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Interval Gauss-Seidel Method for Generalized Solution Sets to Interval Linear Systems

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Abstract. In the paper, we advance a numerical technique for enclosing *generalized AE-solution sets* to interval linear systems. The main result of the paper is an extension of the well-known interval Gauss-Seidel method to the problems of outer estimation of these generalized solution sets. We give a theoretical study of the new method, prove an optimality property for the generalized interval Gauss-Seidel iteration applied to the systems with the interval *M*-matrices.

1. Introduction

The subject matter of our work is interval linear algebraic systems of the form

$$\mathbf{A}x = \mathbf{b}, \tag{1.1}$$

with an interval $m \times n$-matrix $\mathbf{A} = (\mathbf{a}_{ij})$ and interval right-hand side m-vector $\mathbf{b} = (\mathbf{b}_i)$. We will consider the so-called *generalized solution sets* for (1.1), that naturally arise in the situations when the interval parameters of the system express different uncertainty types. We would like to remind that we can take the interval data uncertainty in two ways, in accordance with the two-fold interpretation of the intervals.

DEFINITION 1.1 [15]. *Generalized solution sets* to the interval system of equations $\mathbf{A}x = \mathbf{b}$ are the sets of the form

$$\{x \in \mathbb{R}^n \mid (Q_1 v_{\pi_1} \in \mathbf{v}_{\pi_1})(Q_2 v_{\pi_2} \in \mathbf{v}_{\pi_2}) \ \dots \ (Q_{mn+m} v_{\pi_{mn+m}} \in \mathbf{v}_{\pi_{mn+m}})(Ax = b)\}, \tag{1.2}$$

where

$Q_1, Q_2, \dots, Q_{mn+m}$
are the logical quantifiers $\forall$ or $\exists$,

$(v_1, v_2, \dots, v_{mn+m}) := (a_{11}, \dots, a_{mn}, b_1, \dots, b_m) \in \mathbb{R}^{mn+m}$
is the aggregated (compound) parameter vector of the system of equations considered,

$(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{mn+m}) := (\mathbf{a}_{11}, \ldots, \mathbf{a}_{mn}, \mathbf{b}_1, \ldots, \mathbf{b}_m) \in \mathbb{IR}^{mn+m}$
is the aggregated vector of the intervals of the possible values of these parameters,

$(\pi_1, \pi_2, \ldots, \pi_{mn+m})$
is a permutation of the integers $1, 2, \ldots, mn + m$.

The logical formula written out after the vertical line in the definition of the set (1.2) that determines a distinctive property of its points will be called *selecting predicate* of the corresponding solution set to the interval system of equations.

The interval parameters of the system (1.1) occurring with the universal quantifier "$\forall$" in the selecting predicate of the solution set (1.2) will be referred to as having *interval A-uncertainty*, while the interval parameters standing with the existential quantifier "$\exists$" will be referred to as having *interval E-uncertainty*.

The generalized solution sets to interval equations and systems of equations naturally come into existence in operations research and decision making theory, where they have interesting and significant applications (see e.g. [13]). Further, we shall not consider the most general solution sets to interval equations, with arbitrarily permuted quantifiers at the interval parameters, but confine ourselves only to such solution sets of the form (1.2) that, in their selecting predicates, *all occurrences of the universal quantifier "$\forall$" precede the occurrences of the existential quantifier "$\exists$"*. In other words, we will only deal with the solution sets for which the selecting predicate has *AE-form*. When interpreting in terms of systems analysis, they simulate one stage of a "perturbation-control" action on a system [10], [13].

DEFINITION 1.2. The generalized solution sets to interval systems of equations for which the selecting predicate has AE-form will be termed *AE-solution sets* (or *sets of AE-solutions*).

Definition 1.2 embraces, for instance, the well-known *united solution set*

$$\Xi_{uni}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid (\exists A \in \mathbf{A})(\exists b \in \mathbf{b})(Ax = b)\}$$

i.e., the set of solutions to all point systems $Ax = b$ with the coefficients $A \in \mathbf{A}$ and $b \in \mathbf{b}$, that corresponds to the case when all the entries of both the matrix $\mathbf{A}$ and right-hand side vector $\mathbf{b}$ are of E-uncertainty.

It is known (see e.g. [15]) that, for the AE-solution sets, there exist three different possible ways of describing the interval uncertainty distribution with respect to the elements of the system. As far as the order of the quantifiers is fixed, the simplest of such ways is to directly point out which quantifier is applied to this or that element of the interval system. Namely, we introduce $m \times n$-matrix $\alpha = (\alpha_{ij})$ and m-vector $\beta = (\beta_i)$ made up of the logical quantifiers and such that

$$\alpha_{ij} := \begin{cases} \forall, & \text{if } \mathbf{a}_{ij} \text{ has A-uncertainty,} \\ \exists, & \text{if } \mathbf{a}_{ij} \text{ has E-uncertainty;} \end{cases}$$

$$\beta_i := \begin{cases} \forall, & \text{if } \mathbf{b}_i \text{ has A-uncertainty}, \\ \exists, & \text{if } \mathbf{b}_i \text{ has E-uncertainty}. \end{cases}$$

Another way to represent the uncertainty types corresponding to the elements of the interval linear system (1.1) is to specify partitions of the index sets of both the entries of the matrix **A** and components of the right-hand side **b**. More precisely, let the entire set of the index pairs (i,j) of the entries a_{ij}, that is, the set

$$\{(1,1),(1,2),\dots,(1,n),(2,1),(2,2),\dots,(2,n),\dots,(m,1),(m,2),\dots,(m,n)\},$$

be divided into two nonintersecting parts $\hat{\Upsilon} := \{\hat{\upsilon}_1,\dots,\hat{\upsilon}_p\}$, and $\check{\Upsilon} := \{\check{\upsilon}_1,\dots,\check{\upsilon}_q\}$, $p+q=mn$, such that

the parameter a_{ij} has the interval A-uncertainty for $(i,j)\in\hat{\Upsilon}$, and

the parameter a_{ij} has the interval E-uncertainty for $(i,j)\in\check{\Upsilon}$.

Similarly, we introduce nonintersecting sets of the integer indices $\hat{\Theta} = \{\hat{\vartheta}_1,\dots,\hat{\vartheta}_s\}$ and $\check{\Theta} = \{\check{\vartheta}_1,\dots,\check{\vartheta}_t\}$, $\hat{\Theta}\cup\check{\Theta} = \{1,2,\dots,m\}$, such that, in the right-hand side,

the parameter b_i has the interval A-uncertainty for $i\in\hat{\Theta}$, and

the parameter b_i has the interval E-uncertainty for $i\in\check{\Theta}$.

We allow the natural possibility for some of the sets $\hat{\Upsilon}$, $\check{\Upsilon}$, $\hat{\Theta}$, $\check{\Theta}$ to be empty. It is apparent that

$$\alpha_{ij} = \begin{cases} \forall, & \text{if } (i,j)\in\hat{\Upsilon}, \\ \exists, & \text{if } (i,j)\in\check{\Upsilon}, \end{cases} \qquad \beta_i = \begin{cases} \forall, & \text{if } i\in\hat{\Theta}, \\ \exists, & \text{if } i\in\check{\Theta}. \end{cases}$$

The third way to specify the uncertainty types distribution for the interval linear system is to determine disjoint decompositions of the interval matrix of the system and its right-hand side vector. We define interval matrices $\mathbf{A}^{\forall} = (\mathbf{a}^{\forall}_{ij})$ and $\mathbf{A}^{\exists} = (\mathbf{a}^{\exists}_{ij})$ and interval vectors $\mathbf{b}^{\forall} = (\mathbf{b}^{\forall}_i)$ and $\mathbf{b}^{\exists} = (\mathbf{b}^{\exists}_i)$ of the same sizes as **A** and **b** as follows:

$$\mathbf{a}^{\forall}_{ij} := \begin{cases} \mathbf{a}_{ij}, & \text{if } \alpha_{ij} = \forall, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathbf{a}^{\exists}_{ij} := \begin{cases} \mathbf{a}_{ij}, & \text{if } \alpha_{ij} = \exists, \\ 0, & \text{otherwise}, \end{cases} \tag{1.3}$$

$$\mathbf{b}^{\forall}_{i} := \begin{cases} \mathbf{b}_{i}, & \text{if } \beta_{i} = \forall, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathbf{b}^{\exists}_{i} := \begin{cases} \mathbf{b}_{i}, & \text{if } \beta_{i} = \exists, \\ 0, & \text{otherwise}. \end{cases} \tag{1.4}$$

Thus

$$\mathbf{A} = \mathbf{A}^{\forall} + \mathbf{A}^{\exists}, \qquad \mathbf{a}^{\forall}_{ij}\cdot\mathbf{a}^{\exists}_{ij} = 0,$$
$$\mathbf{b} = \mathbf{b}^{\forall} + \mathbf{b}^{\exists}, \qquad \mathbf{b}^{\forall}_{i}\cdot\mathbf{b}^{\exists}_{i} = 0$$

for all i,j, that is, the matrices $\mathbf{A}^{\forall}$, $\mathbf{A}^{\exists}$ and vectors $\mathbf{b}^{\forall}$, $\mathbf{b}^{\exists}$ really form disjoint decompositions for **A** and **b** respectively.

The matrix $\mathbf{A}^{\forall}$ and vector $\mathbf{b}^{\forall}$ concentrate all the interval elements of the system that correspond to the A-uncertainty, while the matrix $\mathbf{A}^{\exists}$ and vector $\mathbf{b}^{\exists}$ store all the elements that correspond to the interval E-uncertainty.

It should be stressed that the three groups of the objects considered which arise in connection with the interval linear system (1.1), namely

1) the quantifier matrix α and vector β,
2) decompositions of the index sets of the matrix and of the right-hand side vector to the nonintersecting subsets $\hat{\Upsilon}$, $\check{\Upsilon}$, $\hat{\Theta}$, $\check{\Theta}$,
3) disjoint decompositions of the interval matrix $\mathbf{A} = \mathbf{A}^{\forall} + \mathbf{A}^{\exists}$ and of the right-hand side vector $\mathbf{b} = \mathbf{b}^{\forall} + \mathbf{b}^{\exists}$,

are in one-to-one correspondence, so that pointing out any one item of the above triple immediately determines the other two. We will use all three descriptions and change any one for another without special explanations.

We can give the following

DEFINITION 1.3. Let us, for an interval linear system $\mathbf{A}x = \mathbf{b}$, be given

- quantifier $m \times n$-matrix α and m-vector β, as well as the associated
- decompositions of the index sets of the matrix $\mathbf{A}$ and vector $\mathbf{b}$ to nonintersecting subsets $\hat{\Upsilon} = \{\hat{\upsilon}_1, \dots, \hat{\upsilon}_p\}$ and $\check{\Upsilon} = \{\check{\upsilon}_1, \dots, \check{\upsilon}_q\}$, $\hat{\Theta} = \{\hat{\vartheta}_1, \dots, \hat{\vartheta}_s\}$ and $\check{\Theta} = \{\check{\vartheta}_1, \dots, \check{\vartheta}_t\}$, $p + q = mn$, $s + t = m$, and
- the disjoint decompositions $\mathbf{A} = \mathbf{A}^{\forall} + \mathbf{A}^{\exists}$ and $\mathbf{b} = \mathbf{b}^{\forall} + \mathbf{b}^{\exists}$.

We will call the set

$$\begin{aligned} &\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b}) \\ &\quad := \{x \in \mathbb{R}^n \mid (\forall a_{\hat{\upsilon}_1} \in \mathbf{a}_{\hat{\upsilon}_1}) \dots (\forall a_{\hat{\upsilon}_p} \in \mathbf{a}_{\hat{\upsilon}_p})(\forall b_{\hat{\vartheta}_1} \in \mathbf{b}_{\hat{\vartheta}_1}) \dots (\forall b_{\hat{\vartheta}_s} \in \mathbf{b}_{\hat{\vartheta}_s}) \\ &\qquad (\exists a_{\check{\upsilon}_1} \in \mathbf{a}_{\check{\upsilon}_1}) \dots (\exists a_{\check{\upsilon}_q} \in \mathbf{a}_{\check{\upsilon}_q})(\exists b_{\check{\vartheta}_1} \in \mathbf{b}_{\check{\vartheta}_1}) \dots (\exists b_{\check{\vartheta}_t} \in \mathbf{b}_{\check{\vartheta}_t})(Ax = b)\} \end{aligned} \tag{1.5}$$

AE-solution set of the type $\alpha\beta$ to the interval linear $m \times n$-system of equations $\mathbf{A}x = \mathbf{b}$.

THEOREM 1.1. *For any quantifiers* α *and* β*, the intersection of the AE-solution set* $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ *with each orthant of the space* $\mathbb{R}^n$ *is a convex polyhedral set whose vertices are solutions of the “boundary” point linear systems* $Ax = b$*, i.e., such that*

$$\begin{aligned} A &= (a_{ij}), \qquad a_{ij} = \text{either } \underline{\mathbf{a}}_{ij} \text{ or } \overline{\mathbf{a}}_{ij}, \\ b &= (b_i), \qquad b_i = \text{either } \underline{\mathbf{b}}_i \text{ or } \overline{\mathbf{b}}_i. \end{aligned}$$

Proof. The membership of a point vector x to an orthant of $\mathbb{R}^n$ is determined by pointing out the signs of its components. It is worthwhile to note that, for any interval

$m \times n$-matrix $\mathbf{C}$, the components of the product $\mathbf{C}\cdot x = ((\mathbf{C}\cdot x)_1, (\mathbf{C}\cdot x)_2, \ldots, (\mathbf{C}\cdot x)_m)^\top$ can be represented as follows:

$$(\mathbf{C}\cdot x)_i = \sum_{j=1}^{n} \mathbf{c}_{ij} x_j = \left[\sum_{j=1}^{n} \underline{\mathbf{c}_{ij} x_j},\ \sum_{j=1}^{n} \overline{\mathbf{c}_{ij} x_j} \right] = \left[\sum_{j=1}^{n} c'_{ij} x_j,\ \sum_{j=1}^{n} c''_{ij} x_j \right], \tag{1.6}$$

where c'_{ij} and c''_{ij} are some numbers (they may coincide), which belong to the set of endpoints $\{\underline{\mathbf{c}}_{ij}, \overline{\mathbf{c}_{ij}}\}$ and are fixed for each individual orthant.

Writing out the inclusions (2.1) componentwise and changing, on the base of the representation (1.6), each of the one-dimensional inclusions for the pair of inequalities between the endpoints of the intervals, we get a system of $3n$ linear inequalities

$$\begin{cases} A'x \geq b', \\ A''x \leq b'', \\ \text{condition for the signs of } x_i, \ \ i = 1, 2, \ldots, n, \end{cases} \tag{1.7}$$

where A', A'' and b', b'' are made up of the endpoints of $\mathbf{a}_{ij}$ and $\mathbf{b}_i$ respectively. The system of inequalities (1.7) determines a convex polyhedral set. □

At the worst, the description of an AE-solution set thus includes 2^n different linear inequalities systems (its own for each separate orthant). Therefore, for n being merely several tens the exact description may be extremely difficult and practically useless.

Usually, it is sufficient to somehow estimate the AE-solution set, i.e., to approximate it by a simple set. The shape and location of such an estimate with respect to the solution set should depend on the practical sense of the problem under solution. For example, some identification problems require *inner* estimation, that is, computing subsets of the solution set. Conversely, the parametric sensitivity problems for the control systems require *outer* estimation of the solution sets, considering that we must take into account all possible variations of the system state.

The purpose of our work is to advance a numerical technique for outer interval estimation of the generalized AE-solution sets to interval linear systems of equations of the form (1.1), and the problem under solution can be formulated in the following purely interval form:

> *Find (quick and as sharp as possible) an outer interval estimate of the solution set* $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ *to a given interval linear system* $\mathbf{A}x = \mathbf{b}$. (1.8)

For the sake of simplicity, we confine ourselves only to the square systems with the interval $n \times n$-matrices.

2. Preliminaries

In the theory we are constructing below, *complete interval arithmetic* $\mathbb{IR}$ (also termed *Kaucher interval arithmetic*) plays one of the leads. It is a natural completion of the classical interval arithmetic $\mathbb{I}\mathbb{R}$, so $\mathbb{I}\mathbb{R} \subset \mathbb{IR}$. A peculiarity of Kaucher arithmetic is that the basic set of $\mathbb{IR}$ consists of both ordinary *proper* intervals $[\underline{a}, \overline{a}]$ with $\underline{a} \leq \overline{a}$ and *improper* intervals $[\underline{a}, \overline{a}]$ with $\underline{a} > \overline{a}$. As a whole, the complete interval arithmetic has good algebraic properties and is a conditionally complete lattice with respect to the inclusion ordering "$\subseteq$". The reader can find a more detailed description of the complete interval arithmetic both in the original works [4], [5] or in [10]–[12].

For an interval $\mathbf{v} = [\underline{\mathbf{v}}, \overline{\mathbf{v}}] \in \mathbb{IR}$, we define

$$\begin{aligned} \text{dual}\,\mathbf{v} &:= [\overline{\mathbf{v}}, \underline{\mathbf{v}}] && \text{— the dualization operation,} \\ |\mathbf{v}| &:= \max\{|\underline{\mathbf{v}}|, |\overline{\mathbf{v}}|\} && \text{— the absolute value (magnitude).} \end{aligned}$$

"opp" denotes taking the algebraic opposite element in the complete interval arithmetic $\mathbb{IR}$, and "$\ominus$" is the operation which is inverse to the addition, that is,

$$\begin{aligned} \text{opp}\,\mathbf{v} &:= [-\underline{\mathbf{v}}, -\overline{\mathbf{v}}], \\ \mathbf{u} \ominus \mathbf{v} &:= \mathbf{u} + \text{opp}\,\mathbf{v} = [\underline{\mathbf{u}} - \underline{\mathbf{v}},\ \overline{\mathbf{u}} - \overline{\mathbf{v}}]. \end{aligned}$$

For interval vectors and matrices, the operations "dual", "opp" and "$\ominus$" are defined componentwise.

DEFINITION 2.1. [15] The interval matrix and vector

$$\mathbf{A}^c := \mathbf{A}^{\forall} + \text{dual}\,\mathbf{A}^{\exists}, \qquad \mathbf{b}^c := \text{dual}\,\mathbf{b}^{\forall} + \mathbf{b}^{\exists}$$

will be called *characteristic* for the AE-solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ to the interval linear system (1.1) specified by the disjoint decomposition of $\mathbf{A}$ into $\mathbf{A}^{\forall}$ and $\mathbf{A}^{\exists}$, of $\mathbf{b}$ into $\mathbf{b}^{\forall}$ and $\mathbf{b}^{\exists}$.

The following analytic characterization is known [15] for the AE-solution sets of the interval linear systems (1.1):

THEOREM 2.1. *The point $x \in \mathbb{R}^n$ belongs to the solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ if and only if*

$$\mathbf{A}^c \cdot x \subseteq \mathbf{b}^c \tag{2.1}$$

in Kaucher complete interval arithmetic.

DEFINITION 2.2. An interval vector is called an *algebraic solution* to the interval equation (inequality, etc.) if substituting this vector into the equation and executing all interval operations according to the rules of the interval arithmetic result in an equality (inequality, etc.).

The interval [0, 1] is, for example, the algebraic solution to the quadratic interval equation

$$[1,2]\,x^2 + [-1,1]\,x = [-1,3].$$

It has been known that the problems of both inner and outer interval estimation of the generalized solution sets to interval linear systems are successfully solved by the *algebraic approach* [10]–[12], [14], [15], in which we replace the original estimation problem by a problem of computing an algebraic solution to an auxiliary equation in Kaucher complete interval arithmetic $\mathbb{KR}$. In the rest of this section, we summarize our results on the algebraic approach for outer estimation of the AE-solution sets to interval linear systems that will be extensively used further in our study.

An interval matrix $\mathbf{S} \in \mathbb{IR}^{n\times n}$ will be referred to as *invertible* providing that there exists an interval matrix, denoted $\mathbf{S}^{-1}$, such that $\mathbf{S}\,\mathbf{S}^{-1} = \mathbf{S}^{-1}\mathbf{S} = I$—identity matrix. It is fairly simple to realize that diagonal matrices $\mathbf{S} = \mathrm{diag}\{\mathbf{s}_1, \dots, \mathbf{s}_n\}$ are invertible if and only if $\underline{\mathbf{s}}_i\overline{\mathbf{s}}_i > 0$, $i = 1, 2, \dots, n$, and

$$\mathbf{S}^{-1} = \mathrm{diag}\{\mathbf{s}_1^{-1}, \dots, \mathbf{s}_n^{-1}\}.$$

THEOREM 2.2. *Let, for an interval matrix $\mathbf{G} \in \mathbb{IR}^{n\times n}$, there exist an interval diagonal invertible matrix $\mathbf{S} \in \mathbb{IR}^{n\times n}$ such that*

$$\rho(|\mathbf{S}^{-1}|\,|\mathbf{G}|) < 1$$

—the spectral radius of the product $|\mathbf{S}^{-1}|\,|\mathbf{G}|$ of matrices made up of the magnitudes of the entries of $\mathbf{S}^{-1}$ and $\mathbf{G}$ respectively is less than 1. Then, for any vector $\mathbf{h} \in \mathbb{IR}^n$, the algebraic solution to the interval linear system

$$\mathbf{S}x = \mathbf{G}x + \mathbf{h} \tag{2.2}$$

exists, is unique and equal to the limit of the following iteration

$$\mathbf{x}^{(k+1)} = \mathbf{S}^{-1}(\mathbf{G}\mathbf{x}^{(k)} + \mathbf{h}), \qquad k = 0, 1, \dots,$$

from any starting point $\mathbf{x}^{(0)}$.

THEOREM 2.3. *Let, for an interval linear system $\mathbf{A}x = \mathbf{b}$, a set of AE-solutions $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ be nonempty, $\mathbf{A}^{\mathrm{c}}$ and $\mathbf{b}^{\mathrm{c}}$ be the corresponding characteristic matrix and right-hand side vector, and there exists a diagonal invertible interval matrix $\mathbf{S} \in \mathbb{IR}^{n\times n}$, such that*

$$\rho(|\mathbf{S}^{-1}|\,|\mathbf{S} \ominus \mathbf{A}^{\mathrm{c}}|) < 1.$$

Then the algebraic solution to the interval system

$$\mathbf{S}x = (\mathbf{S} \ominus \mathbf{A}^{\mathrm{c}})\,x + \mathbf{b}^{\mathrm{c}} \tag{2.3}$$

(which exists and is unique due to Theorem 2.2) is a proper interval vector enclosing the solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$.

The algebraic solution to the interval system (2.3) thus provides us, under certain conditions, with a solution to the main problem (1.8).

What is the quality of the outer interval estimation of the AE-solution sets to interval linear systems that one can obtain by using the above results? To put it differently, how close is the interval enclosure to the interval hull of the solution set, which is the best possible outer estimate? We are not going to inquire into this interesting question in the general case. One should only bear in mind that the theoretical results in complexity theory [7], [8] show that the problem of estimation turns out to be intractable provided that we do not impose restrictions on the interval matrix of the system. Moreover, as Lakeyev [8] managed to prove recently, if the interval matrix $\mathbf{A}$ has "sufficiently many" entries with E-uncertainty, then the problem of recognition whether the corresponding solution set is empty or not (as well as the problem of its estimation) is NP-hard. Hence, we cannot hope to compute "good" enclosures of the AE-solution sets to interval linear systems in the general case. Still, there holds

THEOREM 2.4 [14]. *Let the interval system*

$$\mathbf{S}x = (\mathbf{S} \ominus \mathbf{A}^{c})x + \mathbf{b}^{c} \tag{2.3}$$

meets the requirements of Theorem 2.3 and, additionally, the matrix

$$\mathbf{S}^{-1}(\mathbf{S} \ominus \mathbf{A}^{c})$$

be nonnegative. Then the algebraic solution to the system (2.3) (which exists, is unique and gives an outer interval estimate for $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ *by virtue of Theorems 2.2–2.3) is the interval hull of* $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$*, i.e., the best possible solution to (1.8).*

3. Generalized Interval Gauss-Seidel Iteration

Interval Gauss-Seidel method is known to be one of the most efficient and popular algorithms for computing of the outer interval estimates (enclosures) of the united solution set to interval linear systems of equations. The method is usually used after preliminary *preconditioning* of the interval systems (see, e.g., [6], [9]). The purpose of this section is to adapt the interval Gauss-Seidel iteration to the problems of outer interval estimation of the generalized AE-solution sets. Below, we suppose that the interval matrix $\mathbf{A}$ is nonsingular, i.e., that all the point matrices $A \in \mathbf{A}$ are nonsingular. One can achieve then, after suitable permutation of the equations (matrix rows), that the diagonal entries $\mathbf{a}_{ii}$, $i = 1, 2, \dots, n$, do not contain zeros.

The basis of the point Gauss-Seidel method is writing out the system of equations $Ax = b$ in the explicit componentwise manner

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad i = 1, \dots, n,$$

and further solving the i-th equation with respect to x_i assuming that $a_{ii} \neq 0$:

$$x_i = a_{ii}^{-1} \left(b_i - \sum_{j \neq i} a_{ij} x_j \right), \qquad i = 1, 2, \ldots, n.$$

When iterating, to find the i-th component of the next $(k+1)$-th approximation to the solution $(k = 0, 1, \ldots)$, we involve the newly computed $(k+1)$-th approximations of the preceding components $1, 2, \ldots, i-1$, along with the old k-th values of the components $i+1, \ldots, n$. The overall formulas of the classical point Gauss-Seidel method look thus as follows:

$$x_i^{(k+1)} := a_{ii}^{-1} \left(b_i - \sum_{j<i} a_{ij} x_j^{(k+1)} - \sum_{j>i} a_{ij} x_j^{(k)} \right), \quad i = 1, 2, \ldots, n, \quad k = 0, 1, \ldots$$

To construct the interval method, we shall act in a similar way.

Let us make use of the characterization of AE-solution sets presented by Theorem 2.1:

$$x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b}) \quad \Longleftrightarrow \quad \mathbf{A}^{\mathrm{c}} x \subseteq \mathbf{b}^{\mathrm{c}}. \tag{3.1}$$

Breaking down the inclusion (3.1) componentwise, we get

$$\sum_{j=1}^{n} \mathbf{a}_{ij}^{\mathrm{c}} x_j \subseteq \mathbf{b}_i^{\mathrm{c}}, \qquad i = 1, \ldots, n,$$

which is equivalent to

$$\mathbf{a}_{ii}^{\mathrm{c}} x_i \subseteq \operatorname{opp} \left(\sum_{j \neq i} \mathbf{a}_{ij}^{\mathrm{c}} x_j \right) + \mathbf{b}_i^{\mathrm{c}}, \qquad i = 1, \ldots, n.$$

If we are already given an interval vector $\mathbf{x}$ containing the solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$, then we have for any $x \in \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ and $i = 1, 2, \ldots, n$

$$\begin{aligned} x_i &\subseteq (\mathbf{a}_{ii}^{\mathrm{c}})^{-1} \left(\operatorname{opp} \sum_{j \neq i} \mathbf{a}_{ij}^{\mathrm{c}} x_j + \mathbf{b}_i^{\mathrm{c}} \right) \\ &= (\mathbf{a}_{ii}^{\mathrm{c}})^{-1} \left(\sum_{j \neq i} \operatorname{opp} (\mathbf{a}_{ij}^{\mathrm{c}} x_j) + \mathbf{b}_i^{\mathrm{c}} \right) \\ &= (\mathbf{a}_{ii}^{\mathrm{c}})^{-1} \left(\sum_{j \neq i} (\operatorname{opp} \mathbf{a}_{ij}^{\mathrm{c}}) x_j + \mathbf{b}_i^{\mathrm{c}} \right) \\ &\subseteq (\mathbf{a}_{ii}^{\mathrm{c}})^{-1} \left(\sum_{j \neq i} (\operatorname{opp} \mathbf{a}_{ij}^{\mathrm{c}}) \mathbf{x}_j + \mathbf{b}_i^{\mathrm{c}} \right) =: \tilde{\mathbf{x}}_i. \end{aligned} \tag{3.2}$$

The interval vector $\tilde{\mathbf{x}}$ determined by (3.2) thereby

Table 1. Generalized interval Gauss-Seidel method.

Input

Characteristic matrix $\mathbf{A}^{c} \in \mathbb{IR}^{n\times n}$ and right-hand side vector $\mathbf{b}^{c} \in \mathbb{IR}^{n}$ corresponding to the AE-solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ under estimation of an interval linear system $\mathbf{A}x = \mathbf{b}$.

An interval vector $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n)^\top \in \mathbb{IR}^n$ bounding the desired portion of the solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$.

A prescribed accuracy $\varepsilon > 0$.

Output

Either the information "the solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ does not intersect the initial vector $\mathbf{x}$" or a new outer estimate $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_n)^\top$ of the solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b}) \cap \mathbf{x}$.

Algorithm

$d := +\infty$;
DO WHILE $(d \geq \varepsilon)$
  DO FOR $i = 1$ TO n
    $\tilde{\mathbf{x}}_i := (\mathbf{a}^{c}_{ii}c)^{-1}\left(\sum_{j=1}^{i-1}(\operatorname{opp}\,\mathbf{a}^{c}_{ij}c)\,\tilde{\mathbf{x}}_j + \sum_{j=i+1}^{n}(\operatorname{opp}\,\mathbf{a}^{c}_{ij}c)\mathbf{x}_j + \mathbf{b}^{c}_{i}c\right)$;
    IF ($\tilde{\mathbf{x}}_i$ is an improper interval) THEN
      STOP, signaling "the solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ does not intersect $\mathbf{x}$";
    $\tilde{\mathbf{x}}_i := \mathbf{x}_i \cap \tilde{\mathbf{x}}_i$;
    IF ($\tilde{\mathbf{x}}_i$ is the empty set $\emptyset$) THEN
      STOP, signaling "the solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ does not intersect $\mathbf{x}$";
  END DO
  $d :=$ distance between $\mathbf{x}$ and $\tilde{\mathbf{x}}$;
  $\mathbf{x} := \tilde{\mathbf{x}}$;
END DO

- must be a proper interval despite the possible presence of improper intervals $\mathbf{a}^{c}_{ij}$ and $\mathbf{b}^{c}_{i}$ and taking the inverses in the expression (3.2),
- provides us with an outer interval estimate of the solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ too.

So, the natural idea is to take the intersection

$$\mathbf{x} \cap \tilde{\mathbf{x}} \supseteq \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b}),$$

which may prove a narrower estimate than each of $\mathbf{x}$ and $\tilde{\mathbf{x}}$ on its own.

Finally, to make the best use of the information available at the runtime we can immediately involve the values of the first components of $\tilde{\mathbf{x}}$, already improved by the algorithm, into the computation of the next components. The overall computational scheme of the interval Gauss-Seidel iteration for computing the enclosures of AE-solution sets to interval linear systems is presented in Table 1.

If $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b}) \cap \mathbf{x} \neq \emptyset$, then the result of the execution of the above algorithm is the sequence $\{\tilde{\mathbf{x}}\}$ of proper nested intervals, which must have a limit in $\mathbb{IR}^n$ (see [1],

[6], [9]). The stopping criteria for the above iteration is, as usual, attaining sufficient closeness (in some interval metric) between the two successive approximations.

To start our generalized interval Gauss-Seidel method we need an initial interval vector $\mathbf{x} \supseteq \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$. For the AE-solution sets, we can always take it as an enclosure of the united solution set $\Xi_{uni}(\mathbf{A}, \mathbf{b})$ for the corresponding interval linear system (since Ξ_{uni} is the widest among the solution sets), applying any one of the numerous techniques that have been elaborated for this purpose [1], [6], [9].

4. Investigating the Method

Barth and Nuding [2] and afterward Neumaier [9] gave a profound investigation of the interval Gauss-Seidel method for the classical case of enclosing the united solution set to interval linear systems. The theory developed by Barth-Nuding and Neumaier can be partly transferred to the generalized interval Gauss-Seidel method we have derived. We are doing that below, although changing accents and interpretation of some results as compared with Neumaier's theory [9].

The key point in the considerations of Barth-Nuding and Neumaier is the notions of *M*-matrix and *H*-matrix:

DEFINITION 4.1 (see [3], [9]). A matrix $A \in \mathbb{R}^{n\times n}$ is called an *M-matrix*, if it satisfies any one of the following equivalent conditions

- $A = sI - P$, where P is a nonnegative matrix and $s > \rho(P)$;
- off-diagonal entries of the matrix A are nonpositive and $A^{-1} \geq 0$;
- …, etc. (For instance, Berman and Plemmons [3] list 50 conditions equivalent to the statement "the matrix A is an *M*-matrix".)

THEOREM 4.1 (Neumaier [9], Proposition 3.6.3). *Let P, Q be point $n \times n$-matrices and suppose that Q is an M-matrix and $P \geq 0$. Then $(Q - P)$ is an M-matrix if and only if $\rho(Q^{-1}P) < 1$.*

DEFINITION 4.2 [2]. An interval matrix $\mathbf{A} \in \mathbb{IR}^{n\times n}$ is termed an *interval M-matrix* if every point matrix $A \in \mathbf{A}$ is an *M*-matrix.

DEFINITION 4.3 [9]. By a *mignitude* $\langle \mathbf{a} \rangle$ of a proper interval $\mathbf{a}$ we mean the smallest distance between the points of $\mathbf{a}$ and zero, i.e.,

$$\langle \mathbf{a} \rangle := \begin{cases} \min\{|\underline{\mathbf{a}}|, |\overline{\mathbf{a}}|\}, & \text{if } \mathbf{a} \not\ni 0, \\ 0, & \text{if } \mathbf{a} \ni 0. \end{cases}$$

For a proper interval matrix $\mathbf{A} = (\mathbf{a}_{ij}) \in \mathbb{IR}^{n\times n}$, by a *comparison matrix* we mean the matrix $\langle \mathbf{A} \rangle \in \mathbb{R}^{n\times n}$ such that

$$\text{the } ij\text{-th entry of } \langle \mathbf{A} \rangle := \begin{cases} \langle \mathbf{a}_{ij} \rangle, & \text{if } i = j, \\ -|\mathbf{a}_{ij}|, & \text{if } i \neq j. \end{cases}$$

DEFINITION 4.4 [9]. A proper interval square matrix $\mathbf{A}$ is called an *H-matrix*, if its comparison matrix is an *M*-matrix.

In particular, *strictly diagonally dominant* interval matrices $\mathbf{A} = (\mathbf{a}_{ij})$ that satisfy

$$\langle \mathbf{a}_{ii} \rangle > \sum_{k \neq i} |\mathbf{a}_{ik}| \qquad \text{for } \ i = 1, 2, \ldots, n, \tag{4.1}$$

are *H*-matrices. Less trivial examples of the interval *H*-matrices are nonsingular upper triangular matrices and lower triangle matrices [9].

THEOREM 4.2. *If* $\mathbf{x}^\star$ *is the limit of the generalized Gauss-Seidel method applied to an interval linear system* $\mathbf{A}x = \mathbf{b}$*, then*

$$\langle \mathbf{A} \rangle \, | \, \mathbf{x}^\star | \leq |\mathbf{b}|. \tag{4.2}$$

If $\mathbf{A}$ *is an interval H-matrix, then*

$$|\mathbf{x}^\star| \leq \langle \mathbf{A} \rangle^{-1} |\mathbf{b}|. \tag{4.3}$$

Proof. We consider only nonsingular interval matrices $\mathbf{A}$, assuming without loss in generality that $0 \notin \mathbf{a}_{ii}$. The formulas specifying the generalized interval Gauss-Seidel method thus imply

$$\mathbf{x}_i^\star \subseteq (\mathbf{a}_{ii}^{\mathrm{c}})^{-1} \left(\sum_{j \neq i} (\mathrm{opp}\ \mathbf{a}_{ij}^{\mathrm{c}}) \mathbf{x}_j^\star + \mathbf{b}_i^{\mathrm{c}} \right),$$

so that

$$|\mathbf{x}_i^\star| \leq \langle \mathbf{a}_{ii} \rangle^{-1} \left(\sum_{j \neq i} |\mathbf{a}_{ij}| \, |\mathbf{x}_j^\star| + |\mathbf{b}_i| \right)$$

since both sides of the above inclusion are proper intervals. We get therefore

$$\langle \mathbf{a}_{ii} \rangle |\mathbf{x}_i^\star| \leq \sum_{j \neq i} |\mathbf{a}_{ij}| \, |\mathbf{x}_j^\star| + |\mathbf{b}_i|,$$

which is equivalent to

$$(\langle \mathbf{A} \rangle |\mathbf{x}^\star|)_i \leq |\mathbf{b}_i|$$

for all $i = 1, 2, \ldots, n$, that is, coincides with (4.2).

If $\mathbf{A}$ is an interval *H*-matrix, then $\langle \mathbf{A} \rangle$ is an *M*-matrix, so $\langle \mathbf{A} \rangle^{-1} \geq 0$. Multiplying both sides of (4.2) by $\langle \mathbf{A} \rangle^{-1}$, we arrive at (4.3). □

It follows from the theorem that an initial box larger than that allowed by the inequality (4.3) is improved (i.e., is decreased in size) by the generalized Gauss-Seidel iteration providing that the matrix $\mathbf{A}$ is an *H*-matrix. On the contrary, if $\mathbf{A}$ is not an *H*-matrix, we cannot draw such a conclusion. Under these circumstances, Neumaier in [9] even proved the following interesting result for the classical version of the interval Gauss-Seidel iteration:

THEOREM 4.3 [9]. *If a proper interval $n \times n$-matrix $\mathbf{A} = (\mathbf{a}_{ij})$ is not an H-matrix, then there exist arbitrary large proper interval vectors that cannot be improved by Gauss-Seidel iteration as applied for outer estimation of the united solution set of the interval system* $\mathbf{A}x = 0$.

For the generalized interval Gauss-Seidel method under study, Neumaier's proof does not work in case the interval matrix (opp $\mathbf{A}^{\mathrm{c}}$) contains at least one improper interval in each row. The reason is simple: magnitude of an interval product is not equal to the product of the factors' magnitudes in Kaucher complete interval arithmetic.

One of the most remarkable facts with the interval Gauss-Seidel iteration as applied to the united solution set is the following optimality property: *if the matrix of the interval linear system is an interval M-matrix, the method produces the interval hull of the solution set*. This fact has been first revealed by Barth and Nuding [2]. We managed to generalize this classical result as the following

THEOREM 4.4. *If, in an interval linear system* $\mathbf{A}x = \mathbf{b}$, *the matrix* $\mathbf{A} = (\mathbf{a}_{ij})$ *is an interval M-matrix, then the generalized interval Gauss-Seidel iteration applied to this system converges to the interval hull of an AE-solution set.*

Proof. We denote by $\mathbf{E} = (\mathbf{e}_{ij})$ the matrix obtained from $\mathbf{A} = (\mathbf{a}_{ij})$ by replacing its diagonal entries by zeros and by $\mathbf{D} = (\mathbf{d}_{ij})$ the diagonal matrix with the diagonal entries $\mathbf{d}_{ii} = \mathbf{a}_{ii}$, $i = 1, 2, \dots, n$. Similarly, we denote by $\mathbf{E}^{\mathrm{c}} = (\mathbf{e}^{\mathrm{c}}_{ij})$ the matrix obtained from $\mathbf{A}^{\mathrm{c}} = (\mathbf{a}^{\mathrm{c}}_{ij})$ by replacing its diagonal entries by zeros and by $\mathbf{D}^{\mathrm{c}} = (\mathbf{d}^{\mathrm{c}}_{ij})$ the diagonal matrix with the diagonal entries $\mathbf{d}^{\mathrm{c}}_{ii} = \mathbf{a}^{\mathrm{c}}_{ii}$, $i = 1, 2, \dots, n$. Then

$$\mathbf{A} = \mathbf{D} + \mathbf{E}, \qquad \mathbf{A}^{\mathrm{c}} = \mathbf{D}^{\mathrm{c}} + \mathbf{E}^{\mathrm{c}},$$

with $\mathbf{d}_{ii} = \mathbf{d}^{\mathrm{c}}_{ij} = 0$ for $i \neq j$, and $\mathbf{e}_{ij} = \mathbf{e}^{\mathrm{c}}_{ij} = 0$ for $i = j$.

If $\mathbf{x}^{\star}$ is the limit of the generalized interval Gauss-Seidel method, then evidently

$$\mathbf{x}^{\star} = \mathbf{x}^{\star} \cap (\mathbf{D}^{\mathrm{c}})^{-1}\big((\operatorname{opp} \mathbf{E}^{\mathrm{c}})\mathbf{x}^{\star} + \mathbf{b}^{\mathrm{c}}\big),$$

and therefore

$$\mathbf{x}^{\star} \subseteq (\mathbf{D}^{\mathrm{c}})^{-1}\big((\operatorname{opp} \mathbf{E}^{\mathrm{c}})\mathbf{x}^{\star} + \mathbf{b}^{\mathrm{c}}\big). \tag{4.4}$$

Next, if $\mathbf{A}$ is an M-matrix, then its main diagonal consists of positive entries, $|(\mathbf{D}^{\mathrm{c}})^{-1}| = \langle \mathbf{D} \rangle^{-1}$, so

$$|(\mathbf{D}^{\mathrm{c}})^{-1}|\,|\operatorname{opp} \mathbf{E}^{\mathrm{c}}| = \langle \mathbf{D} \rangle^{-1}|\mathbf{E}|. \tag{4.5}$$

Additionally, $\mathbf{D}$ is an M-matrix too.

But the comparison matrix $\langle \mathbf{A} \rangle$ is also an M-matrix, being a point matrix within $\mathbf{A}$. Moreover, since $\langle \mathbf{A} \rangle = \langle \mathbf{D} \rangle - |\mathbf{E}|$, Neumaier's result (Theorem 4.1) implies $\rho(\langle \mathbf{D} \rangle^{-1}|\mathbf{E}|) < 1$, which yields, together with (4.5), the inequality

$$\rho\big(|(\mathbf{D}^{\mathrm{c}})^{-1}|\,|\operatorname{opp} \mathbf{E}^{\mathrm{c}}|\big) < 1.$$

We can thus conclude that the iteration in $\mathbb{IR}^n$ defined by

$$\mathbf{x}^{(0)} := \mathbf{x}^\star,$$
$$\mathbf{x}^{(k+1)} := (\mathbf{D}^c)^{-1}\big((\text{opp}\ \mathbf{E}^c)\mathbf{x}^{(k)} + \mathbf{b}^c\big), \qquad k = 0, 1, \dots,$$

converges to a unique algebraic solution of the interval linear system

$$x = (\mathbf{D}^c)^{-1}\big((\text{opp}\ \mathbf{E}^c)x + \mathbf{b}^c\big).$$

Furthermore, the inclusion (4.4) implies that

$$\mathbf{x}^\star \subseteq \mathbf{x}^*, \tag{4.6}$$

which can be substantiated by induction. Indeed, $\mathbf{x}^\star \subseteq \mathbf{x}^{(0)}$, and if $\mathbf{x}^\star \subseteq \mathbf{x}^{(k)}$, then

$$\begin{aligned}\mathbf{x}^\star &\subseteq (\mathbf{D}^c)^{-1}\big((\text{opp}\ \mathbf{E}^c)\mathbf{x}^\star + \mathbf{b}^c\big)\\ &\subseteq (\mathbf{D}^c)^{-1}\big((\text{opp}\ \mathbf{E}^c)\mathbf{x}^{(k)} + \mathbf{b}^c\big) = \mathbf{x}^{(k+1)}.\end{aligned}$$

Passing to the limit $k \to \infty$ yields (4.6).

To complete the proof, one should only refer to Theorem 2.4: $\mathbf{x}^*$ is the interval hull of the solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$, so does $\mathbf{x}^\star$ too, inasmuch as $\mathbf{x}^\star \supseteq \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ and (4.6) holds. □

Overall, the theory of this section shows that the generalized interval Gauss-Seidel method works well only for the interval linear systems with *H*-matrices. How can we find enclosures for the AE-solution sets to interval linear systems in the general case? An answer to this question is the so-called *generalized preconditioning* proposed in [15].

Another important issue is treating the interval linear systems with general rectangular matrices (typically arising in real life identification and control problems). The technique we have elaborated is not directly applicable to the non-square case, and we recommend as a natural (although not universal) outcome *extracting square subproblems* of the initial problem. Specifically, any interval linear $m \times n$-system with $m > n$ can be represented as a system of several square $n \times n$-subsystems, generally overlapping. In its turn, each of the elementary subsystems may be solved by our generalized Gauss-Seidel iteration, while the overall outer estimate of the solution set to the original system is thus obtained by intersecting the enclosures for the separate subsystems.

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