Interval matrices: Regularity generates singularity*

Jiri Rohn [1, †], Sergey P. Shary [2, ‡]

[1]Institute of Computer Science, Czech Academy of Sciences, Prague, Czech Republic
E-mail: rohn@cs.cas.cz, URL: http://uivtx.cs.cas.cz/~rohn

[2]Institute of Computational Technologies SB RAS, Novosibirsk, Russia
E-mail: shary@ict.nsc.ru, URL: http://www.nsc.ru/interval/shary

Abstract

It is proved that regularity of an interval matrix implies singularity of four related interval matrices. The result is used to prove that for each nonsingular point matrix A, either Aor A^{-1} can be brought to a singular matrix by perturbing only the diagonal entries by an amount of at most 1 each. As a consequence, the notion of a diagonally singularizable matrix is introduced.

Keywords: interval matrix, regularity, singularity, P-matrix, absolute value equation, diagonally singularizable matrix.

Mathematical Subject Classification 2010: 15A09, 65G40

1 Introduction and notation

Throughout this paper matrix and vector inequalities, as well as the absolute value, are understood entrywise. Also, intervals and other interval objects are denoted by bold letters in accordance with the informal standard [1].

As is well known, a square interval matrix

$$\mathbf{A} = [A - D,\, A + D] = \{\, B \mid |B - A| \le D \,\},$$

where $D \geq 0$, is called *singular* if it contains a singular matrix, and it is said to be *regular* otherwise. Regularity/singularity is an important concept in the classical matrix theory, and this is true for interval matrices too. Interested readers can get acquainted with

†This author's work was supported with institutional support RVO:67985807.

‡Corresponding author.

*Published in *Linear Algebra and its Applications*, vol. 540 (2018), pp. 149–159.
doi:10.1016/j.laa.2017.11.020.

the details of research on this topic in the works [2, 3, 4, 5, 6] as well as in the surveys [7, 8, 9]. Some generalizations to rectangular interval matrices are given in [10].

Common sense dictates that singularity and regularity exclude each other. Yet in this paper we are going to show that regularity of an interval matrix implies singularity of four interval matrices constructed from it in a nontrivial way (we add the word "nontrivial" to emphasize that we do not take into account interval matrices like $[A-A,\, D-D]$ that are trivially singular). For the proof of these results we need four auxiliary theorems that are seemingly not generally known and that are listed in Section 2. Our main result is then formulated in Theorem 3 which says that regularity of $\boldsymbol{A}$ implies singularity of four interval matrices

$$[D-|A|,\, D+|A|],$$
$$[A^{-1}D-I,\, A^{-1}D+I],$$
$$[DA^{-1}-I,\, DA^{-1}+I],$$
$$[A^{-1}-|D^{-1}|,\, A^{-1}+|D^{-1}|],$$

I being the identity matrix. From this result, we draw in Section 4 a purely linear algebraic (i. e., non-interval) consequence: for each nonsingular square matrix A either there exists a singular matrix S_1 satisfying

$$|A-S_1| \le I,$$

or there exists a singular matrix S_2 satisfying

$$|A^{-1}-S_2| \le I.$$

In addition, we introduce the concept of *diagonally singularizable matrices* and give its practical motivations. Last Section 5 brings some examples.

2 Auxiliary results

In this section we sum up four auxiliary results that will be used in the proof of the main Theorem 3 in Section 3. The first two come from [4], [11] and concern some properties of P-matrices. Let us recall that a square matrix is called a P-matrix if all its principal minors are positive.

Theorem 1. *If an interval matrix $\boldsymbol{A}$ is regular, then for each $A_1, A_2 \in \boldsymbol{A}$, $A_1^{-1}A_2$ is a P-matrix.*

Theorem 2. *If A is a P-matrix, then $Ax > 0$ for some $x > 0$.*

The next result which can be found in [12] characterizes singular interval matrices. It shows that in order to demonstrate existence of a singular matrix in an interval matrix it is sufficient to find a solution of a nonlinear vector inequality. In fact, singularity of all four interval matrices introduced in Theorem 3 will be established in this way.

Theorem 3. *An interval matrix* $[A - D, A + D]$ *is singular if and only if the inequality*

$$|Ax| \leq D|x| \tag{1}$$

has a nontrivial (i.e., nonzero) solution. Moreover, for each $x \neq 0$ *satisfying* (1) *there exists a singular matrix* $S \in [A - D, A + D]$ *with* $Sx = 0$.

The last auxiliary result proved in [12] shows a connection between regularity of interval matrices and unique solvability of absolute value equations.

Theorem 4. *If the interval matrix* $[A - |B|, A + |B|]$ *is regular, then the absolute value equation*

$$Ax + B|x| = b$$

has a unique solution for each right-hand side b.

3 Regularity generates singularity

In the main result of this paper we show how regularity "generates" (or, implies) singularity.

Theorem 5. *If* $\boldsymbol{A} = [A - D, A + D]$ *is regular, then each of the interval matrices*

$$\begin{aligned}
\boldsymbol{A}_1 &= [D - |A|, D + |A|], &(2)\\
\boldsymbol{A}_2 &= [A^{-1}D - I, A^{-1}D + I], &(3)\\
\boldsymbol{A}_3 &= [DA^{-1} - I, DA^{-1} + I], &(4)\\
\boldsymbol{A}_4 &= [A^{-1} - |D^{-1}|, A^{-1} + |D^{-1}|] &(5)
\end{aligned}$$

is singular, the last of them under an additional assumption of invertibility of D.

Proof. (a) First we prove that under regularity assumption the interval matrix $\boldsymbol{A}_2$ is singular. Denote $C = A^{-1}D$. Notice that $I - C = A^{-1}(A - D)$, as a product of two nonsingular matrices, is nonsingular. Consider the matrix

$$(I - C)^{-1}(I + C) = (A - D)^{-1}(A + D).$$

By Theorem 2 the matrix $(A - D)^{-1}(A + D)$ is a P-matrix, hence so is the product $(I - C)^{-1}(I + C)$, and Theorem 2 implies existence of an $\tilde{x} > 0$ satisfying

$$(I - C)^{-1}(I + C)\tilde{x} > 0. \tag{6}$$

Put $x = (I - C)^{-1}\tilde{x}$. Then $(I - C)x = \tilde{x} > 0$, hence

$$Cx < x, \tag{7}$$

and because of the identity

$$(I + C)(I - C) = (I - C^2) = (I - C)(I + C),$$

from (6) we have

$$0 < (I - C)^{-1}(I + C)\tilde{x} = (I - C)^{-1}(I + C)(I - C)x = (I + C)x.$$

The latter gives $-x < Cx$ and, together with (7), yields

$$-x < Cx < x, \tag{8}$$

that is

$$|Cx| < x. \tag{9}$$

This inequality immediately implies $x > 0$, so that we can rewrite (9) as

$$|Cx| < I|x|,$$

which in the light of Theorem 2 means that the interval matrix $[C - I, C + I]$ is singular.

(b) Next we prove that the interval matrix $\boldsymbol{A}_1$ is singular. From (9) we know that regularity of $[A - D, A + D]$ implies existence of a positive vector x satisfying

$$|A^{-1}Dx| < x.$$

Hence we have

$$|Dx| = |AA^{-1}Dx| \le |A||A^{-1}Dx| < |A|x = |A||x|$$

and Theorem 2 implies singularity of $[D - |A|, D + |A|]$.

(c) To prove singularity of $\boldsymbol{A}_3$, we use the fact that by Theorem 2 regularity of $[A - D, A + D]$ implies existence of a unique solution x^* of the absolute value equation

$$Ax - D|x| = e,$$

where e denotes the vector of all ones. Then

$$Ax^* = D|x^*| + e > D|x^*|,$$

so that $x = Ax^*$ satisfies

$$|DA^{-1}x| \le D|A^{-1}x| < x = I|x|, \tag{10}$$

and Theorem 2 proves singularity of $\boldsymbol{A}_3$.

(d) Finally, to prove singularity of $\boldsymbol{A}_4$, we use the previously established inequality (10) to show that under the assumed invertibility of D there holds

$$|A^{-1}x| = |D^{-1}DA^{-1}x| \le |D^{-1}|D|A^{-1}x| \le |D^{-1}||x|.$$

The latter enables us to employ again Theorem 2 to conclude that the interval matrix $\boldsymbol{A}_4$ is singular. ■

In fact, we have proved a little more.

Theorem 6. *If $\boldsymbol{A}$ is regular, then each of the interval matrices $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{A}_3$, $\boldsymbol{A}_4$ contains a singular matrix S satisfying $Sx = 0$ for some $x > 0$.*

Proof. Positive vectors satisfying $|A^{-1}Dx| < |x|$ or $|DA^{-1}x| < |x|$ were constructed in parts (a), (c) of the previous proof and were then utilized in parts (b), (d) as well. ■

We have this consequence.

Theorem 7. *If $\boldsymbol{A}$ is regular, then for each $j = 1, \ldots, 4$ no row of the lower bound of the interval matrix $\boldsymbol{A}_j$ is positive and no row of its upper bound is negative.*

Proof. Let $1 \leq j \leq 4$. According to Theorem 3 there exists a singular matrix $S \in \boldsymbol{A}_j$ such that $Sx = 0$ for some $x > 0$. Assume to the contrary that the ith row of the lower bound of $\boldsymbol{A}_j$ is positive for some i. Then the ith row of S is positive, hence $(Sx)_i > 0$ contrary to $Sx = 0$, a contradiction. The second assertion concerning the upper bound can be proved in a similar way. ■

It turns out that in many cases singularity of $\boldsymbol{A}_1$ or $\boldsymbol{A}_4$ can be observed immediately because of presence of the zero matrix.

Theorem 8. *If $[A - D, A + D]$ is regular and*

$$D \leq |A| \tag{11}$$

holds, then $0 \in \boldsymbol{A}_1$. Similarly, if

$$|A^{-1}| \leq |D^{-1}| \tag{12}$$

holds, then $0 \in \boldsymbol{A}_4$.

Proof. Indeed, if (11) holds, then $-|A| \leq -D \leq |A|$ which implies $D - |A| \leq 0 \leq D + |A|$ so that $0 \in \boldsymbol{A}_1$. Similarly for $\boldsymbol{A}_4$. ■

4 Diagonally singularizable matrices

We shall say that a square point matrix A is *diagonally singularizable* if there exists a singular matrix S of the same size satisfying

$$|A - S| \leq I. \tag{13}$$

This means that A can be brought to singularity by shifting (only!) its diagonal entries by an amount of at most 1 each.

The definition (13) corresponds to the diagonal perturbation of the unit magnitude. The point is that, with the help of a suitable perturbation of only diagonal elements, any matrix can be made singular, but the magnitude of the perturbation required may be arbitrarily large.

In fact, let an $n \times n$-matrix A be given, and consider the diagonal matrices

$$I = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \qquad \text{and} \qquad J = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix},$$

of the same size as A. It is clear that $\det I = 1$ and $\det J = -1$. If the number $\varkappa > 1$ is sufficiently large, then the matrix $A + \varkappa I$ is almost equal to $\varkappa I$, and $\det(A + \varkappa I)$ should be close to $\det(\varkappa I) = \varkappa^n > 0$. Similarly, if the number $\theta > 1$ is sufficiently large, then the matrix $A + \theta J$ is almost equal to θJ, and $\det(A + \theta J)$ should be close to $\det(\theta J) = -\theta^n < 0$. Therefore, the continuous real function

$$g(t) = \det\bigl(A + t\varkappa I + (1-t)\theta J\bigr)$$

changes its sign at the interval $t \in [0, 1]$ insofar as

$$g(0) = \det\bigl(A + \theta J\bigr) \qquad \text{and} \qquad g(1) = \det\bigl(A + \varkappa J\bigr).$$

In view of the intermediate value theorem, we can conclude that there must exist a number $t^\star \in [0, 1]$ such that the matrix $A + t^\star\varkappa I + (1-t^\star)\theta J$ has zero determinant, i. e., it is singular. Moreover, it is made of the matrix A by the perturbation $t^\star\varkappa I + (1-t^\star)\theta J$ affecting only diagonal elements of A.

So, any matrix can be made singular by a suitable perturbation of its diagonal, but, of course, the magnitude of this perturbation plays an important role in practice. In our approach, it makes sense to confine ourselves to a fixed finite level of the perturbation magnitude, and the unit is the most natural choice. Then, being interested in investigating the specific magnitude of the diagonal perturbation, we can scale it up to a perturbation of a unit value and do the same with the matrix under study.

Investigation of the effect that perturbations of only diagonal elements exert on a matrix can be very important for practical reasons. One of the most popular interpretations of a square matrix is known to be a table of numbers that characterize the interrelation (interaction) between elements of a finite set, which can be, e. g., material flows from one element to another, mutual forces or impacts and such like. At the intersection of the ith row and the jth column of the matrix, we specify the measure of the mutual connection (interaction) of the ith and jth elements of the considered set. Then the diagonal elements correspond to the connections (interactions) of the elements with themselves. The question under study about the influence of diagonal perturbations on the matrix is thus equivalent to the question of how the connections of the elements with themselves affects the system on the whole. The concept of diagonal singularizability of a matrix helps studying this issue.

The next theorem indicates that diagonally singularizable matrices occur more frequently than one could expect.

Theorem 9. *For each nonsingular square matrix A, either A or A^{-1} is diagonally singularizable.*

Proof. Consider the auxiliary interval matrix

$$\boldsymbol{B} = [A^{-1} - I,\ A^{-1} + I].$$

If $\boldsymbol{B}$ is singular, then there exists a singular matrix satisfying $|A^{-1} - S| \leq I$ which means that A^{-1} is diagonally singularizable. If $\boldsymbol{B}$ is regular, then by Theorem 3, case (3), the interval matrix $[A - I,\ A + I]$ is singular, hence A is diagonally singularizable. ∎

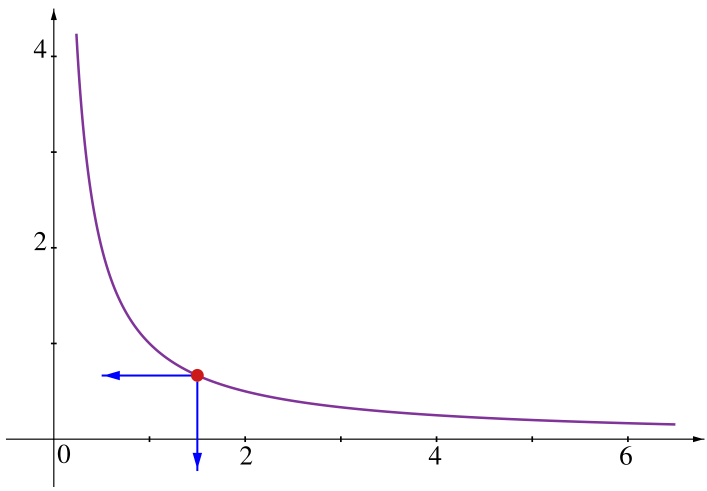

Figure 1: Points of the hyperbola can be reduced to singularity by unit shifting

This is a generalization of a phenomenon that is almost obvious in the one-dimensional case, when we have just 1×1-matrix, i. e., a scalar. Fig. 1 shows an illustration of the effect. Points of the hyperbola $y = 1/x$, having the coordinates (x, x^{-1}), can be reduced to singularity, when one of the coordinates becomes infinite, by shifting by at most 1 along any coordinate direction.

5 Examples

In this section we give two examples. Regularity or singularity of interval matrices was checked by the MATLAB file `regising.m` available at `http://uivtx.cs.cas.cz/~rohn/other/regising.m`. It is invoked by

```
>> S = regising(A,D)
```

If the output argument S is nonempty, then S is a singular matrix within $[A-D,\ A+D]$; if it is empty, then $[A-D,\ A+D]$ is regular.

Consider a randomly generated regular 3×3 interval matrix

```
A =
   [ 4.5489    4.7230]  [-8.7167  -8.3588]  [-3.9384  -3.8631]
   [-1.6413   -1.5949]  [-8.4549  -8.2373]  [-9.6674  -9.6564]
   [ 6.2962    6.6055]  [-3.0349  -2.7066]  [ 1.1286   1.3471]
```

As it satisfies (11), the interval matrix $\boldsymbol{A}_1$ (denoted here by `A1`) contains the zero matrix by Theorem 3, but our file `regising.m` finds instead another singular matrix `S1` in $\boldsymbol{A}_1$. To confirm that `S1` is singular, we computed, by MATLAB, its rank under the variable `rnk1`:

```
A1 =
   [-4.5489   4.7230]  [ -8.3588   8.7167]  [-3.8631   3.9384]
   [-1.5949   1.6413]  [ -8.2373   8.4549]  [-9.6564   9.6674]
   [-6.2962   6.6055]  [ -2.7066   3.0349]  [-1.1286   1.3471]

S1 =
    0.0870    0.1789    0.0377
    0.0247    0.1013   -0.0032
    0.1547    0.1642    0.1092

rnk1 =
     2
```

Next, all off-diagonal entries of the interval matrices $\boldsymbol{A}_2$ and $\boldsymbol{A}_3$ are point intervals by (3), (4) and the file finds a singular matrix in each of them:

```
A2 =
  [-0.9555   1.0445]   [ 0.0340   0.0340]   [ 0.0375   0.0375]
  [ 0.0305   0.0305]   [-0.9913   1.0087]   [ 0.0315   0.0315]
  [-0.0362  -0.0362]   [-0.0244  -0.0244]   [-1.0341   0.9659]

S2 =
    0.0445    0.0340    0.0375
    0.0305    0.0087    0.0315
   -0.0362   -0.0244   -0.0318
```

```
rnk2 =
     2

A3 =
  [-1.1078   0.8922]   [ 0.0530   0.0530]   [ 0.1043   0.1043]
  [-0.0645  -0.0645]   [-0.9671   1.0329]   [ 0.0582   0.0582]
  [-0.0852  -0.0852]   [ 0.0351   0.0351]   [-0.9060   1.0940]

S3 =
   -0.1078    0.0530    0.1043
   -0.0645    0.0341    0.0582
   -0.0852    0.0351    0.0940

rnk3 =
     2
```

Next, the interval matrix $\boldsymbol{A}_4$ again contains the zero matrix, and again the file finds another singular matrix `S4`:

```
A4 =
  [-67.3660  66.6764] [-81.3642  81.7584] [-18.5441  19.4486]
  [-10.8322   9.7395] [-22.1849  22.7446] [ -1.9505   2.8761]
  [-78.9143  79.9736] [-82.1085  81.3520] [-32.9024  31.9514]

S4 =
   -0.3448    0.1971    0.4522
   -0.5463    0.2799    0.4628
    0.5994   -0.3066   -0.5040

rnk4 =
     2
```

Finally, to illustrate Theorem 4, consider a random 6×6 point matrix

```
A =
   -1.2807   -1.7660    1.4231    0.0250    4.2500    3.5528
    6.4249    6.6608   -7.1955   -5.3853   -5.8594   -3.7186
    3.2943   -0.3531   -7.5444    2.5959   -5.6015    8.3778
    2.9109   -9.7564    0.7534   -2.0974    8.7650   -7.4650
    6.8797    7.9521   -0.2415    8.6383    9.1531    8.8403
    8.9885   -4.8639   -0.4357    0.5714    3.6866    7.9766
```

Here the interval matrix $[A-I,\ A+I]$ is regular, hence A is not diagonally singularizable, and we know from Theorem 4 that A^{-1} (denoted here by `Ainv`) must be diagonally singularizable.

```
Ainv =
   -0.1380    0.0093   -0.0410   -0.0039    0.0115    0.0925
    0.0390    0.0457   -0.0395   -0.0372    0.0330   -0.0258
   -0.1137   -0.0496   -0.1197   -0.0770   -0.0202    0.1035
   -0.2096   -0.0929    0.0526    0.0322    0.0527   -0.0334
    0.1500    0.0344    0.0018    0.0483    0.0353   -0.0466
    0.1187    0.0054    0.0110   -0.0470   -0.0140    0.0350

S =
   -0.1380    0.0093   -0.0410   -0.0039    0.0115    0.0925
    0.0390    0.0457   -0.0395   -0.0372    0.0330   -0.0258
   -0.1137   -0.0496   -0.1197   -0.0770   -0.0202    0.1035
   -0.2096   -0.0929    0.0526    0.0322    0.0527   -0.0334
    0.1500    0.0344    0.0018    0.0483   -0.0740   -0.0466
    0.1187    0.0054    0.0110   -0.0470   -0.0140    0.0350

rnk =
     5
```

We can see that in fact a single change at the $(5,5)$th entry of `Ainv` was sufficient to turn it into a singular matrix.

Acknowledgment

The authors thank an anonymous referee for helpful suggestions.

References

[1] R. Kearfott, M. Nakao, A. Neumaier, S. Rump, S. Shary, P. van Hentenryck, Standardized notation in interval analysis, Computational Technologies 15 (1) (2010) 7–13. www.ict.nsc.ru/jct/getfile.php?id=1345.

[2] M. Baumann, A regularity criterion for interval matrices, in: J. Garloff (Ed.), Collection of Scientific Papers Honouring Prof. Dr. K. Nickel on Occasion of his 60th Birthday, Part I, Albert-Ludwigs-Universität, Freiburg, 1984, pp. 45–50.

[3] H. Beeck, Zur Problematik der Hüllenbestimmung von Intervallgleichungssystemen, in: K. Nickel (Ed.), Interval Mathematics. Proceedings of the International Symposium, Karlsruhe, West Germany, May 20-24, 1975, Vol. 29 of Lecture Notes in Computer Science, Springer-Verlag, Berlin, 1975, pp. 150–159. doi:10.1007/3-540-07170-9_12.

[4] J. Rohn, Systems of linear interval equations, Linear Algebra and Its Applications 126 (1989) 39–78. doi:10.1016/0024-3795(89)90004-9.

[5] S. Rump, Verification methods for dense and sparse systems of equations, in: J. Herzberger (Ed.), Topics in Validated Computations, North-Holland, Amsterdam, 1994, pp. 63–135.

[6] E. Hansen, A theorem on regularity of interval matrices, Reliable Computing 11 (6) (2005) 495–497. doi:10.1007/s11155-005-0405-9.

[7] J. Rohn, Forty necessary and sufficient conditions for regularity of interval matrices: A survey, Electronic Journal of Linear Algebra 18 (2009) 500–512.

[8] A. Neumaier, Interval Methods for Systems of Equations, Cambridge University Press, Cambridge, 1990.

[9] S. Shary, Finite-Dimensional Interval Analysis, XYZ Publishers, Novosibirsk, 2017 (in Russian) www.nsc.ru/interval/Library/InteBooks/SharyBook.pdf.

[10] S. Shary, On full-rank interval matrices, Numerical Analysis and Applications 7 (3) (2014) 241–254. doi:10.1134/S1995423914030069.

[11] D. Gale, H. Nikaido, The Jacobian matrix and global univalence of mappings, Mathematische Annalen 159 (1965) 81–93. doi:10.1007/BF01360282.

[12] J. Rohn, An algorithm for solving the absolute value equation, Electronic Journal of Linear Algebra 18 (2009) 589–599.