vartheta}_1,\dots,\check{\vartheta}_t\}$, $\hat{\Theta} \cup \check{\Theta} = \{1,2,\dots,m\}$, such that, in the right-hand side vector,

b_i is of the interval A-uncertainty for $i \in \hat{\Theta}$,

b_i is of the interval E-uncertainty for $i \in \check{\Theta}$.

Also, we allow the natural possibility for some of the sets $\hat{\Omega}$, $\check{\Omega}$, $\hat{\Theta}$, $\check{\Theta}$ to be empty. It is evident that

$$\alpha_{ij} = \begin{cases} \forall, & \text{if } (i,j) \in \hat{\Omega}, \\ \exists, & \text{if } (i,j) \in \check{\Omega}, \end{cases} \qquad \beta_i = \begin{cases} \forall, & \text{if } i \in \hat{\Theta}, \\ \exists, & \text{if } i \in \check{\Theta}, \end{cases}$$

and, again, determining $\hat{\Omega}$, $\check{\Omega}$, $\hat{\Theta}$, $\check{\Theta}$ results in a complete specification of an AE-solution set to the interval linear system (1.1).

3. The third way to describe the uncertainty types distribution for an interval linear system is to fix disjoint decompositions of both the interval matrix of the system and its right-hand side. Namely, we define interval matrices $\mathbf{A}^{\forall} = (\mathbf{a}_{ij}^{\forall})$ and $\mathbf{A}^{\exists} = (\mathbf{a}_{ij}^{\exists})$ and interval vectors $\mathbf{b}^{\forall} = (\mathbf{b}_i^{\forall})$ and $\mathbf{b}^{\exists} = (\mathbf{b}_i^{\exists})$, of the same sizes

as $\mathbf{A}$ and $\mathbf{b}$, as follows:

$$\mathbf{a}_{ij}^{\forall} := \begin{cases} \mathbf{a}_{ij}, & \text{if } \alpha_{ij} = \forall, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathbf{a}_{ij}^{\exists} := \begin{cases} \mathbf{a}_{ij}, & \text{if } \alpha_{ij} = \exists, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

$$\mathbf{b}_{i}^{\forall} := \begin{cases} \mathbf{b}_{i}, & \text{if } \beta_{i} = \forall, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathbf{b}_{i}^{\exists} := \begin{cases} \mathbf{b}_{i}, & \text{if } \beta_{i} = \exists, \\ 0, & \text{otherwise}. \end{cases} \tag{2.2}$$

Thus

$$\mathbf{A} = \mathbf{A}^{\forall} + \mathbf{A}^{\exists}, \qquad \mathbf{a}_{ij}^{\forall} \cdot \mathbf{a}_{ij}^{\exists} = 0,$$
$$\mathbf{b} = \mathbf{b}^{\forall} + \mathbf{b}^{\exists}, \qquad \mathbf{b}_{i}^{\forall} \cdot \mathbf{b}_{i}^{\exists} = 0$$

for all i, j. The matrix $\mathbf{A}^{\forall}$ and vector $\mathbf{b}^{\forall}$ concentrate all the interval elements of the system that corresponds to the A-uncertainty, while the matrix $\mathbf{A}^{\exists}$ and vector $\mathbf{b}^{\exists}$ stores all the elements that correspond to the interval E-uncertainty.

It should be stressed that the three groups of the objects considered which arise in connection with an AE-solution set of an interval linear system (1.1), namely

1) the quantifier matrix α and vector β,
2) decompositions of the index sets of the matrix $\mathbf{A}$ and of the right-hand side vector $\mathbf{b}$ to the nonintersecting subsets $\hat{\Omega}$, $\check{\Omega}$, $\hat{\Theta}$, $\check{\Theta}$,
3) disjoint decompositions of the interval matrix $\mathbf{A} = \mathbf{A}^{\forall} + \mathbf{A}^{\exists}$ and of the right-hand side vector $\mathbf{b} = \mathbf{b}^{\forall} + \mathbf{b}^{\exists}$,

are in a one-to-one correspondence, so that pointing out any one item of the above triple immediately determines the other two. We will extensively use all three descriptions and change any one for another without special explanations.

Summarizing, we can give the following

DEFINITION 2.1. Let us, for an interval linear system $\mathbf{A}x = \mathbf{b}$, be given a quantifier $m \times n$-matrix α and an m-vector β as well as the associated decompositions of the index sets of the matrix $\mathbf{A}$ and vector $\mathbf{b}$ to nonintersecting subsets $\hat{\Omega} = \{\hat{\omega}_1, \dots, \hat{\omega}_p\}$ and $\check{\Omega} = \{\check{\omega}_1, \dots, \check{\omega}_q\}$, $\hat{\Theta} = \{\hat{\vartheta}_1, \dots, \hat{\vartheta}_s\}$ and $\check{\Theta} = \{\check{\vartheta}_1, \dots, \check{\vartheta}_t\}$, $p + q = mn$, $s + t = m$, and disjoint decompositions $\mathbf{A} = \mathbf{A}^{\forall} + \mathbf{A}^{\exists}$ and $\mathbf{b} = \mathbf{b}^{\forall} + \mathbf{b}^{\exists}$.

AE-solution set of the type $\alpha\beta$ to the interval linear system $\mathbf{A}x = \mathbf{b}$ is the set

$$\begin{aligned} \Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b}) := \{x \in \mathbb{R}^n \mid \\ (\forall a_{\hat{\omega}_1} \in \mathbf{a}_{\hat{\omega}_1}) \cdots (\forall a_{\hat{\omega}_p} \in \mathbf{a}_{\hat{\omega}_p})\ (\forall b_{\hat{\vartheta}_1} \in \mathbf{b}_{\hat{\vartheta}_1}) \cdots (\forall b_{\hat{\vartheta}_s} \in \mathbf{b}_{\hat{\vartheta}_s}) \\ (\exists a_{\check{\omega}_1} \in \mathbf{a}_{\check{\omega}_1}) \cdots (\exists a_{\check{\omega}_q} \in \mathbf{a}_{\check{\omega}_q})\ (\exists b_{\check{\vartheta}_1} \in \mathbf{b}_{\check{\vartheta}_1}) \cdots (\exists b_{\check{\vartheta}_t} \in \mathbf{b}_{\check{\vartheta}_t}) \\ (Ax = b)\} \end{aligned} \tag{2.3}$$

or, which is equivalent, the set

$$\Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b}) := \{x \in \mathbb{R}^n \mid (\forall \hat{A} \in \mathbf{A}^{\forall})(\forall \hat{b} \in \mathbf{b}^{\forall})(\exists \check{A} \in \mathbf{A}^{\exists})(\exists \check{b} \in \mathbf{b}^{\exists})((\hat{A} + \check{A})\, x = \hat{b} + \check{b})\}.$$

3. Outer Estimation Problem

The intersections of the AE-solution sets to interval linear systems with each orthant of the space $\mathbb{R}^n$ are easily proved to be convex polyhedral sets (see [15]). They are defined by systems of linear inequalities whose coefficients are the endpoints of the interval elements of the system (1.1). In principle, one could give a direct description of an AE-solution set by writing out the equations of all its bounding hyperplanes in each orthant, etc. But in general the complexity of such a process may grow not slower than the total number of orthants, i.e., exponentially with the dimension of the space $\mathbb{R}^n$. The direct explicit description of the solution sets becomes, as a result, extremely difficult, tedious, and practically even useless as the dimension of the system under consideration increases*.

On the other hand, a full description of the solution set usually is not even necessary in real-life situations. It suffices to change the exact solution set for some approximation (estimate) of it which is sufficient for practical purposes. For example, viability analysis and some system identification problems require *inner* estimation of the solution sets to interval equations, that is, computing simple subsets of the solution sets (see e.g. [21]). Alternatively, when analyzing the parametric sensitivity of a control system, one is often required to know guaranteed estimates of the state set within which our compensating control actions are able to hold the system in spite of the presence of uncontrolled perturbations. This is the case when *outer* estimates are needed, and the corresponding problem is usually formulated as follows:

> Find (quick and as sharp as possible) outer coordinate estimates of the solution set $\Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ or, another way, evaluate $\inf\{x_k \mid x \in \Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})\}$ from below and $\sup\{x_k \mid x \in \Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})\}$ from above, $k = 1, \ldots, n$. (3.1)

In point of fact, the problem statement (3.1) prescribes seeking a box—rectangular parallelotope with the axis-aligned faces—that contains the solution set. The boxes are geometrical images of the interval vectors, so that we shall term a box enclosing the solution set as an *outer interval estimate* of this solution set. To sum up, it is convenient to reformulate the problem (3.1) in the following purely interval form:

> Find (quick and as sharp as possible) an outer interval estimate of the solution set $\Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ to a given interval linear system $\mathbf{A}x = \mathbf{b}$.

* Lakeyev managed to prove recently [9] that the complexity of recognition whether an AE-solution set to a given interval linear system is empty or not is NP-hard, that is, computationally intractable problem, provided that sufficiently many entries of $\mathbf{A}$ have the interval E-uncertainty.

The above problem is the main object under study in the present work. More precisely, we aim at developing a numerical technique for outer interval estimation of the generalized AE-solution sets to the interval systems (1.1). For simplicity, in the rest of the paper we consider only the interval linear systems $\mathbf{A}x = \mathbf{b}$ with the square $n \times n$-matrices.

In the theory that we are presenting below, *Kaucher complete interval arithmetic* $\mathbb{KR}$ plays a crucial role. This arithmetic is a natural completion of the classical interval arithmetic $\mathbb{IR}$, so that $\mathbb{IR} \subset \mathbb{KR}$. The distinctive feature of the arithmetic $\mathbb{KR}$ is the presence of *improper* intervals $[\underline{x}, \overline{x}]$, $\underline{x} > \overline{x}$, apart from the ordinary *proper* intervals $[\underline{x}, \overline{x}]$ with $\underline{x} \leq \overline{x}$ forming the classical interval arithmetic. As a whole, the complete interval arithmetic has good algebraic and inclusion order properties, which facilitates easier symbolic manipulations, etc.

We remind that the *dualization* of an interval $\mathbf{v} \in \mathbb{KR}$ is

$$\text{dual}\ \mathbf{v} := [\overline{\mathbf{v}}, \underline{\mathbf{v}}],$$

i.e., reversing its endpoints. For interval vectors and matrices, the dualization operation is taken componentwise. *Modulus* (magnitude) of an interval $\mathbf{v} \in \mathbb{KR}$ is defined as

$$|\mathbf{v}| := \max\{|\underline{\mathbf{v}}|, |\overline{\mathbf{v}}|\}.$$

By "opp", we will denote taking the opposite element in the complete arithmetic $\mathbb{KR}$, while "$\ominus$" is the inverse operation to the addition:

$$\begin{aligned} \text{opp}\ \mathbf{v} &:= [-\underline{\mathbf{v}}, -\overline{\mathbf{v}}], \\ \mathbf{u} \ominus \mathbf{v} &:= \mathbf{u} + \text{opp}\ \mathbf{v} = [\underline{\mathbf{u}} - \underline{\mathbf{v}}, \overline{\mathbf{u}} - \overline{\mathbf{v}}]. \end{aligned}$$

The definition of the inclusion ordering on $\mathbb{KR}$ is as follows:

$$\mathbf{u} \subseteq \mathbf{v} \quad \Longleftrightarrow \quad \underline{\mathbf{u}} \geq \underline{\mathbf{v}} \quad \text{and} \quad \overline{\mathbf{u}} \leq \overline{\mathbf{v}}.$$

The detailed description of Kaucher complete interval arithmetic can be found e.g. in the original works [3]–[5], or in [14], [16].

4. Characterizations of AE-Solution Sets

For the generalized AE-solution sets to interval linear systems (1.1), the following analytic characterization is known [15], [17]:

THEOREM 4.1. *A point $x \in \mathbb{R}^n$ belongs to the solution set $\Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ if and only if*

$$\mathbf{A}^{\forall} \cdot x - \mathbf{b}^{\forall} \subseteq \mathbf{b}^{\exists} - \mathbf{A}^{\exists} \cdot x, \tag{4.1}$$

where all the operations and relations are those of the classical interval arithmetic.

We introduce

DEFINITION 4.1. The interval matrix and interval vector

$$\mathbf{A}^c := \mathbf{A}^\forall + \text{dual } \mathbf{A}^\exists, \qquad \mathbf{b}^c := \text{dual } \mathbf{b}^\forall + \mathbf{b}^\exists$$

are called *characteristic* for the AE-solution set $\Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ to the interval linear system (1.1) specified by the disjoint decomposition of $\mathbf{A}$ into $\mathbf{A}^\forall$ and $\mathbf{A}^\exists$, and of $\mathbf{b}$ into $\mathbf{b}^\forall$ and $\mathbf{b}^\exists$.

The new language Definition 4.1 suggests enables us to speak of a *solution set that corresponds to the characteristic matrix* $\mathbf{A}^c$ *and right-hand side vector* $\mathbf{b}^c$ (in the last Section). $\mathbf{A}^c$ and $\mathbf{b}^c$ actually express, in a concentrated form, both the types of interval uncertainty of all parameters and their intervals proper. In addition, the new concepts facilitate rewriting the result of Theorem 4.2 in a more concise form:

THEOREM 4.2. *A point* $x \in \mathbb{R}^n$ *belongs to the solution set* $\Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ *if and only if*

$$\mathbf{A}^c \cdot x \subseteq \mathbf{b}^c \tag{4.2}$$

in complete interval arithmetic.

Proof. Notice that

$$\text{opp } (-\mathbf{v}) = \text{dual } \mathbf{v}$$

for any interval $\mathbf{v} \in \mathbb{IR}$. Therefore, adding (dual $\mathbf{b}^\forall$ + dual $(\mathbf{A}^\exists \cdot x)$) to both sides of (4.1) yields the following equivalent inclusion in the complete interval arithmetic

$$\mathbf{A}^\forall \cdot x + \text{dual } (\mathbf{A}^\exists \cdot x) \subseteq \text{dual } \mathbf{b}^\forall + \mathbf{b}^\exists. \tag{4.3}$$

Further, dual $(\mathbf{A}^\exists \cdot x) = (\text{dual } \mathbf{A}^\exists) \cdot x$, since x is a point. So, (4.3) is equivalent to

$$\mathbf{A}^\forall \cdot x + (\text{dual } \mathbf{A}^\exists) \cdot x \subseteq \text{dual } \mathbf{b}^\forall + \mathbf{b}^\exists.$$

In the left-hand side, we can avail ourselves of the distributivity with respect to the point variable x, which results in

$$(\mathbf{A}^\forall + \text{dual } \mathbf{A}^\exists) \cdot x \subseteq \text{dual } \mathbf{b}^\forall + \mathbf{b}^\exists,$$

and that coincides with (4.2). □

In our work, we will need a "fixed-point form characterization" of the AE-solution sets. To derive it, we add $(x \ominus \mathbf{A}^c x)$ to both sides of the inclusion (4.2), thus getting the equivalent relation

$$x \subseteq x + \text{opp } (\mathbf{A}^c x) + \mathbf{b}^c.$$

But opp $(\mathbf{A}^c x)$ = opp $(\mathbf{A}^c)x$ for the point x, we have therefore

$$x \subseteq x + (\text{opp } \mathbf{A}^c)x + \mathbf{b}^c,$$

Again, we can make use of the fact that x is a point and factor it out in the right-hand side due to the distributivity. Overall,

$$x \in \Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b}) \quad \Longleftrightarrow \quad x \subseteq (I \ominus \mathbf{A}^c)x + \mathbf{b}^c.$$

It should be stressed that for $x \in \Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b}) \neq \emptyset$ the above proof implies the interval vector $(I \ominus \mathbf{A}^c)x + \mathbf{b}^c$ being proper.

To summarize, we get the following

THEOREM 4.3. *A point $x \in \mathbb{R}^n$ belongs to the solution set $\Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ if and only if*

$$x \in (I \ominus \mathbf{A}^c)x + \mathbf{b}^c.$$

5. "Algebraic Approach" in Outer Estimation Problem

The problems of inner interval estimation of the generalized solution sets to interval systems of equations are known to be successfully solved by *algebraic approach* [13], [15], [17], a technique that changes the original estimation problem for a problem of computing an algebraic solution to an auxiliary equation in Kaucher complete interval arithmetic $\mathbb{KR}$. Recall

DEFINITION 5.1. An interval vector is called an *algebraic solution* to an interval equation (inequality, etc.) if substituting this vector into the equation and executing all interval operations according to the rules of the interval arithmetic result in an equality (inequality, etc.).

We are going to show how a similar approach (which we also shall call *algebraic*) may be applied to the problems of outer interval estimation of the AE-solution sets. An alternative technique that solves the same problem—generalized interval Gauss-Seidel iteration—has been presented in [18].

THEOREM 5.1. *Let an interval matrix $\mathbf{C} \in \mathbb{KR}^{n \times n}$ be such that the spectral radius $\rho(|\mathbf{C}|)$ of the matrix made up of the moduli of its entries is less than 1. Then, for any vector $\mathbf{d} \in \mathbb{KR}^n$, the algebraic solution to the interval linear system*

$$x = \mathbf{C}x + \mathbf{d} \tag{5.1}$$

exists and is unique.

Proof. In complete interval arithmetic $\mathbb{KR}$, the distance dist$(\cdot, \cdot)$ between the elements is known to be introduced as follows [5]:

$$\text{dist}(\mathbf{u}, \mathbf{v}) := \max\{|\underline{\mathbf{u}} - \underline{\mathbf{v}}|, |\overline{\mathbf{u}} - \overline{\mathbf{v}}|\} = |\mathbf{u} \ominus \mathbf{v}|.$$

It is worth noting as well that for any intervals $\mathbf{c}, \mathbf{u}, \mathbf{v} \in \mathbb{IR}$ the inequality

$$\operatorname{dist}(\mathbf{cu}, \mathbf{cv}) \le |\mathbf{c}| \cdot \operatorname{dist}(\mathbf{u}, \mathbf{v})$$

is valid (see also [5]). This estimate holds true for the multidimensional case too if the distance between $\mathbf{u}, \mathbf{v} \in \mathbb{IR}^n$ is understood as the componentwise vector-valued metric (*pseudometric* according to the terminology by Collatz [2]). More precisely, for the interval vectors $\mathbf{u}$, $\mathbf{v}$ we define

$$\operatorname{dist}(\mathbf{u}, \mathbf{v}) := \begin{pmatrix} \operatorname{dist}(\mathbf{u}_1, \mathbf{v}_1) \\ \vdots \\ \operatorname{dist}(\mathbf{u}_n, \mathbf{v}_n) \end{pmatrix} \in \mathbb{R}^n.$$

Then, for any interval matrix $\mathbf{C}$ with the elements $\mathbf{c}_{ij} \in \mathbb{IR}$ and any interval vectors $\mathbf{u}$, $\mathbf{v}$ of the corresponding size, we have

$$\operatorname{dist}(\mathbf{Cu}, \mathbf{Cv}) \le |\mathbf{C}| \cdot \operatorname{dist}(\mathbf{u}, \mathbf{v}). \tag{5.2}$$

To prove the inequality (5.2), let us remind that

$$\operatorname{dist}(\mathbf{y} + \mathbf{z}, \mathbf{y}' + \mathbf{z}') \le \operatorname{dist}(\mathbf{y}, \mathbf{y}') + \operatorname{dist}(\mathbf{z}, \mathbf{z}')$$

for any one-dimensional intervals $\mathbf{y}, \mathbf{y}', \mathbf{z}, \mathbf{z}' \in \mathbb{IR}$ (see [5]). We can therefore conclude that

$$\begin{aligned} \operatorname{dist}\big((\mathbf{Cu})_i, (\mathbf{Cv})_i\big) &= \operatorname{dist}\Big(\sum_{j=1}^{n} \mathbf{c}_{ij}\mathbf{u}_j, \sum_{j=1}^{n} \mathbf{c}_{ij}\mathbf{v}_j\Big) \\ &\le \sum_{j=1}^{n} \operatorname{dist}(\mathbf{c}_{ij}\mathbf{u}_j, \mathbf{c}_{ij}\mathbf{v}_j) \\ &\le \sum_{j=1}^{n} |\mathbf{c}_{ij}| \cdot \operatorname{dist}(\mathbf{u}_j, \mathbf{v}_j) \end{aligned}$$

for all $i = 1, 2, \dots, n$, which proves the multidimensional estimate (5.2).

In the situation under study, for any $\mathbf{d} \in \mathbb{IR}^n$

$$\operatorname{dist}(\mathbf{Cu} + \mathbf{d}, \mathbf{Cv} + \mathbf{d}) = \operatorname{dist}(\mathbf{Cu}, \mathbf{Cv}) \le |\mathbf{C}| \cdot \operatorname{dist}(\mathbf{u}, \mathbf{v}).$$

If the spectral radius of the matrix $|\mathbf{C}|$ is less than 1, then we can apply the finite-dimensional version of Schröder's fixed-point theorem (see e.g. [1], [2], [11], [12]). Namely, the map $\mathbb{IR}^n \to \mathbb{IR}^n$ which acts

$$\mathbf{x} \mapsto \mathbf{Cx} + \mathbf{d}$$

is a contraction with respect to the pseudometric "dist" and has thus a unique fixed-point that is an algebraic solution to the interval linear system (5.1). □

THEOREM 5.2. *Let an AE-solution set* $\Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ *of the interval linear system (1.1) be nonempty and* $\rho(|I \ominus \mathbf{A}^c|) < 1$. *Then the algebraic solution to the interval linear system*

$$x = (I \ominus \mathbf{A}^c)x + \mathbf{b}^c \tag{5.3}$$

(which exists and is unique by virtue of Theorem 5.1) is a proper interval vector enclosing the solution set $\Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$.

Proof. Assume that $\mathbf{x}^*$ is an algebraic solution to the interval linear system (5.3). We are going to show that for any point $\tilde{x} \in \Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ there holds $\tilde{x} \in \mathbf{x}^*$.

Due to Theorem 4.1, the membership $\tilde{x} \in \Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ is equivalent to the inclusion

$$\tilde{x} \in (I \ominus \mathbf{A}^c)\tilde{x} + \mathbf{b}^c. \tag{5.4}$$

Let us launch an iteration process in $\mathbb{IR}^n$ according to the following formulas:

$$\mathbf{x}^{(0)} := \tilde{x}, \tag{5.5}$$

$$\mathbf{x}^{(k+1)} := (I \ominus \mathbf{A}^c)\mathbf{x}^{(k)} + \mathbf{b}^c. \tag{5.6}$$

Using induction, it is fairly simple to prove that all the vectors generated by this process contain $\tilde{x}$. Indeed, for $\mathbf{x}^{(0)}$ it is true by construction. If $\tilde{x} \in \mathbf{x}^{(k)}$, then in view of (5.4) and inclusion monotonicity of the interval arithmetic operations in $\mathbb{IR}$ we arrive at

$$\tilde{x} \in (I \ominus \mathbf{A}^c)\tilde{x} + \mathbf{b}^c \subseteq (I \ominus \mathbf{A}^c)\mathbf{x}^{(k)} + \mathbf{b}^c = \mathbf{x}^{(k+1)}.$$

Therefore, $\tilde{x} \in \mathbf{x}^{(k)}$ for all integer k. In particular, the above means that all the interval vectors $\mathbf{x}^{(k)}$ must be *proper*.

Furthermore, the condition $\rho(|I \ominus \mathbf{A}^c|) < 1$ implies the convergence of the iteration process defined in the pseudometric space $\mathbb{IR}^n$ by the formulas (5.5)–(5.6) (see, e.g., [1], [2], [11], [12]). There is no difficulty realizing that the sequence $\mathbf{x}^{(k)}$, $k = 1, 2, \ldots$, converges to a fixed point of the map

$$\mathbf{x} \mapsto (I \ominus \mathbf{A}^c)\mathbf{x} + \mathbf{b}^c,$$

that is, to the unique algebraic solution $\mathbf{x}^*$ of the equation (5.3). Since the membership $\tilde{x} \in \mathbf{x}^{(k)}$ is equivalent to a system of $2n$ nonstrict inequalities, then it must hold in the limit as well,

$$\tilde{x} \in \lim_{k \to \infty} \mathbf{x}^{(k)} = \mathbf{x}^*,$$

while this limit interval $\mathbf{x}^*$ is proper too. □

6. Implementation

We conclude the paper with some comments on practical implementation of the technique developed, i.e., on the methods for computing algebraic solutions to the main equation (5.3) and overall applicability of our approach. Notice that Theorems 5.1–5.2 give the necessary theoretical basis for constructing stationary iteration algorithms relying upon Schröder's contracting mapping theorem.

Another possibility is the subdifferential Newton method (see e.g. [14]) whose convergence is substantiated rigorously for the interval linear systems (5.3) with the matrices $\mathbf{A}^c$ in which, along every row, the entries are either all proper or all improper. Empirically, it has been revealed that the method works well even for the general interval linear systems of the form (5.3), with arbitrarily mixed proper and improper entries in the matrix $\mathbf{A}^c$ (although then the algorithm is no longer subdifferential, it is *quasidifferential Newton method*).

The key point of the applicability of our algebraic approach is the reduction of the original interval system (1.1) to the form (5.3) so that the requirement $\rho(|I \ominus \mathbf{A}^c|) < 1$ is met. This cannot always be done.

In the classical problem of computing enclosures for the united solution set, one traditionally makes use of the so-called *preconditioning*—multiplying both sides of the system, from the left, by a point matrix. Such a transformation leads to widening of the united solution set, but a careful choice of the preconditioning point matrix improves the properties of the interval matrix of the system we thus obtain [6], [11]. Unfortunately, the above prescription fails when we turn to outer estimation of the generalized solution sets: they do not necessarily extend after the preconditioning, changing in a more complex way. Still, an outcome from our difficulty exists and it amounts to that we should precondition *the characteristic matrix and right-hand side vector* rather than the original interval system itself.

Let us turn to the analytical characterization of AE-solution sets that Theorem 4.2 gives:

$$x \in \Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b}) \qquad \Longleftrightarrow \qquad \mathbf{A}^c x \subseteq \mathbf{b}^c.$$

If Λ is a point $n \times n$-matrix, then $\mathbf{A}^c x \subseteq \mathbf{b}^c$ implies the inclusion $\Lambda(\mathbf{A}^c x) \subseteq \Lambda \mathbf{b}^c$. The interval matrix product is known to be non-associative in the general case, but for point Λ and x there holds the equality $\Lambda(\mathbf{A}^c x) = (\Lambda \mathbf{A}^c) x$. Finally, we get

$$x \in \Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b}) \qquad \Longrightarrow \qquad (\Lambda \mathbf{A}^c) x \subseteq \Lambda \mathbf{b}^c. \tag{6.1}$$

We can interpret $\Lambda \mathbf{A}^c$ and $\Lambda \mathbf{b}^c$ as characteristic matrix and right-hand side vector of another interval linear system, so the implication (6.1) establishes

THEOREM 6.1. *If Λ is a point $n \times n$-matrix, then the AE-solution set corresponding to the characteristic matrix $\mathbf{A}^c$ and right-hand side vector $\mathbf{b}^c$ is contained in the AE-solution set corresponding to the characteristic matrix $\Lambda \mathbf{A}^c$ and right-hand side vector $\Lambda \mathbf{b}^c$.*

We can therefore replace our main problem (3.1) with the outer estimation of an AE-solution set defined by the new characteristic matrix $\Lambda \mathbf{A}^c$ and right-hand side vector $\Lambda \mathbf{b}^c$. If the interval matrix of the original system is not "too large," one may hope that a suitable choice of Λ will cause the spectral radius $\rho(|I \ominus \Lambda \mathbf{A}^c|)$

to become actually less than one. It is worth noting that, similar to the traditional case, taking Λ as the inverse to the middle of **A** works reasonably well. Overall, we can consider the procedure summarized in Theorem 6.1 as a kind of *generalized preconditioning* of the interval linear system (1.1). Its detailed analysis is going to be presented in an expanded version of this short note.

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