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Algebraic Solutions to Interval Linear Equations and their Applications

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1 Introduction

Historically, interval analysis originated from sensitivity problems and it is no wonder that at its early years the solution set to a problem with interval coefficients was understood as the set of all possible solutions to point problems with the coefficients within the given intervals. In our paper, the main object under study is the interval linear system

$$\mathbf{A}x = \mathbf{b} \tag{1}$$

with an interval $m \times n$-matrix $\mathbf{A}$ and interval right-hand side m-vector $\mathbf{b}$ and for many years the only solution set to the system (1) was the *united solution set*

$$\Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \mid (\exists A \in \mathbf{A})(\exists\, b \in \mathbf{b})(Ax = b) \,\}, \tag{2}$$

formed by the solutions of all systems $Ax = b$ with $A \in \mathbf{A}$ and $b \in \mathbf{b}$ (see [1] et al.). The above definition, the most correct mathematically, is arranged according to the *selection axiom* of the formal set theory [3]. Namely, a point $\tilde{x}$ belongs to the set if and only if substituting it for the variable x in the predicate written out after the vertical line results in a true proposition. So, from now on we shall call the predicates written out after the vertical line in the records of the form (2) the *selecting predicates* for the respective sets.

In 1972, Nuding [4] introduced the other solution set to interval linear systems, which we shall refer to as the *tolerable solution set*:

$$\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \mid (\forall\, A \in \mathbf{A})(\exists\, b \in \mathbf{b})(Ax = b) \,\},$$

formed by all vectors $x \in \mathbb{R}^n$, such that the product Ax falls into $\mathbf{b}$ for any $A \in \mathbf{A}$ (see also [6]). To my mind, the work [4] was a remarkable advance, whose significance has not been appreciated at its true value so far. In point of fact, Nuding demonstrated us the possibility to vary quantifiers in the selecting predicate of the definition of the solution set. Anyway, the next step on that way was made only in 1991–92 when several Russians independently and almost simultaneously faced with the necessity to introduce the solution set

$$\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \mid (\forall\, b \in \mathbf{b})(\exists\, A \in \mathbf{A})(Ax = b) \,\},$$

formed by vectors $x \in \mathbb{R}^n$ such that for any desired $b \in \mathbf{b}$ we can find an appropriate $A \in \mathbf{A}$ satisfying $Ax = b$. Shary [5] proposed to call it *controllable solution set* and the term seems to have been adopted.

In principle, since the quantifiers $\forall$ and $\exists$ do not commute with each other [3], the next rightful question to ask is as follows: what about the solution sets $\{\, x \in \mathbb{R}^n \mid (\forall\, A \in \mathbf{A})(\forall\, b \in \mathbf{b})(Ax = b) \,\}$, $\{\, x \in \mathbb{R}^n \mid (\exists\, A \in \mathbf{A})(\forall\, b \in \mathbf{b})(Ax = b) \,\}$ and $\{\, x \in \mathbb{R}^n \mid (\exists\, b \in \mathbf{b})(\forall\, A \in \mathbf{A})(Ax = b) \,\}$? Considering them is not senseless, but we can go even farther.

We would like to remind that the symbolic designation $(\forall\, A \in \mathbf{A})$ means nothing but $(\forall\, a_{11} \in \mathbf{a}_{11})(\forall\, a_{12} \in \mathbf{a}_{12}) \ldots (\forall\, a_{mn} \in \mathbf{a}_{mn})$. This is true for $(\exists\, A \in \mathbf{A})$, $(\forall\, b \in \mathbf{b})$ and $(\exists\, b \in \mathbf{b})$, too. Hence, to further generalize the concept of the solution set to interval systems, we can split the action of quantifiers as applied to various elements of the matrix and right-hand side: *we can form the other solution sets to interval equations through combining* $\forall$ *and* $\exists$ *with the parameters of the equation and changing their order!* Since the quantifier that corresponds to each interval element may have two values $\{\forall, \exists\}$ and the order of entries in the selecting predicate is also essential for the definition, the total number of the solution sets we can thus define for the interval linear $m \times n$-system far exceeds 2^{mn+m}. Generally, these solution sets can be practically interpreted as solutions of some games or multistep decision-making processes under interval uncertainty, as well as solutions to some minimax operations research problems (see e.g. [8]).

2 $\alpha\beta$-solution sets and their characterization

Before developing the proper mathematics, we need some formal definitions for the objects we discussed in the introduction. In this work, we shall restrict ourselves only to the solution sets of interval linear systems (1), in which *all occurrences of the universal quantifier* $\forall$ *(if any) precede the occurrences of the existential quantifier* $\exists$ in the selecting predicate, or, put it differently, only those solution sets for which the selecting predicate has *AE-form.*

Let the entire set of the index pairs (i, j) of the elements a_{ij} be divided into two nonintersecting parts $\Omega' = \{\omega'_1, \ldots, \omega'_p\}$ and $\Omega'' = \{\omega''_1, \ldots, \omega''_r\}$, $p+r = mn$, such that the parameter a_{ij} belongs to the $\forall$-type of uncertainty for $(i, j) \in \Omega'$ and to the $\exists$-type of uncertainty for $(i, j) \in \Omega''$. Similarly, we introduce nonintersecting sets of integer indices $\Theta' = \{\vartheta'_1, \ldots, \vartheta'_s\}$ and $\Theta'' = \{\vartheta''_1, \ldots, \vartheta''_t\}$, $\Theta' \cup \Theta'' = \{1, 2, \ldots, m\}$, such that the element b_i of the right-hand side is subsumed under the interval $\forall$-uncertainty for $i \in \Theta'$ and of the $\exists$-type for $i \in \Theta''$. Also, we allow the natural possibility for some of the sets $\Omega', \Omega'', \Theta', \Theta''$ to be empty.

Definition 1. We define the *set of $\alpha\beta$-solutions* to the interval linear system (1) as the set

$$
\begin{aligned}
&\Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b}) \\
&= \{\ x \in \mathbb{R}^n \mid \\
&\quad (\forall a_{\omega'_1} \in \mathbf{a}_{\omega'_1})\ \ldots\ (\forall a_{\omega'_p} \in \mathbf{a}_{\omega'_p})\ (\forall b_{\vartheta'_1} \in \mathbf{b}_{\vartheta'_1})\ \ldots\ (\forall b_{\vartheta'_s} \in \mathbf{b}_{\vartheta'_s}) \\
&\quad (\exists a_{\omega''_1} \in \mathbf{a}_{\omega''_1})\ \ldots\ (\exists a_{\omega''_r} \in \mathbf{a}_{\omega''_r})\ (\exists b_{\vartheta''_1} \in \mathbf{b}_{\vartheta''_1})\ \ldots\ (\exists b_{\vartheta''_t} \in \mathbf{b}_{\vartheta''_t}) \\
&\qquad (Ax = b)\ \},
\end{aligned}
\tag{3}
$$

where the quantifier $m \times n$-matrix $\alpha = (\,\alpha_{ij})$ and m-vector $\beta = (\,\beta_i)$ are such that

$$
\alpha_{ij} = \begin{cases} \forall, \text{ if } \ (i,j) \in \Omega', \\ \exists, \text{ if } \ (i,j) \in \Omega'', \end{cases} \qquad \beta_i = \begin{cases} \forall, \text{ if } \ i \in \Theta', \\ \exists, \text{ if } \ i \in \Theta''. \end{cases}
$$

Other suitable terms to denote (3) are *$\alpha\beta$-solution set* or *solution set of the type $\alpha\beta$*.

For the interval linear system $\mathbf{A}x = \mathbf{b}$, we define interval matrices $\mathbf{A}^\forall = (\mathbf{a}_{ij}^\forall)$ and $\mathbf{A}^\exists = (\mathbf{a}_{ij}^\exists)$ and interval vectors $\mathbf{b}^\forall = (\mathbf{b}_i^\forall)$ and $\mathbf{b}^\exists = (\mathbf{b}_i^\exists)$ of the same size as $\mathbf{A}$ and $\mathbf{b}$ as follows:

$$\mathbf{a}_{ij}^\forall = \begin{cases} \mathbf{a}_{ij}, & \text{if } \alpha_{ij} = \forall, \\ 0, & \text{otherwise,} \end{cases} \qquad \mathbf{a}_{ij}^\exists = \begin{cases} \mathbf{a}_{ij}, & \text{if } \alpha_{ij} = \exists, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathbf{b}_i^\forall = \begin{cases} \mathbf{b}_i, & \text{if } \beta_i = \forall, \\ 0, & \text{otherwise,} \end{cases} \qquad \mathbf{b}_i^\exists = \begin{cases} \mathbf{b}_i, & \text{if } \beta_i = \exists, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $\mathbf{A} = \mathbf{A}^\forall + \mathbf{A}^\exists$, $\mathbf{b} = \mathbf{b}^\forall + \mathbf{b}^\exists$ and $\mathbf{a}_{ij}^\forall\, \mathbf{a}_{ij}^\exists = 0$, $\mathbf{b}_i^\forall\, \mathbf{b}_i^\exists = 0$ for all i, j.

Theorem 1. The point x belongs to the solution set $\Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ if and only if

$$\mathbf{A}^\forall \cdot x - \mathbf{b}^\forall \subseteq \mathbf{b}^\exists - \mathbf{A}^\exists \cdot x. \tag{4}$$

Proof. Using the matrices $\mathbf{A}^\forall$, $\mathbf{A}^\exists$ and vectors $\mathbf{b}^\forall$, $\mathbf{b}^\exists$ introduced above, we can rewrite the definition of the solution set $\Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ in the following equivalent form:

$$\Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \mid (\forall\, A' \in \mathbf{A}^\forall)(\forall b' \in \mathbf{b}^\forall)(\exists A'' \in \mathbf{A}^\exists)(\exists b'' \in \mathbf{b}^\exists) \\ (\,(A' + A'')x = (b' + b'')\,)\,\}.$$

It is not hard to complete the proof of the theorem now, transforming equivalently the selecting predicate of the solution set. We have

$$\begin{aligned} \Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b}) &= \{\, x \in \mathbb{R}^n \mid (\forall A' \in \mathbf{A}^\forall)(\forall b' \in \mathbf{b}^\forall)(\exists A'' \in \mathbf{A}^\exists)(\exists b'' \in \mathbf{b}^\exists) \\ &\qquad (A'x - b' = b'' - A''x)\,\} \\ &= \{\, x \in \mathbb{R}^n \mid (\forall A' \in \mathbf{A}^\forall)(\forall b' \in \mathbf{b}^\forall)(A'x - b' \in \mathbf{b}^\exists - \mathbf{A}^\exists \cdot x)\,\} \\ &= \{\, x \in \mathbb{R}^n \mid \mathbf{A}^\forall \cdot x - \mathbf{b}^\forall \subseteq \mathbf{b}^\exists - \mathbf{A}^\exists \cdot x\,\}, \end{aligned}$$

since for any interval matrix $\mathbf{C}$ and a real point vector x the result of the multiplication $\mathbf{C}\cdot x$ always coincides with $\{\, Cx \mid C \in \mathbf{C}\,\}$ [1]. ■

Corollary. For any quantifiers α and β, the intersection of the solution set $\Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ with each orthant of the space $\mathbb{R}^n$ is a convex polyhedral set.

3 Problem statement

Now, that we have defined what the solution sets to interval systems are, it is time to decide what to do with them. In spite of the fine characterization Theorem 1, the complexity of the direct description of $\Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ grows exponentially with n. Such a description thus becomes extremely laborious and practically useless already for moderate dimension of the system.

There are various ways to estimate solution sets, to change their direct and complete description for an approximate one, which is suitable or sufficient in some sense. According to the practical interpretation of the generalized solution sets and taking into account the ability of the mathematical technique, we shall confine ourselves to finding some interval subsets of $\Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$, since the selecting predicate from the definition (3) remains valid for all their points. In other words, we change the solution set $\Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ for some inner approximation, formulating the problem to be solved in the following form:

$$\boxed{\begin{array}{c}\textit{Find an interval vector that is contained in the solution set}\\ \Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})\ \textit{(if it is nonempty) of the interval linear system.}\end{array}} \tag{5}$$

If the tolerable solution set $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ is taken as a case in point in the above definition, then the problem (5) is the classical *linear tolerance problem* with numerous and fruitful practical applications [6]. Actually, the linear tolerance problem is a problem of stabilization within the required output state corridor $\mathbf{b}$ for the system in which *all* parameters a_{ij} are subject to bounded perturbations. If some a_{ij} have $\forall$-uncertainty while the others have $\exists$-uncertainty and all $\beta_i = \exists$, $i = 1, 2, \ldots, m$, then we arrive at the stabilization problem with a control possibility, which some of the researchers call "the problem of insuring survival of the system". Alternately, if part of a_{ij}'s are $\forall$-parameters and a part of them are $\exists$-parameters while all $\beta_i = \forall$, $i = 1, 2, \ldots, m$, then we have the control problem under bounded perturbations.

The aim of this work is not only to introduce new solution sets, but to present algorithmically efficient approach to work with them, i.e., to solve the problem (5), and we are based upon the concept of *algebraic solution* to the interval equation.

Definition 2. An interval vector is said to be *algebraic solution* to the interval system if substitution of it to the equation and execution of all interval arithmetic operations results in a valid equality.

More precisely, we change the problem (5) for the problem of finding algebraic solution to a special systems of equations in the extended *Kaucher interval arithmetic* $\mathbb{IR}$, thus reducing the original problem to a purely algebraic problem of the numerical analysis.

4 Kaucher interval arithmetic

The extended interval arithmetic $\mathbb{IR}$ proposed by Kaucher [2] is a natural algebraic completion of the common interval arithmetic. The elements of $\mathbb{IR}$ are the pairs $[\,\underline{x}, \overline{x}\,]$ of reals, that are not connected by the obligatory condition $\underline{x} \leq \overline{x}$. Thus, $\mathbb{IR}$ is obtained by adding *improper* intervals $[\,\underline{x}, \overline{x}\,]$, $\underline{x} > \overline{x}$, to the set $\{\,[\,\underline{x}, \overline{x}\,] \mid \underline{x}, \overline{x} \in \mathbb{R},\ \underline{x} \leq \overline{x}\,\}$ of the *proper* intervals and the real numbers. The proper and improper intervals, the two halves of $\mathbb{IR}$, can change places as the result of the *dualization* mapping $\operatorname{dual}\, : \mathbb{IR} \to \mathbb{IR}$, such that

$$\operatorname{dual}\,[\,\underline{x}, \overline{x}\,] = [\,\overline{x}, \underline{x}\,].$$

As in classical interval arithmetic,

$$\mathbf{x} \subseteq \mathbf{y} \quad \overset{\text{def}}{\Longleftrightarrow} \quad \underline{\mathbf{x}} \geq \underline{\mathbf{y}} \ \text{ and } \ \overline{\mathbf{x}} \leq \overline{\mathbf{y}},$$

but Kaucher interval arithmetic $\mathbb{IR}$ is a distributive conditionally complete lattice with respect to this inclusion order, in contrast to classical interval arithmetic. In other words,

$$\bigvee_{\gamma\in\Gamma} \mathbf{x}_\gamma \ := \ \Big[\, \inf_{\leq}\{\, \underline{\mathbf{x}}_\gamma \mid \gamma \in \Gamma \,\},\ \sup_{\leq}\{\, \overline{\mathbf{x}}_\gamma \mid \gamma \in \Gamma \,\} \,\Big] \text{ — } \begin{array}{c}\text{maximum with}\\ \text{respect to "}\subseteq\text{"},\end{array}$$

$$\bigwedge_{\gamma\in\Gamma} \mathbf{x}_\gamma \;:=\; \Bigl[\, \sup_{\le}\{\, \underline{\mathbf{x}}_\gamma \mid \gamma\in\Gamma \,\},\; \inf_{\le}\{\, \overline{\mathbf{x}}_\gamma \mid \gamma\in\Gamma \,\} \,\Bigr] \text{ — } \begin{array}{l}\text{minimum with}\\ \text{respect to “}\subseteq\text{”}\end{array}$$

are elements from $\mathbb{IR}$ now, if $\{\, \mathbf{x}_\gamma \mid \gamma \in \text{index set } \Gamma \,\}$ is a bounded family of intervals from $\mathbb{IR}$.

The addition is defined upon $\mathbb{IR}$ by

$$\mathbf{x} + \mathbf{y} := \bigl[\, \underline{\mathbf{x}} + \underline{\mathbf{y}}, \overline{\mathbf{x}} + \overline{\mathbf{y}} \,\bigr].$$

Each element $\mathbf{x}$ from $\mathbb{IR}$ has thus the only opposite element $[\,-\underline{\mathbf{x}}, -\overline{\mathbf{x}}\,]$, the consequence of this being the usual possibility to rearrange terms from one side of equation (or inequality, or inclusion) to the other side “with the opposite sign”. The following lattice operation distributivity will be useful for us:

$$\mathbf{x} + (\mathbf{y} \vee \mathbf{z}) = (\mathbf{x} + \mathbf{y}) \vee (\mathbf{x} + \mathbf{z}). \tag{6}$$

The connection between the result of the interval arithmetical operation $\mathbf{x} * \mathbf{y}$, $* \in \{+, -, \cdot, /\}$, and the results of separate point operations $x * y$ for x “from” $\mathbf{x}$ and y “from” $\mathbf{y}$ is expressed in Kaucher arithmetic by the fundamental representation:

$$\mathbf{x} * \mathbf{y} = \mathop{\text{И}^{\mathbf{x}}}_{x\in \operatorname{pro}\mathbf{x}} \; \mathop{\text{И}^{\mathbf{y}}}_{y\in \operatorname{pro}\mathbf{y}} (x * y), \tag{7}$$

where

$$\text{И}^{\mathbf{x}} \;:=\; \begin{cases} \bigvee, & \text{if } \mathbf{x} \text{ is proper,} \\ \bigwedge, & \text{otherwise,} \end{cases} \qquad \text{—} \quad \begin{array}{l}\text{conditional}\\ \text{lattice operation,}\end{array}$$

$$\operatorname{pro}\mathbf{x} \;:=\; \begin{cases} \mathbf{x}, & \text{if } \mathbf{x} \text{ is proper,} \\ \operatorname{dual}\mathbf{x}, & \text{otherwise,} \end{cases} \qquad \text{—} \quad \begin{array}{l}\text{proper projection}\\ \text{of the interval.}\end{array}$$

In Kaucher interval arithmetic, the operations with vectors and matrices are defined similar to those in classical interval arithmetic. Further, inclusion ordering on the sets of interval vectors and matrices are direct products of one-dimensional inclusion orders, so we shall understood the operations $\vee$ and $\wedge$ applied to interval vectors in a componentwise manner. This will be valid for the operations “dual” and “pro” too.

5 Inner estimation by algebraic solutions

The main result of the algebraic approach developed is the following

Theorem 2. If the proper interval vector $\mathbf{x}$ is an algebraic solution to the equation

$$(\mathbf{A}^{\forall} + \text{dual}\ \mathbf{A}^{\exists})\, x = \text{dual}\ \mathbf{b}^{\forall} + \mathbf{b}^{\exists}, \tag{8}$$

then $\mathbf{x} \subseteq \Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$, that is, this interval vector is a solution to the problem (5).

Definition 3. For the interval system $\mathbf{A}x = \mathbf{b}$, we will call the equation (8) the *dualization equation* that corresponds to its $\alpha\beta$-solution set.

Proof. Let a proper interval vector $\mathbf{x}$ be algebraic solution to the system (8) and $\tilde{x} \in \mathbf{x}$. Then, in view of inclusion monotonicity of interval arithmetical operations in $\mathbb{IR}$, we have

$$(\mathbf{A}^{\forall} + \text{dual}\ \mathbf{A}^{\exists}) \cdot \tilde{x} \subseteq (\mathbf{A}^{\forall} + \text{dual}\ \mathbf{A}^{\exists}) \cdot \mathbf{x} = \text{dual}\ \mathbf{b}^{\forall} + \mathbf{b}^{\exists}.$$

Since the matrices $\mathbf{A}^{\forall}$ and $\mathbf{A}^{\exists}$ form a disjunct decomposition of $\mathbf{A}$, we may avail ourselves by distributivity (6) in the above formula:

$$\mathbf{A}^{\forall} \cdot \tilde{x} + \text{dual}\ \mathbf{A}^{\exists} \cdot \tilde{x} \ \subseteq\ \text{dual}\ \mathbf{b}^{\forall} + \mathbf{b}^{\exists}.$$

After rearranging the terms with dualizations to the opposite sides of the inclusion we finally get

$$\mathbf{A}^{\forall} \cdot \tilde{x} - \mathbf{b}^{\forall} \ \subseteq\ \mathbf{b}^{\exists} - \mathbf{A}^{\exists} \cdot \tilde{x},$$

that is, $\tilde{x} \in \Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ by Theorem 1. ■

Theorem 3. If the proper interval vector $\mathbf{x}$ is an inclusion-maximal algebraic interval solution to the dualization equation (8), then it is also an inclusion-maximal interval vector contained in $\Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$, i.e., gives an inclusion-maximal solution to the problem (5).

Proof. We need the following auxiliary representation: if $\mathbf{v}$ is a proper interval n-vector and $\mathbf{A}$ is an (arbitrary) interval $m \times n$-matrix, then

$$\mathbf{A}\cdot\mathbf{v} = \bigvee_{v\in\mathbf{v}} \mathbf{A}\cdot v. \tag{9}$$

Indeed, if $\mathbf{A}\cdot\mathbf{v} = \big(\,(\mathbf{A}\cdot\mathbf{v})_1, (\mathbf{A}\cdot\mathbf{v})_2, \ldots, (\mathbf{A}\cdot\mathbf{v})_m\big)^\top$, then, using (7) and distributivity of the operation "$\vee$" with respect to addition (6), we get

$$\begin{aligned}(\mathbf{A}\cdot\mathbf{v})_i &= \sum_{j=1}^{n} \mathbf{a}_{ij}\mathbf{v}_j = \sum_{j=1}^{n} \bigvee_{v_j\in\mathbf{v}_j} \mathbf{a}_{ij}v_j = \bigvee_{v_1\in\mathbf{v}_1}\bigvee_{v_2\in\mathbf{v}_2}\cdots\bigvee_{v_n\in\mathbf{v}_n}\sum_{j=1}^{n}\mathbf{a}_{ij}v_j \\ &= \bigvee_{v\in\mathbf{v}}\sum_{j=1}^{n}\mathbf{a}_{ij}v_j = \bigvee_{v\in\mathbf{v}}(\mathbf{A}\cdot v)_i.\end{aligned}$$

Let us turn to the proof of the theorem. Denote the proper maximal algebraic solution of (8) by $\mathbf{x}$ and assume that, contrary to the assertion of the theorem, there exists a proper interval vector $\mathbf{y}$, such that

$$\Sigma_{\alpha\beta}(\mathbf{A},\mathbf{b}) \supseteq \mathbf{y} \supset \mathbf{x}.$$

Utilizing inclusion monotonicity of the arithmetic $\mathbb{IR}$ one obtains

$$(\mathbf{A}^\forall + \mathrm{dual}\,\mathbf{A}^\exists)\cdot\mathbf{y} \supset (\mathbf{A}^\forall + \mathrm{dual}\,\mathbf{A}^\exists)\cdot\mathbf{x} = \mathrm{dual}\,\mathbf{b}^\forall + \mathbf{b}^\exists,$$

the exact equality instead of inclusion being impossible due to the maximality of $\mathbf{x}$. Further, the representation (9) results in

$$\bigvee_{y\in\mathbf{y}} (\mathbf{A}^\forall + \mathrm{dual}\,\mathbf{A}^\exists)\cdot y \;\supset\; \mathrm{dual}\,\mathbf{b}^\forall + \mathbf{b}^\exists, \tag{10}$$

and we can conclude that there must be

$$(\mathbf{A}^\forall + \mathrm{dual}\,\mathbf{A}^\exists)\cdot\tilde{y} \;\not\subseteq\; \mathrm{dual}\,\mathbf{b}^\forall + \mathbf{b}^\exists$$

for some (at least one) $\tilde{y}\in\mathbf{y}$. Otherwise, if we had $(\mathbf{A}^\forall + \mathrm{dual}\,\mathbf{A}^\exists)\cdot y \subseteq \mathrm{dual}\,\mathbf{b}^\forall + \mathbf{b}^\exists$ for all $y\in\mathbf{y}$, then the inclusion that is opposite to (10) would be valid. However, owing to Theorem 1, the relation (10) is equivalent to $\tilde{y}\notin\Sigma_{\alpha\beta}(\mathbf{A},\mathbf{b})$, so $\mathbf{y}\not\subseteq\Sigma_{\alpha\beta}(\mathbf{A},\mathbf{b})$. ■

6 Numerical methods

The summary of what we have expounded may be formulated as follows: it is very useful to find algebraic solutions to interval linear systems in Kaucher arithmetic. But how can one get them in practice? The paper [7] gives partial answers to these questions. In that work, we investigate existence and uniqueness of the algebraic solutions to interval linear systems, propose an efficient numerical algorithm for their computation — the subdifferential Newton method — and prove its convergence. In the nearest future, the author hopes to issue a separate detailed paper on the computational aspects of the approach proposed.

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