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# Standardized notation in interval analysis

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**Abstract**

A standard for the notation of the most used quantities and operators in interval analysis is proposed.

# 1 Introduction

Publications in interval analysis currently suffer from a multitude of incompatible notational styles. There are obvious advantages in having a standardized notation, especially for those peripheral to our field who only want to read an occasional paper to see whether the field offers something for the solution of their problems. It is important for the future of interval analysis to reach out to these colleagues; a standardized notation contributes to limit the burden of learning new notation to a minimum.

In much of mathematics, standardization happens automatically because people use the notation introduced by the first influential papers on an issue. In interval analysis, this unfortunately did not happen. Worse, because there was no consensus in the past literature, new authors of work in interval analysis created their own notational habits, and produced further variants that added to the confusion. The time seems ripe to attempt to correct this unpleasant situation.

The purpose of this paper is to propose a standard that hopefully persuades the entire community to use it for publishing their work. Emphasis is on easy usage and easy comprehensibility. To facilitate the acceptance of the standard, a LaTeX style file is provided [5] that makes it easy to create documents conforming to the standard.

The proposed standard is based on the following guiding lines: The notation should blend seamlessly with traditional notation in mathematics, in particular numerical analysis and optimization. It should also result in formulas that look as simple as possible, while conveying the meaning clearly even to readers not working in the field. And it should create a minimal burden in preparing manuscripts for authors wanting to conform to the standard. In particular, standardization is restricted to the most basic aspects of interval terminology.

We hope that the suggested notation will appear persuasive to authors in interval analysis and its applications, convincing them that using it is likely to improve the communication of ideas in interval analysis to colleagues and potential users.

# 2 Standard notation

**Noninterval quantities.** General recommendations on the mathematical style are in the authoritative *Handbook of Writing for the Mathematical Sciences* by Higham [3].

In order for the notation to be consistent with traditional usage in other fields of mathematics, in particular optimization and numerical analysis, letters defining scalars and vectors should be lower case, and those defining matrices should be upper case. Sets should be capitals not in bold, unless they are intervals or boxes (see below). Similarly, letters denoting scalar-valued functions should be lower case, and letters denoting vector-valued functions should be upper case.

*Upper case vector-valued functions are advisable because these are nonlinear operators, generalizations of linear operators and matrices. This is the dominant usage, cf., e.g.,* Ortega & Rheinboldt *[13, p. 20],* Dennis & Schnabel *[1], although not universally followed (e.g.,* Nocedal & Wright *[12] use lower case).*

Arithmetic *expressions* are formulas (or more general programs) composed of a finite sequence of operations and elementary functions applied to constants, the components of an argument vector, or intermediate results. Letters denoting expressions should be sansserif lower case for scalar results, and sansserif upper case for vector results. Expressions are evaluated on arguments of a given class using class specific operations and elementary functions.

*A notational distinction of arithmetic expressions and the function they represent is important because equivalent expressions give different results when evaluated in nonstandard (e.g., floating point) arithmetic. This has been blurred in the past, sometimes leading to confusion, especially for people outside interval analysis who are likely to interpret* $f([-1,1])$ *as the image of* $[-1,1]$ *under* $f$*;* $\mathsf{f}([-1,1])$ *specifies it as the result of applying the operations in* $\mathsf{f}$ *to the interval* $[-1,1]$ *in its intrinsic arithmetic.*

*The sloppy usage of "the function* $f(x) = x^2-x+1$*" is accepted in mathematics, but is discouraged and should be replaced by either "the expression* $\mathsf{f}(x) = x^2-x+1$*" or "the function* $f$ *defined by* $f(x) = x^2-x+1$ *for all* $x \in \mathbb{R}$*", depending on the intended usage: The first form emphasizes the syntactic form to be used for evaluation and is defined for all arguments for which the operations are defined, while the second form emphasizes the mapping aspect and has no meaning with a nonreal argument.*

Given a list $x$ of arguments, the sublist consisting of components $x_k$ with indices $k$ in a subset $K$ of indices is denoted by $x_K$, and the complementary sublist by $x_{\notin K}$, or by $x_{\neq k}$ if $K = \{k\}$; the ordering is that of the natural ordering of the index set. If $\mathsf{f}$ is an expression in $x$ then

$$\mathsf{f}(x_K, y_{\notin K}) := \mathsf{f}(z), \quad z_k = \begin{cases} x_k & \text{if } k \in K, \\ y_k & \text{if } k \notin K, \end{cases}$$

denotes the value of $\mathsf{f}$ at the argument with components in $K$ taken from $x$ and the others taken from $y$. Similarly, $\mathsf{f}(x_k, y_{\neq k})$ denotes the value of $\mathsf{f}$ at the argument with components $k$ taken from $x$ and the others taken from $y$.

*This is the simplest of various notations used in some versions of slopes, and in constraint propagation for slicing, where some arguments in an expression are intervals, and others are components of centers or endpoints of intervals.*

$\mathbb{R}$ denotes the field of real numbers, $\mathbb{R}^n$ the vector space of column vectors of length $n$ with real entries, and $\mathbb{R}^{m\times n}$ the vector space of $m \times n$-matrices with real coefficients.

*This is the standard notation in numerical analysis; cf.* Golub & van Loan *[2],* Higham *[4],* Neumaier *[11]. In optimization, it is usually avoided to refer explicitly to a space of matrices but if it occurs, such as in* Nocedal & Wright *[12, p. 255], the above notation is used there, too.*

It is recommended to write $S \subseteq S'$ if $S$ is a subset of $S'$, and to avoid the use of the ambiguous symbol $\subset$.

*To say unambiguously that* $S$ *is a proper subset of* $S'$ *(rarely needed in our field), it is best to use words, or* $S \subseteq S'$*,* $S \neq S'$*.*

The interior of a subset $S \subseteq \mathbb{R}^n$ is denoted by $\operatorname{int} S$, the boundary by $\partial S$. The convex hull (closed convex hull) of a set $S$ is denoted by $\operatorname{ch} S$ ($\operatorname{cch} S$).

*Note that the above includes the possibility of writing* $\operatorname{int}(S)$*,* $\operatorname{ch}(S)$*, etc., where appropriate.*

The relations $=, <, \leq, >, \geq$ between vectors or matrices, and the supremum $\sup S$ and infimum $\inf S$ of a set $S$ of vectors or matrices are interpreted componentwise.

*This conforms with standard usage in lattice theory, and is essential in arguments based on the theory of nonnegative matrices, M-matrices, and H-matrices, where vectors with all components* $> 0$ *figure prominently.*

The transpose of a vector $x$ (a matrix $A$) is written as $x^T$ ($A^T$). The transposed inverse of a nonsingular square matrix $A$ is denoted by

$$A^{-T} = (A^T)^{-1} = (A^{-1})^T.$$

*Using $x^T$ and $A^T$ is standard in numerical analysis and optimization. The notation $A^{-T}$ is now also frequently used in numerical analysis and very convenient. Many pure mathematicians prefer $x^\top$, $A^\top$, and statisticians (and Matlab) use $x'$, $A'$; our choice is guided by the closeness of interval analysis to numerical analysis and optimization.*

Components of a matrix $A$ are denoted by $A_{ik}$ (preferably) or $a_{ik}$ (if done consistently); the $i$th row of $A$ is denoted by $A_{i:}$, and the $k$th column by $A_{:k}$.

$$\mathrm{diag}(A) = (A_{11}, \ldots, A_{nn})^T$$

denotes the diagonal of a square matrix $A$, $\mathrm{Diag}\,(a) = \mathrm{Diag}\,(a_1, \ldots, a_n)$ the diagonal matrix with diagonal entries $a_k$, and $\mathrm{Diag}\,(A) = \mathrm{Diag}\,(\mathrm{diag}(A))$ the diagonal part of $A$.

*This is a compromise between mathematical notation and Matlab notation, consistent with traditional notation.*

There is no common notation in mathematics for the identity matrix; numerical analysts usually use $I$; other authors use $E$ or $Id$. Many mathematicians use 1 as the unit in any ring and hence also in the ring of matrices, and this has its advantages. Our recommendation is to use $I$, and 1 in contexts where $I$ is used as an index set.

The preferred (but not the only useful) norm in interval computations is the maximum norm, $\|x\|_\infty = \max_k |x_k|$; in papers where $\|x\|$ shall denote a distinguished norm, it should be defined so explicitly in the paper.

**Intervals and boxes.** A *box* of dimension $n$ is a pair $\boldsymbol{x} = [\underline{x}, \overline{x}]$ consisting of two real column vectors $\underline{x}$ and $\overline{x}$ of length $n$ with $\underline{x} \le \overline{x}$. The set of all boxes of dimension $n$ is denoted by $\mathbb{IR}^n$.

*$\mathbb{IR}^n$ has been used in four recent books,* NEUMAIER *[10, 11],* KEARFOTT *[9],* JAULIN *et al. [6]. Boldface for intervals is in [9, 11]; [10] did not distinguish between reals and intervals notationally; [6] used $\boldsymbol{x}$ for real vectors and $[\boldsymbol{x}]$ for interval vectors, which seems unnecessarily complicated, given that mathematicians do not specially mark vectors; [9] used capital boldface letters for interval vectors, with the disadvantage that formulas of linear algebra have a different appearance when written for intervals.*

A box $\boldsymbol{x}$ is generally identified with the (nonempty) set of points between its lower and upper bound,

$$\boldsymbol{x} = \{x \in \mathbb{R}^n \mid \underline{x} \le x \le \overline{x}\},$$

so that a vector $x \in \mathbb{R}^n$ is *contained* in a box $\boldsymbol{x}$, i.e., $x \in \boldsymbol{x}$, iff $\underline{x} \le x \le \overline{x}$. Similarly, a *thin* box $\boldsymbol{x} = [x, x]$ (i.e., a box of zero width) is usually identified with the unique point $x$ it contains. A generic (arbitrary) point in a box $\boldsymbol{x}$ is often denoted by $x$ or $\tilde{x}$. The set of vertices of a box $\boldsymbol{x}$ (or more generally a polytope $S$) is denoted by $\mathrm{vert}\,\boldsymbol{x}$ ($\mathrm{vert}\,S$).

We write $\inf \boldsymbol{x} := \underline{x}$ for the *lower bound*, $\sup \boldsymbol{x} := \overline{x}$ for the *upper bound*, and $\dim x = n$ for the dimension of $\boldsymbol{x}$. The *width* of a box $\boldsymbol{x}$ is

$$\mathrm{wid}\,\boldsymbol{x} = \overline{x} - \underline{x} \ge 0;$$

its *radius* is

$$\mathrm{rad}\,\boldsymbol{x} = \frac{1}{2}\mathrm{wid}\,\boldsymbol{x} = \frac{1}{2}(\overline{x} - \underline{x}),$$

and its *midpoint* is

$$\mathrm{mid}\,\boldsymbol{x} = \frac{1}{2}(\overline{x} + \underline{x}).$$

*$\check{x}$ was reserved for the midpoint in [10], but is elsewhere used more generally for a* center*, i.e., a representative point (not necessarily the midpoint) used in centered forms.*

A (real, closed, nonempty) *interval* is a 1-dimensional box, i.e., a pair $\boldsymbol{x} = [\underline{x}, \overline{x}]$ consisting of two real numbers $\underline{x}$ and $\overline{x}$ with $\underline{x} \le \overline{x}$. The set of all intervals is denoted by $\mathbb{IR}$. A box $\boldsymbol{x}$ may be considered as an *interval vector* $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)^T$ with *components* $\boldsymbol{x}_k = [\underline{x}_k, \overline{x}_k]$. For example, if $\underline{x} = \binom{1}{3}$ and $\overline{x} = \binom{2}{4}$ then $\boldsymbol{x} = \binom{[1,2]}{[3,4]}$.

The *deviation* of an interval $\boldsymbol{x}$ is the real number $\operatorname{dev} \boldsymbol{x}$ defined by $\operatorname{dev} \boldsymbol{x} = \underline{x}$ if $|\underline{x}| \ge |\overline{x}|$, and $\operatorname{dev} \boldsymbol{x} = \overline{x}$ otherwise. The *mignitude* of an interval $\boldsymbol{x}$ is the number

$$\langle \boldsymbol{x} \rangle = \min\{|x| \mid x \in \boldsymbol{x}\}.$$

*Using $< \boldsymbol{x} >$ for the mignitude would make formulas more difficult to interpret, especially if used together with inequality signs.*

The interval-valued *absolute value function* is defined on intervals by

$$\operatorname{abs}(\boldsymbol{x}) = \{|x| \mid x \in \boldsymbol{x}\},$$

The *absolute value* of a box $\boldsymbol{x}$ is the real vector

$$|\boldsymbol{x}| = \max\{|x| \mid x \in \boldsymbol{x}\} = \sup\{-\underline{x}, \overline{x}\}.$$

In particular,

$$\operatorname{abs}(\boldsymbol{x}) = [\langle \boldsymbol{x} \rangle, |\boldsymbol{x}|] \quad \text{for intervals } \boldsymbol{x}.$$

The vector valued hypermetric between $\boldsymbol{x}$ and $\boldsymbol{y}$ is denoted by

$$\operatorname{dist}(\boldsymbol{x}, \boldsymbol{y}) = \sup\{|\underline{x} - \underline{y}|, |\overline{x} - \overline{y}|\};$$

the Hausdorff distance between two boxes $\boldsymbol{x}$ and $\boldsymbol{y}$ in the metric given by the maximum norm is then

$$\operatorname{dist}_\infty(\boldsymbol{x}, \boldsymbol{y}) = \|\operatorname{dist}(\boldsymbol{x}, \boldsymbol{y})\|_\infty,$$

and similarly for other specific norms.

*[10] used the traditional $q(\boldsymbol{x}, \boldsymbol{y})$ for $\operatorname{dist}(\boldsymbol{x}, \boldsymbol{y})$, which is less easy to understand.*

**Interval matrices.** An $m \times n$ *interval matrix* is a $m \times n$ matrix $\boldsymbol{A}$ whose entries $\boldsymbol{A}_{jk} = [\underline{A}_{jk}, \overline{A}_{jk}]$ $(j = 1, \dots, m,\ k = 1, \dots, n)$ are intervals. An interval matrix $\boldsymbol{A}$ is generally identified with the (nonempty) set of matrices $A$ with $A_{jk} \in \boldsymbol{A}_{jk}$ for all $j, k$, equivalently with $\underline{A} \le A \le \overline{A}$. The notation for boxes is adapted to interval matrices in the natural componentwise way. An exception is the mignitude, which is undefined for non-square matrices, and becomes the *comparison matrix* $\langle \boldsymbol{A} \rangle$ of a square matrix $\boldsymbol{A}$, defined as the matrix with diagonal components $\langle \boldsymbol{A} \rangle_{kk} = \langle \boldsymbol{A}_{kk} \rangle$ and off-diagonal components $\langle \boldsymbol{A} \rangle_{jk} = -|\boldsymbol{A}_{jk}|$ for $j \ne k$.

*This is needed for consistent usage in the context of H-matrices; see* Neumaier *[10, Chapter 3]. $\langle \boldsymbol{A} \rangle$ is undefined for matrices that are not square and for vectors of length $> 1$.*

**Operations and expressions.** A relation $\boldsymbol{x} \omega \boldsymbol{y}$ (with $\omega \in \{=, <, \le, >, \ge\}$ between two boxes $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined to be true iff $x \omega y$ for all $x \in \boldsymbol{x}, y \in \boldsymbol{y}$.

*This is the traditional, oldest interpretation, and is established. Of course, other interpretations are possible but should be designated differently. For example, the statement "$x\omega y$ for some $x \in \boldsymbol{x}, y \in \boldsymbol{y}$" could perhaps be denoted by $\boldsymbol{x}\omega^*\boldsymbol{y}$; but this should be defined in each paper using it.*

Operations (and elementary functions) are automatically interpreted as the natural operations on the class of objects involved; i.e., real for real (vector, matrix) arguments, interval if an argument is interval (vector, matrix).

In case of finite precision arithmetic, interval operations are assumed outward rounded. In case of conflicting interpretations (exact vs. rounded, or interval versus set operations), the recommended notation is fl(`expression`) for the floating point evaluation of an explicitly given expression `expression`, $\mathrm{fl}_\triangle$(`expression`) and $\mathrm{fl}_\nabla$(`expression`) for upward and downward directed rounding, respectively, flint(`expression`) for the outward rounded interval evaluation, and set(`expression`) for the set (Minkowski) evaluation.

fl(`expression`) *is commonly used in numerical analysis, and generalizes naturally in the form stated.*

Since expressions define unique functions from (part of) $\mathbb{IR}$ to $\mathbb{IR}$, other functions from $\mathbb{IR}$ to $\mathbb{IR}$ may also be written in bold if desired.

The image of a set $S$ under a mapping $f$ (which equals the range of $f$ for arguments in $S$) is denoted by

$$\mathrm{range}(f, S) = \{f(x) \mid x \in S\},$$

and the range of an expression $\mathsf{f}$ over a box $\boldsymbol{x}$ is

$$\mathrm{range}(\mathsf{f}, \boldsymbol{x}) = \{\mathsf{f}(x) \mid x \in \boldsymbol{x}\}.$$

*The use of $f(S)$ for the image of $S$ under $f$ is discouraged since, for boxes $S = \boldsymbol{x}$, a confusion with an interval evaluation may occur. Alternatives such as* $\mathrm{range}_{x\in S} f(x)$ *formed in analogy to the use of* min *or* lim *are acceptable.*

The interval hull of a set $S$ is denoted by $\square S$, and the interval evaluation of an expression $\mathsf{f}$ is $\mathsf{f}(\boldsymbol{x})$, so that $\mathrm{range}(\mathsf{f}, \boldsymbol{x}) \subseteq \mathsf{f}(\boldsymbol{x})$.

*$[\,]S$ is a much-used alternative symbol for interval hull, but has the disatvantage that its LaTeX definition does not adapt to different fonts and sizes.*

**Generalizations of intervals.** Complex intervals exist either as rectangles or as disks. If only one sort of complex intervals is used, the set of such intervals should be denoted by $\mathbb{IC}$, otherwise use $\mathbb{IC}_{\mathrm{rect}}$ for the set of complex rectangles and $\mathbb{IC}_{\mathrm{disc}}$ for the set of complex discs.

An *extended box* of dimension $n$ is either the empty set $\boldsymbol{x} = \emptyset$, or a pair $\boldsymbol{x} = [\underline{x}, \overline{x}]$ consisting of two column vectors $\underline{x} \in (\mathbb{R} \cup \{-\infty\})^n$ and $\overline{x} \in (\mathbb{R} \cup \{\infty\})^n$ with $\underline{x} \le \overline{x}$. ${}^*\mathbb{IR}^n$ denotes the set of extended boxes of dimension $n$. An *extended interval* is an extended box of dimension 1.

Kaucher [7, 8] completed interval arithmetic by introducing anti-intervals where the upper bound is smaller than the lower bound, and converse operations to the standard interval arithmetic operations. In particular, we have *inner* addition and *inner* subtraction,

$$\boldsymbol{x} \oplus \boldsymbol{y} = [\underline{x} + \overline{y}, \overline{x} + \underline{y}], \quad \boldsymbol{x} \ominus \boldsymbol{y} = [\underline{x} - \underline{y}, \overline{x} - \overline{y}],$$

and more compicated formulas for inner multiplication $\odot$ and inner division $\oslash$. The support of the resulting algebraic system — *Kaucher complete interval arithmetic* — consisting of both intervals and anti-intervals is denoted by $\mathbb{KR}$.

*Kaucher's notation $\mathbb{IR}$ conflicts with the later consensus usage of this for the set of intervals.*

All notation for intervals and boxes is extended to these generalizations in a straightforward way.

## 3 The `intmacros.sty` style file

To facilitate the acceptance of the proposed standard, a LaTeX style file is provided [5] (together with the LaTeX source of the present paper as an example of its use) that makes it easy to create documents conforming to the standard. The style file is designed to keep the LaTeX notation for intervals as simple as possible.

The style file uses `\a` for $\boldsymbol{a}$, `\A` for $\boldsymbol{A}$, etc., to denote bold face (i.e., interval) quantities. This notation is very short and quite convenient; even on the blackboard, `\A` is better than $[A]$ since it is shorter, and can be read naturally as "*interval A*".

`\mathsf f` gives the sansserif letter $\mathsf{f}$, etc., denoting an expression.

The style file uses `\Rz` for $\mathbb{R}$, and similarly for other open-faced upper case letters. `\ul x` and `\ol x` encode the lower bound $\underline{x}$ and the upper bound $\overline{x}$ of $\boldsymbol{x}$.

In some cases, the existence of already frequently used macros prevented the natural name for an abbreviation, and we modified it according to the annotation in the style file. For example, both `\vert` and `\Vert` are reserved in LaTeX for single and double vertical lines, respectively. So, the vertex set gets the abbreviation `\ivert` instead of `\vert`. Similarly, the interior is typed as `\iint` instead of `\int`, the midpoint as `\imid` instead of `\mid`, an interval or box $\boldsymbol{i}$ is typed as `\ii` instead of `\i`, $\boldsymbol{v}$ as `\vv` instead of `\v`, and an interval matrix $\boldsymbol{D}$ as `\DD` instead of `\D`.

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