New characterizations for the solution set to interval linear systems of equations

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Abstract. New characterizations of the points from the solution set to interval linear systems of equations are proposed, alternatives to the well-known result by W. Oettli and W. Prager. We also introduce recognizing functionals of the solution sets that determine, at a given point, aggregated quantitative measures of compatibility (consistency) between the interval data of the system.

Keywords: interval linear equations, solution set, characterization

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1 Introduction

We consider interval linear systems of equations of the form

$$\begin{cases} \boldsymbol{a}_{11}x_1 + \boldsymbol{a}_{12}x_2 + \ldots + \boldsymbol{a}_{1n}x_n = \boldsymbol{b}_1, \\ \boldsymbol{a}_{21}x_1 + \boldsymbol{a}_{22}x_2 + \ldots + \boldsymbol{a}_{2n}x_n = \boldsymbol{b}_2, \\ \quad \vdots \qquad\qquad \ddots \qquad\qquad \vdots \\ \boldsymbol{a}_{m1}x_1 + \boldsymbol{a}_{m2}x_2 + \ldots + \boldsymbol{a}_{mn}x_n = \boldsymbol{b}_m, \end{cases} \tag{1}$$

or, briefly,

$$\boldsymbol{A}x = \boldsymbol{b} \tag{2}$$

with an interval $m \times n$-matrix $\boldsymbol{A} = (\,\boldsymbol{a}_{ij}\,)$ and an interval m-vector $\boldsymbol{b} = (\,\boldsymbol{b}_i)$.

Below, we use the notation proposed in the informal standard [1]. In particular, intervals and interval values are denoted by bold letters, whereas noninterval (point) values are not specified in any special manner. Underlining $\underline{\boldsymbol{a}}$ and overlining $\overline{\boldsymbol{a}}$ denote the lower and upper endpoints of the interval $\boldsymbol{a}$, so that we have $\boldsymbol{a} = [\underline{\boldsymbol{a}}, \overline{\boldsymbol{a}}] = \{\, a \in \mathbb{R} \mid \underline{\boldsymbol{a}} \leqslant a \leqslant \overline{\boldsymbol{a}}\}$. Additionally,

$\operatorname{mid} \boldsymbol{a} = \frac{1}{2}(\overline{\boldsymbol{a}} + \underline{\boldsymbol{a}})$ is the midpoint of the interval,

$\operatorname{rad} \boldsymbol{a} = \frac{1}{2}(\overline{\boldsymbol{a}} - \underline{\boldsymbol{a}})$ is the radius of the interval,

$\langle \boldsymbol{a} \rangle = \begin{cases} \min\{\,|\overline{\boldsymbol{a}}|, |\underline{\boldsymbol{a}}|\,\}, & \text{if } 0 \notin \boldsymbol{a}, \\ 0, & \text{otherwise,} \end{cases}$ is the mignitude of the interval, i. e., the smallest distance from its points to zero.

The mignitude is, in a sense, an antipode of the absolute value (magnitude) of the interval, defined as $|\boldsymbol{a}| = \max_{a \in \boldsymbol{a}} |a| = \max\{\,|\underline{\boldsymbol{a}}|, |\overline{\boldsymbol{a}}|\,\}$.

Interval linear system of equations of the form (1)–(2),

$$\boldsymbol{A}x = \boldsymbol{b},$$

is a family of point (non-interval) linear systems $Ax = b$ of the same structure with $A = (a_{ij}) \in \boldsymbol{A}$ and $b = (b_i) \in \boldsymbol{b}$, i. e., such that $a_{ij} \in \boldsymbol{a}_{ij}$ and $b_i \in \boldsymbol{b}_i$ for all the indices i, j. The *solution set* to the interval linear system is defined as the set

$$\varXi(\boldsymbol{A}, \boldsymbol{b}) \;=\; \bigl\{\, x \in \mathbb{R}^n \mid (\exists\, A \in \boldsymbol{A})(\exists\, b \in \boldsymbol{b})(\, Ax = b\,)\,\bigr\},$$

that is, the set of all solutions to every point linear system $Ax = b$ whose coefficients and right-hand-sides belong to $\boldsymbol{A}$ and $\boldsymbol{b}$ respectively. The set $\varXi(\boldsymbol{A}, \boldsymbol{b})$ is often referred to as *the united solution set*, since there exist the other solution sets to the systems (1)–(2) (see [2]). We do not consider them in our text, thus using the brief term "solution set".

An analytic description of the solution set to interval linear systems is given by the result obtained by W. Oettli and W. Prager in 1964 [3]:

$$x \in \varXi(\boldsymbol{A}, \boldsymbol{b}) \quad\Leftrightarrow\quad \bigl|\,(\operatorname{mid} \boldsymbol{A})\, x - \operatorname{mid} \boldsymbol{b}\bigr| \leq (\operatorname{rad} \boldsymbol{A})\, |x| + \operatorname{rad} \boldsymbol{b}, \tag{3}$$

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where the operations mid, rad and $|\cdot|$ are applied to interval vectors and matrices in componentwise and elementwise manner, while the vector inequality is understood as the same inequality between every component. The purpose of our paper is to present new characterizations of the solution set $\varXi(\boldsymbol{A},\boldsymbol{b})$, different from the result by Oettli-Prager.

2 New characterizations

Our starting point is the following assertion, which is due to H. Beeck [4]:

The Beeck characterization. *A point $x \in \mathbb{R}^n$ belongs to the solution set $\varXi(\boldsymbol{A},\boldsymbol{b})$ if and only if $\boldsymbol{A}\cdot x \cap \boldsymbol{b} \neq \varnothing$, i. e., the interval vectors $\boldsymbol{A}\cdot x$ and $\boldsymbol{b}$ have nonempty intersection.*

In the formulation of the Beeck characterization, the product $\boldsymbol{A}\cdot x$ is understood as an interval-arithmetic product. Henceforth, we will omit the multiplication sign "$\cdot$", writing just $\boldsymbol{A}x$ instead of $\boldsymbol{A}\cdot x$. Specifically, the i-th component of the vector $\boldsymbol{A}x$ is, by definition, $\sum_{j=1}^{n} \boldsymbol{a}_{ij}x_j$, where all the operations are those from the classical interval arithmetic (see e. g., the books [5, 6]).

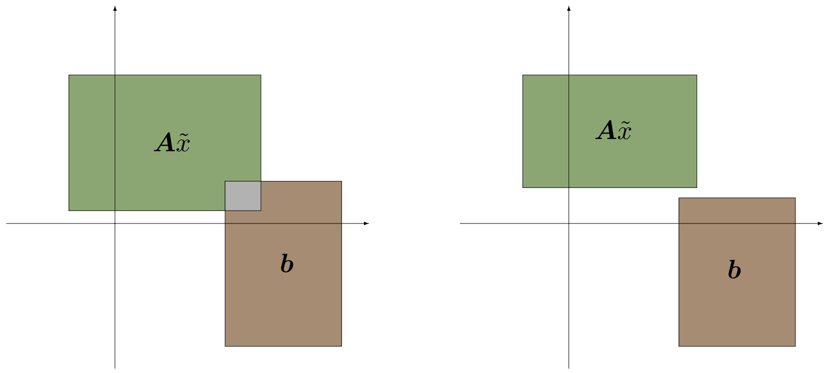

Figure 1: Mutual disposition of the boxes $\boldsymbol{A}\tilde{x}$ and $\boldsymbol{b}$

Testing the Beeck characterization for a point $\tilde{x} \in \mathbb{R}^n$ amounts to examination whether the interval boxes $\boldsymbol{A}\tilde{x}$ and $\boldsymbol{b}$ intersect with each other in the space $\mathbb{R}^n$ (see Fig. 1). To render the geometric ideas of the Beeck characterization into analytical language, let us consider the one-dimensional case first, i. e., intersection of two one-dimensional real intervals $\boldsymbol{p}$ and $\boldsymbol{q}$.

Let us shift the whole situation by mid $\boldsymbol{q}$, that is, let us consider the intervals $(\boldsymbol{p} - \operatorname{mid}\boldsymbol{q})$ and $(\boldsymbol{q} - \operatorname{mid}\boldsymbol{q})$ instead of the original $\boldsymbol{p}$ and $\boldsymbol{q}$. It is obvious that the intervals $(\boldsymbol{p} - \operatorname{mid}\boldsymbol{q})$ and $(\boldsymbol{q} - \operatorname{mid}\boldsymbol{q})$ intersect each other if and only if this is true for the intervals $\boldsymbol{p}$ and $\boldsymbol{q}$.

The intervals $(\boldsymbol{p} - \operatorname{mid}\boldsymbol{q})$ and $(\boldsymbol{q} - \operatorname{mid}\boldsymbol{q})$ have nonempty intersection if and only if the mignitude of the first interval does not exceed the absolute value of the second one. Since the absolute value of $(\boldsymbol{q} - \operatorname{mid}\boldsymbol{q})$ is evidently rad $\boldsymbol{q}$, then the above means that

$$\boldsymbol{p}\cap\boldsymbol{q} \neq \varnothing \qquad \Leftrightarrow \qquad \langle\, \boldsymbol{p} - \operatorname{mid}\boldsymbol{q}\,\rangle \leq \operatorname{rad}\boldsymbol{q} \tag{4}$$

(see Fig. 2). Also, the last inequality can be rewritten as

$$\operatorname{rad}\boldsymbol{q} - \langle\, \boldsymbol{p} - \operatorname{mid}\boldsymbol{q}\,\rangle \;\geq\; 0. \tag{5}$$

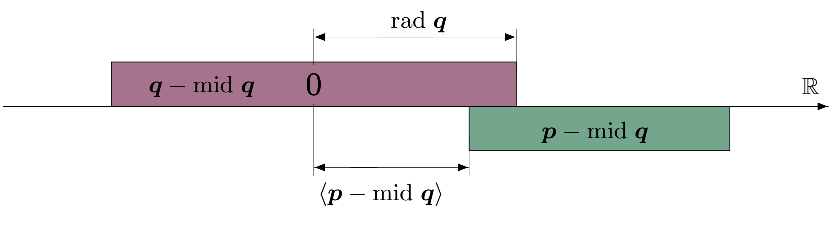

Figure 2: Intersection of shifted intervals (zero is the midpoint of $\boldsymbol{q} - \operatorname{mid}\boldsymbol{q}$)

Insofar as the multidimensional interval boxes are direct products of one-dimensional intervals, then, testing the Beeck characterization and examining intersection of $\boldsymbol{A}\tilde{x}$ and $\boldsymbol{b}$, we can claim that

$$\boldsymbol{A}\tilde{x} \cap \boldsymbol{b} \neq \varnothing \quad \Leftrightarrow \quad \big\langle (\boldsymbol{A}\tilde{x})_i - \operatorname{mid} \boldsymbol{b}_i \big\rangle \leq \operatorname{rad} \boldsymbol{b}_i, \qquad i = 1, 2, \dots, m. \tag{6}$$

If we fix the understanding that the operations $\langle\,\cdot\,\rangle$, $\operatorname{rad}(\cdot)$ and the inequality "$\leq$" are applied to interval vectors in a component-wise manner, then the system (6) can be reduced to a concise form:

$$\boldsymbol{A}\tilde{x} \cap \boldsymbol{b} \neq \varnothing \quad \Leftrightarrow \quad \big\langle \boldsymbol{A}\tilde{x} - \operatorname{mid} \boldsymbol{b} \big\rangle \leq \operatorname{rad} \boldsymbol{b}.$$

Overall,

$$\boxed{\begin{array}{l}\text{a point } \tilde{x} \text{ is in the solution set to an interval linear system} \\ \text{of equations } \boldsymbol{A}x = \boldsymbol{b} \text{ if and only if } \big\langle \boldsymbol{A}\tilde{x} - \operatorname{mid} \boldsymbol{b} \big\rangle \leq \operatorname{rad} \boldsymbol{b}.\end{array}} \tag{7}$$

On the other hand, we can revert the situation and take $\boldsymbol{p} = \boldsymbol{b}_i$ and $\boldsymbol{q} = (\boldsymbol{A}\tilde{x})_i$, $i = 1, 2, \dots, m$, in the equivalence (4). Then we get

$$\boldsymbol{A}\tilde{x} \cap \boldsymbol{b} \neq \varnothing \quad \Leftrightarrow \quad \big\langle \operatorname{mid}(\boldsymbol{A}\tilde{x}) - \boldsymbol{b} \big\rangle \leq \operatorname{rad}(\boldsymbol{A}\tilde{x}).$$

To simplify the above formula, we can use the equalities (see e. g., [5])

$$\operatorname{mid}(\boldsymbol{A}\tilde{x}) = (\operatorname{mid}\boldsymbol{A})\,\tilde{x}, \qquad \operatorname{rad}(\boldsymbol{A}\tilde{x}) = (\operatorname{rad}\boldsymbol{A})\,|\tilde{x}|.$$

Overall,

$$\boxed{\begin{array}{l}\text{a point } \tilde{x} \text{ belongs to the solution set to an interval linear system} \\ \text{of equations } \boldsymbol{A}x = \boldsymbol{b} \text{ if and only if } \big\langle (\operatorname{mid}\boldsymbol{A})\,\tilde{x} - \boldsymbol{b} \big\rangle \leq (\operatorname{rad}\boldsymbol{A})\,|\tilde{x}|.\end{array}} \tag{8}$$

The equivalences (7) and (8) are new analytic characterizations of the points from the solution sets to interval linear systems, alternatives to the well-known Oettli-Prager inequality (3).

3 Recognizing functionals

In fact, we can proceed further, constructing recognizing functionals of the solution set on the basis of (7) and (8). Let us show that on the example of the equivalence (7).

The main idea is to convolve the separate componentwise inequalities (5) to a single relation. This can be done by various ways, but several natural requirements should be fulfilled. The constructed expression must be negative in case at least one of the one-dimensional expressions is negative (which corresponds to empty intersection of $\boldsymbol{A}x$ and $\boldsymbol{b}$). The expression must be non-negative (positive) if non-negative (positive) are all the one-dimensional expressions (when the interval boxes $\boldsymbol{A}x$ and $\boldsymbol{b}$ have nonempty intersection).

It is undesirable that, in the resulting expression, the values of the one-dimensional expressions are combined with each other in such a way that one of them could compensate the values of the others. The contributions of all the subexpressions should be regarded on an equal basis.

For our purposes, "min" operation suits well, since it takes into account the separate expressions "uniformly". To summarize, we can take, as a generalized "compatibility measure" of a point $\tilde{x}$, the expression

$$\min_{1 \leq i \leq m} \big\{ \operatorname{rad} \boldsymbol{b}_i - \big\langle (\boldsymbol{A}\tilde{x})_i - \operatorname{mid} \boldsymbol{b}_i \big\rangle \big\}, \tag{9}$$

which can be written out in expanded form as

$$\min_{1 \leq i \leq m} \left\{ \operatorname{rad} \boldsymbol{b}_i - \left\langle \sum_{j=1}^{n} \boldsymbol{a}_{ij}\tilde{x}_j - \operatorname{mid} \boldsymbol{b}_i \right\rangle \right\}.$$

We thus arrive at the following result, first formulated in [7]:

Theorem. *Let $\boldsymbol{A}$ be an interval $m\times n$-matrix and $\boldsymbol{b}$ be an interval m-vector. Then the expression*

$$\operatorname{Uni}(x, \boldsymbol{A}, \boldsymbol{b}) \;=\; \min_{1 \leq i \leq m} \left\{ \operatorname{rad} \boldsymbol{b}_i - \left\langle \sum_{j=1}^{n} \boldsymbol{a}_{ij}x_j - \operatorname{mid} \boldsymbol{b}_i \right\rangle \right\}$$

defines a functional $\operatorname{Uni} : \mathbb{R}^n \to \mathbb{R}$*, such that the membership of a point $x \in \mathbb{R}^n$ in the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ to the interval linear system $\boldsymbol{A}x = \boldsymbol{b}$ is equivalent to non-negativity of the functional* Uni *at the point x. In other words, $x \in \Xi(\boldsymbol{A}, \boldsymbol{b})$ if and only if* $\operatorname{Uni}(x, \boldsymbol{A}, \boldsymbol{b}) \geq 0$.

As a consequence, the solution set $\varXi(\boldsymbol{A}, \boldsymbol{b})$ to the interval linear system is a "level set"

$$\{\ x \in \mathbb{R}^n \mid \mathrm{Uni}\,(x, \boldsymbol{A}, \boldsymbol{b}) \geq 0\ \}$$

for the functional Uni . It turns out that the functional Uni "recognizes", by the sign of its values, the membership of a point in the solution set $\varXi(\boldsymbol{A}, \boldsymbol{b})$. This is why we call Uni *recognizing functional.*

For the characterization (8), the corresponding recognizing functional is

$$\min_{1 \leq i \leq m} \left\{ \sum_{j=1}^{n} (\mathrm{rad}\ \boldsymbol{a}_{ij})\,|x_j| - \left\langle \sum_{j=1}^{n} (\mathrm{mid}\ \boldsymbol{a}_{ij})\,x_j - \boldsymbol{b}_i \right\rangle \right\}.$$

Again, it takes non-negative values if and only if its argument is a point from the solution set to the interval linear system $\boldsymbol{A}x = \boldsymbol{b}$. Finally, the Oettli-Prager characterization (3) also can be folded into the recognizing functional [9, 10]

$$\min_{1 \leq i \leq m} \left\{ \sum_{j=1}^{n} (\mathrm{rad}\ \boldsymbol{a}_{ij})\,|x_j| + \mathrm{rad}\ \boldsymbol{b}_i - \left| \sum_{j=1}^{n} (\mathrm{mid}\ \boldsymbol{a}_{ij})\,x_j - \mathrm{mid}\ \boldsymbol{b}_i \right| \right\}.$$

Notice that the values of every recognizing functional provide us with an aggregated quantitative measure of how the point x is compatible with the interval data $\boldsymbol{A}$ and $\boldsymbol{b}$ of the linear system. Maximizing this measure over all x's gives compatibility measure for the interval linear system $\boldsymbol{A}x = \boldsymbol{b}$ in general.

What could be the practical application of our theoretical results? The above observation can be used as the basis for a new approach to the solution of data fitting problems under unterval uncertainty. For linear regression models, solving such problems reduces to interval linear systems of the form $\boldsymbol{A}x = \boldsymbol{b}$, where $\boldsymbol{A}$ represents input data and $\boldsymbol{b}$ stands for outputs. Then we can take, as parameters of the best fit regression line, the point that maximizes the recognizing functional, i. e., corresponds to maximal compatibility (consistency) with the interval data $\boldsymbol{A}$ and $\boldsymbol{b}$. See details in [7, 8, 10].

Further possible development of the above theory that results in a promising solvability examination technique for interval linear systems may be found, for instance, in the papers [7, 9, 10].

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