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Chapter 2
A New Method for Inner Estimation of Solution Sets to Interval Linear Systems

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Abstract For an interval system of linear equations $\boldsymbol{A}x = \boldsymbol{b}$, we consider the problem of inner estimation of its solution set, formed by all the solutions to point systems $Ax = b$ with $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. The so-called "center approach" to the problem is developed when the inner interval box is constructed around an a priori known center point from the solution set. Determining the size of the inner box is shown to be reduced to a maximization problem for a special quasiconcave objective function.

2.1 Introduction

In our work, we consider interval linear equations systems of the form

$$\left\{\begin{array}{ccccccccc} \boldsymbol{a}_{11}x_1 & + & \boldsymbol{a}_{12}x_2 & + & \ldots & + & \boldsymbol{a}_{1n}x_n & = & \boldsymbol{b}_1, \\ \boldsymbol{a}_{21}x_1 & + & \boldsymbol{a}_{22}x_2 & + & \ldots & + & \boldsymbol{a}_{2n}x_n & = & \boldsymbol{b}_2, \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ \boldsymbol{a}_{m1}x_1 & + & \boldsymbol{a}_{n2}x_2 & + & \ldots & + & \boldsymbol{a}_{mn}x_n & = & \boldsymbol{b}_m, \end{array}\right. \tag{2.1}$$

or, briefly,

$$\boldsymbol{A}x = \boldsymbol{b}, \tag{2.2}$$

where $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ is an interval $m \times n$-matrix and $\boldsymbol{b} = (\boldsymbol{b}_i)$ is an interval m-vector. The above interval systems are understood as families of usual "point" linear systems $Ax = b$ with the same structure, while the matrices A are taken from $\boldsymbol{A}$ and the vectors b are taken from $\boldsymbol{b}$.

It is well-known that, for interval equations systems, solutions and solution sets can be defined in various ways (see e.g. [15–18]), but below we confine ourselves

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only to the so-called *united solution set* for (2.1)–(2.2), the set formed by solutions x to the point systems $Ax = b$ with the matrix A and right-hand side vector b independently varying through $\boldsymbol{A}$ and $\boldsymbol{b}$ respectively. The united solution set is rigorously defined as

$$\Xi(\boldsymbol{A},\boldsymbol{b}) = \{\, x \in \mathbb{R}^n \mid (\exists A \in \boldsymbol{A})(\exists b \in \boldsymbol{b})(Ax = b)\,\}, \tag{2.3}$$

and it is called just *solution set* for (2.1)–(2.2) in the rest of the paper, insofar as the other solution sets are not treated herein.

The solution set $\Xi(\boldsymbol{A},\boldsymbol{b})$ is known to be a polyhedral set, generally nonconvex, while its intersection with each orthant of the space $\mathbb{R}^n$ is convex. An exact description of the solution set may grow exponentially as the dimension n increases, thus being practically impossible even for several tens of unknowns. On the other hand, in most real-life problem statements such an exact description of the solution set is not necessary. The practice is usually satisfied by an *estimate* of the solution set, i.e. an approximate description that meets the requirements of the problem under solution.

In this work, we are interested in computing *inner* interval estimates (subsets) for the solution set $\Xi(\boldsymbol{A},\boldsymbol{b})$, i. e. we solve the following problem:

$$\boxed{\begin{array}{c}\text{Find a box } \boldsymbol{U} \text{ (as wide as possible)}\\ \text{contained in the solution set } \Xi(\boldsymbol{A},\boldsymbol{b})\\ \text{of the interval linear system } \boldsymbol{A}x = \boldsymbol{b}.\end{array}} \tag{2.4}$$

There are several known approaches to solving the problem of inner interval estimation of the solution sets to interval linear systems proposed in the literature. Among those, the so-called formal (algebraic) approach is especially efficient for square (i.e., with $m = n$) interval linear systems, developed in [8, 15, 17, 18]. Nonetheless, for arbitrary interval linear systems with rectangular (non-square) matrices, i.e. when $m \neq n$, inner interval estimation of the solution sets is an actual and significant problem. Relying on vivid geometric considerations, we propose a simple and quite general technique for constructing a box inscribed into $\Xi(\boldsymbol{A},\boldsymbol{b})$ around an a priori known point from this set (see Fig. 2.1). It is shown that the considered problem reduces to computing maximum of a special quasiconcave function, and its approximate value can be obtained by elementary means.

In the rest of the paper, we do not require regularity properties for $\boldsymbol{A}$ and even admit the case of unbounded solution set $\Xi(\boldsymbol{A},\boldsymbol{b})$. The only mild condition on $\boldsymbol{A}$ is that it must not have entirely zero rows.

Our notation follows the well-known project of informal international standard [6]. In particular, intervals and interval quantities are denoted by boldface letters — $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z},$ — while non-interval (point) objects are not distinguished in any way. Arithmetical operations with the interval quantities are those of the classical interval arithmetic $\mathbb{IR}$ (see, e.g., [1, 9, 10]). Underlining and overlining — $\underline{\boldsymbol{a}}$, $\overline{\boldsymbol{a}}$ — denote lower and upper endpoints of the interval $\boldsymbol{a}$, and, additionally,

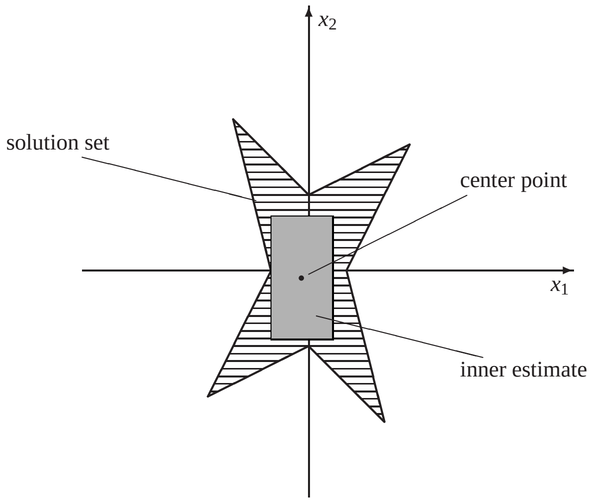

Fig. 2.1: Inner estimation of the solution set.

$\text{mid}\,\boldsymbol{a} = \frac{1}{2}(\overline{\boldsymbol{a}} + \underline{\boldsymbol{a}})$ — midpoint (center) of the interval,

$\text{rad}\,\boldsymbol{a} = \frac{1}{2}(\overline{\boldsymbol{a}} - \underline{\boldsymbol{a}})$ — radius of the interval,

$|\boldsymbol{a}| = \max\{\,|\overline{\boldsymbol{a}}|, |\underline{\boldsymbol{a}}|\,\}$ — absolute value (modulus) of the interval,

$\langle \boldsymbol{a} \rangle = \begin{cases} \min\{\,|\overline{\boldsymbol{a}}|, |\underline{\boldsymbol{a}}|\,\}, & \text{if } 0 \notin \boldsymbol{a}, \\ 0, & \text{otherwise}, \end{cases}$ — mignitude of the interval (antipode of the absolute value), the smallest distance between its points and zero.

With respect to interval vectors and matrices, the operations of taking the midpoint, radius and absolute value are applied in component-wise and element-wise manner.

We expect that the reader is familiar with fundamentals of interval analysis, e.g. from the books [1, 9, 10].

2.2 Refinement of Problem Statement

In applications, the problem statement (2.4) often contains additional information about the desired form of the box $\boldsymbol{U} = (\boldsymbol{U}_1, \boldsymbol{U}_2, \ldots, \boldsymbol{U}_n)^\top$ that has to estimate $\Xi(\boldsymbol{A}, \boldsymbol{b})$ from inside: the widths of the components of $\boldsymbol{U}$ are supposed to be proportional to the respective components of a real positive vector

$$w = (w_1, w_2, \ldots, w_n), \qquad w_i > 0\,.$$

In other words, the formulation (2.4) is additionally supplied with the weight coefficients w_i for the widths (or radii) of the components of the inner box $\boldsymbol{U}$, such that

$$\mathrm{rad}\,\boldsymbol{U}_i / \mathrm{rad}\,\boldsymbol{U}_j = w_i / w_j, \qquad i, j = 1, 2, \ldots, n.$$

Scaling the interval system (2.1)–(2.2) by the nonsingular diagonal matrix

$$W = \mathrm{diag}\{w_1, w_2, \ldots, w_n\}$$

with the entries w_1, w_2, ..., w_n along the main diagonal can reduce the problem to the simplest case when $w = (1, 1, \ldots, 1)$ and the box $\boldsymbol{U}$ turns to a cube that we have to inscribe into the solution set of a modified interval equations system. Moreover, we have

Proposition. *Let $\tilde{\boldsymbol{A}} = \boldsymbol{A}W$. The interval vector $\tilde{\boldsymbol{U}}$ with equal component widths, i.e. such that*

$$\mathrm{rad}\,\tilde{\boldsymbol{U}}_i = \mathrm{rad}\,\tilde{\boldsymbol{U}}_j, \qquad i, j = 1, 2, \ldots, n,$$

is a solution of the inner estimation problem (2.4) *for the modified interval system $\tilde{\boldsymbol{A}}x = \boldsymbol{b}$ if and only if the interval vector $\boldsymbol{U} = W\tilde{\boldsymbol{U}}$ with the desired ratios of the component widths is a solution to the inner estimation problem* (2.4) *for the original system $\boldsymbol{A}x = \boldsymbol{b}$.*

Proof. We use Beeck's characterization [10] of the solution set to the interval linear system (2.1)–(2.2): for $x \in \mathbb{R}^n$

$$x \in \varXi(\boldsymbol{A}, \boldsymbol{b}) \qquad \Longleftrightarrow \qquad \boldsymbol{A}x \cap \boldsymbol{b} \neq \varnothing. \tag{2.5}$$

In particular, for the modified equations system

$$\tilde{x} \in \varXi(\tilde{\boldsymbol{A}}, \boldsymbol{b}) \qquad \Longleftrightarrow \qquad \tilde{\boldsymbol{A}}\tilde{x} \cap \boldsymbol{b} \neq \varnothing. \tag{2.6}$$

Multiplication by the matrix W defines a one-to-one correspondence between the points of the boxes $\boldsymbol{U}$ and $\tilde{\boldsymbol{U}}$ according to the rule

$$x \;\rightleftarrows\; \tilde{x} = Wx$$

for $x \in \boldsymbol{U}$ and $\tilde{x} \in \tilde{\boldsymbol{U}}$. Further, for every pair of the mutually corresponding x and $\tilde{x}$, there holds

$$\boldsymbol{A}x = \boldsymbol{A}WW^{-1}x = \tilde{\boldsymbol{A}}\tilde{x},$$

so that the relations from the right-hand sides of the equivalences (2.5) and (2.6) either fulfill or not fulfill simultaneously. Moreover, for each $i, j = 1, 2, \ldots, n$, we really have

$$\mathrm{rad}\,\boldsymbol{U}_i / \mathrm{rad}\,\boldsymbol{U}_j = w_i / w_j,$$

as was required.

To sum up, in the rest of the paper we can consider the inner estimation problem (2.4) with the additional requirement that the interval vector $\boldsymbol{U}$ should have equal component widths.

2.3 Idea of our Approach

If we find a point from the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$, then it can be further used as a "center" around which the interval solution to the problem (2.4) is to be constructed somehow, by "inflation" etc. (see Fig. 2.1). This is the main idea of the approach developed, so that one can call it "center approach" in analogy to what has been done in [4, 16] for the inner estimation of the *tolerable solution set*. So,

- we look for a point $t \in \Xi(\boldsymbol{A}, \boldsymbol{b})$ first,
- then we use the coordinates of t for the computation of
  the size of the inner estimating cube with the center in t.

The formula for the size of the interval solution of the problem (2.4) is going to be derived later (see Section 2.5). Computation according to this formula involves taking maximum of a rational expression with moduli over a box, so that the entire solution of the inner estimation problem (2.4) boils down to an optimization over a box provided that a point $t \in \Xi(\boldsymbol{A}, \boldsymbol{b})$ is known. We consider this in Section 2.6 in details.

2.4 Choosing Center of Inner Estimate

The problem of recognition of whether the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ is empty or not and the problem of finding a point from the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ are known to be NP-hard in general [7]. A universal method for solving these problems can exploit the fact that intersections of the solution sets to interval linear systems with every orthant of the space $\mathbb{R}^n$ are convex polyhedral sets whose boundary planes are described by equations one can easily write out from the interval matrix and right-hand side vector of the system (see, e.g., [3, 11]). Therefore, finding out whether the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ has empty or nonempty intersection with each orthant of $\mathbb{R}^n$ can be revealed by developed linear programming techniques. Overall, the recognition of the solution sets to interval linear systems and finding a point from it requires no more than 2^n solutions of linear inequalities systems, and this result cannot be principally improved.

Therefore, in the general situation, finding a point from the solution set and its adjustment are not easy tasks. It makes sense to give a list of particular prescriptions for the solution of the above problems in some specific cases.

We consider first a square interval system with an $n \times n$-matrix $\boldsymbol{A}$. If it is regular (i.e., all $A \in \boldsymbol{A}$ are not singular), then the point t from $\Xi(\boldsymbol{A},\boldsymbol{b})$ can be obtained as the result of solving a point linear system $At = b$ with A from $\boldsymbol{A}$ and b from $\boldsymbol{b}$, say, the "middle system"

$$(\operatorname{mid} \boldsymbol{A})\, t = \operatorname{mid} \boldsymbol{b}.$$

Checking regularity of the interval matrix $\boldsymbol{A}$ can be performed by the techniques proposed e.g. in [12].

Let us consider now the case of a singular interval matrix $\boldsymbol{A}$, that is, when it contains a singular point matrix. It is well-known that the set of singular matrices forms a smooth manifold with co-dimension 1 in the set of all $n \times n$-matrices, thus being quite a meager set with zero Lebesgue measure in $\mathbb{R}^{n\times n}$. Hence, if all the entries of the matrix $\boldsymbol{A}$ have nonzero widths, then we can always hope to arrive at a regular point matrix A as the result of proper varying entries of the point $n \times n$-matrix within $\boldsymbol{A}$. Again, it suffices to solve the system $At = b$ with any $b \in \boldsymbol{b}$ in order to find the "center" point t.

What should we do in case of rectangular equation systems? Sometimes, the technique based on the so-called *recognizing functional* may help in this case, which has been elaborated by the author in [14, 16]. We would remind some facts and concepts.

Theorem 2.1. *Let $\boldsymbol{A}$ be an interval $m\times n$-matrix, $\boldsymbol{b}$ be an interval m-vector, and the expression*

$$\operatorname{Uni}(x,\boldsymbol{A},\boldsymbol{b}) = \min_{1\le i\le m} \left\{ \operatorname{rad} \boldsymbol{b}_i - \left\langle \operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} \boldsymbol{a}_{ij}\, x_j \right\rangle \right\}$$

defines a functional $\operatorname{Uni} : \mathbb{R}^n \to \mathbb{R}$. *The membership of a point x in the solution set to an interval linear system $\boldsymbol{A}x = \boldsymbol{b}$ is equivalent to nonnegativity of the functional* Uni *in x,*

$$x \in \Xi(\boldsymbol{A},\boldsymbol{b}) \qquad \Longleftrightarrow \qquad \operatorname{Uni}(x,\boldsymbol{A},\boldsymbol{b}) \ge 0,$$

i. e., the solution set $\Xi(\boldsymbol{A},\boldsymbol{b})$ of the interval linear system is Lebesgue set $\{\, x \in \mathbb{R}^n \mid \operatorname{Uni}(x,\boldsymbol{A},\boldsymbol{b}) \ge 0 \,\}$ of the functional Uni.

If it is clear from the context which interval system is meant, then we shall write simply $\operatorname{Uni}(x)$ instead of $\operatorname{Uni}(x,\boldsymbol{A},\boldsymbol{b})$.

Proof. A point x belongs to the solution set $\Xi(\boldsymbol{A},\boldsymbol{b})$ if and only if there exists a matrix $\tilde{A} = (\tilde{a}_{ij}) \in \boldsymbol{A}$, such that

$$\tilde{A}x \in \boldsymbol{b}.$$

After writing out the matrix-vector product and representing the right-hand side intervals in the center-radius form, this membership takes the from

$$\sum_{j=1}^{n} \tilde{a}_{ij} x_j \in \operatorname{mid} \boldsymbol{b}_i + \bigl[-\operatorname{rad} \boldsymbol{b}_i, \operatorname{rad} \boldsymbol{b}_i \bigr], \qquad i = 1,2,\ldots,m.$$

Adding $(-\mathrm{mid}\,\boldsymbol{b}_i)$ to both sides of the above inclusions, we get the equivalent relations

$$\sum_{j=1}^{n} \tilde{a}_{ij}x_j \;-\mathrm{mid}\,\boldsymbol{b}_i \in \left[-\mathrm{rad}\,\boldsymbol{b}_i, \mathrm{rad}\,\boldsymbol{b}_i\right], \qquad i = 1,2,\dots,m,$$

which are, in its turn, equivalent to

$$\left| \sum_{j=1}^{n} \tilde{a}_{ij}x_j \;-\mathrm{mid}\,\boldsymbol{b}_i \right| \le \mathrm{rad}\,\boldsymbol{b}_i\ ,$$

and therefore

$$\mathrm{rad}\,\boldsymbol{b}_i - \left| \mathrm{mid}\,\boldsymbol{b}_i - \sum_{j=1}^{n} \tilde{a}_{ij}x_j \right| \ge 0 \tag{2.7}$$

for every $i = 1,2,\dots,m$.

Hence, $x \in \varXi(\boldsymbol{A},\boldsymbol{b})$ if and only if for each index i there exist such $\tilde{a}_{ij} \in \boldsymbol{a}_{ij}$, $j = 1,2,\dots,n$, that the inequalities (2.17) are true. This amounts to the fulfillment of

$$\max_{\substack{\tilde{a}_{ij}\in\boldsymbol{a}_{ij},\\ j=1,2,\dots,n}} \left\{ \mathrm{rad}\,\boldsymbol{b}_i - \left| \mathrm{mid}\,\boldsymbol{b}_i - \sum_{j=1}^{n} \tilde{a}_{ij}x_j \right| \right\} \ \ge\ 0 \tag{2.8}$$

for $i = 1,2,\dots,m$. Bringing the maximum into the brackets and taking into account that the natural interval extension of the expression under module coincides with its range of values, we get for $i = 1,2,\dots,m$

$$\left\{ \mathrm{rad}\,\boldsymbol{b}_i - \left\langle \mathrm{mid}\,\boldsymbol{b}_i - \sum_{j=1}^{n} \boldsymbol{a}_{ij}x_j \right\rangle \right\} \ \ge\ 0 \tag{2.9}$$

instead of (2.8). Finally, taking the minimum, we can reduce m conditions (2.9) into one, to get that the point x belongs to the set $\varXi(\boldsymbol{A},\boldsymbol{b})$ only in the case when

$$\min_{1\le i\le m} \left\{ \mathrm{rad}\,\boldsymbol{b}_i - \left\langle \mathrm{mid}\,\boldsymbol{b}_i - \sum_{j=1}^{n} \boldsymbol{a}_{ij}x_j \right\rangle \right\} \ \ge\ 0,$$

as required.

One may see that the functional Uni "recognizes", through the sign of its values, whether the point is in the solution set $\varXi(\boldsymbol{A},\boldsymbol{b})$ or not. This is why we use the term "recognizing" with respect to it. Additionally, the following properties hold [14]:

1) The functional Uni is concave in each orthant of $\mathbb{R}^n$, and if the matrix $\boldsymbol{A}$ has entirely noninterval (point) columns, then $\mathrm{Uni}(x,\boldsymbol{A},\boldsymbol{b})$ is concave on unions of several orthants.
2) The functional $\mathrm{Uni}(x,\boldsymbol{A},\boldsymbol{b})$ is continuous and attains a finite maximum over the whole space $\mathbb{R}^n$.

3) If $\mathrm{Uni}(x,\boldsymbol{A},\boldsymbol{b}) > 0$, then x is a point from the topological interior $\mathrm{int}\,\Xi(\boldsymbol{A},\boldsymbol{b})$ of the solution set.
4) Under some additional conditions on $\boldsymbol{A}$, $\boldsymbol{b}$ and x, the reverse is also true: the membership $x \in \mathrm{int}\,\Xi(\boldsymbol{A},\boldsymbol{b})$ implies $\mathrm{Uni}(x,\boldsymbol{A},\boldsymbol{b}) > 0$.

The last two properties of the recognizing functional enables us to use it for deciding whether a point belongs to the interior of the solution set. This is especially important inasmuch as our technique can construct a solid inner estimate of the solution set only around the center point t that lies in the interior of the solution set $\mathrm{int}\,\Xi(\boldsymbol{A},\boldsymbol{b})$.

As a consequence of the results obtained, we arrive at the following practical prescription for the correction of the point t in our "center" approach to the solution of the problem (2.4): find a starting guess and then, using gradient ascent, try reaching better value of the recognizing functional Uni. If the value found is strictly greater than zero, then we are in the interior of the solution set.

We do not discuss the question of optimization (the best choice) of the center of the inner interval box, since it is closely related to specific needs of the customers that solve a practical problem statement.

2.5 Formula for Size of Inner Estimate

Theorem 2.2. *If a point $t \in \mathbb{R}^n$ belongs to the solution set of an interval linear system $\boldsymbol{A}x = \boldsymbol{b}$, i.e. $t \in \Xi(\boldsymbol{A},\boldsymbol{b})$, then*

$$\rho = \min_{1\leq i\leq m} \max_{A\in\boldsymbol{A}} \left\{ \frac{\mathrm{rad}\,\boldsymbol{b}_i - \left| \mathrm{mid}\,\boldsymbol{b}_i - \sum_{j=1}^{n} a_{ij}t_j \right|}{\sum_{j=1}^{n} |a_{ij}|} \right\} \geq 0 \qquad (2.10)$$

and the interval vector $\boldsymbol{U} = (t + \rho\boldsymbol{e})$, $\boldsymbol{e} = ([-1,1],\ldots,[-1,1])^\top$, with the center t is entirely contained in the solution set $\Xi(\boldsymbol{A},\boldsymbol{b})$.

The expression under extrema in (2.10) looks very impressive, but it has a clear sense which is worth mentioning. The vector $|\,\mathrm{mid}\,\boldsymbol{b} - At\,|$ is composed of absolute values of the deviations of the product At components from the center of the right-hand side of the interval linear system considered. The signs of the differences between the radii of the right-hand side and such deviations, given by the components of $(\mathrm{rad}\,\boldsymbol{b} - |\,\mathrm{mid}\,\boldsymbol{b} - At\,|)$, show whether the image At of the point t under the linear transformation A belongs to the right-hand side vector $\boldsymbol{b}$. This all is familiar to us from the previous section, where we used the same technique to derive the recognizing functional Uni. However, when divided by the sums $\sum_j |a_{ij}|$ of the moduli of the entries in the respective rows of A, the components of the vector $(\mathrm{rad}\,\boldsymbol{b} - |\,\mathrm{mid}\,\boldsymbol{b} - At\,|)$ produce a new characteristic, namely, sensitivity of the recognizing functional with respect to variations of its first argument. More precisely, the minimum of such ratios over all the rows of A gives a "perturbation robustness" that shows how much we can shift the point t in order not to leave the solution set of the interval linear system $Ax = \boldsymbol{b}$.

Proof. Since the matrix of the interval linear system does not have zero rows, then

$$\sum_{j=1}^{n} |a_{ij}| \;>\; 0$$

for every $i = 1, 2, \ldots, m$, and $\rho \geq 0$ is equivalent to nonnegativity of the expression

$$\min_{1\leq i\leq m} \max_{A\in\boldsymbol{A}} \left\{ \operatorname{rad} \boldsymbol{b}_i - \left| \operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} a_{ij} t_j \right| \right\},$$

which defines the values of the recognizing functional Uni in the point $t \in \mathbb{R}^n$ due to the theorem of Section 2.4. It is indeed nonnegative for $t \in \varXi(\boldsymbol{A}, \boldsymbol{b})$.

Starting the substantiation of the second statement of the theorem, suppose first that the matrix $\boldsymbol{A}$ in the problem (2.4) has zero width, i.e. is noninterval, $\boldsymbol{A} = A = (a_{ij})$. Denoting then

$$\rho_A \;=\; \min_{1\leq i\leq m} \left\{ \frac{\operatorname{rad} \boldsymbol{b}_i - \left| \operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} a_{ij} t_j \right|}{\sum_{j=1}^{n} |a_{ij}|} \right\}, \tag{2.11}$$

we represent every $x \in \boldsymbol{U}$ in the form $x = t + y$, where $y \in \mathbb{R}^n$ and

$$\max_{1\leq k\leq n} |y_k| \leq \rho_A.$$

In view of the fact that

$$|y_i| \;\leq\; \rho_A \;\leq\; \frac{\operatorname{rad} \boldsymbol{b}_i - \left| \operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} a_{ij} t_j \right|}{\sum_{j=1}^{n} |a_{ij}|}, \qquad i = 1, 2, \ldots, m,$$

the following inequalities chain is valid for each $i = 1, 2, \ldots, m$:

$$
\begin{aligned}
|\,(Ay)_i| \;&=\; \left|\,\sum_{j=1}^{n} a_{ij} y_j \right| \;\leq\; \sum_{j=1}^{n} |\,a_{ij}|\,|y_j| \;\leq\; \rho_A \cdot \sum_{j=1}^{n} |\,a_{ij}| \\
&\leq\; \operatorname{rad} \boldsymbol{b}_i - \left|\, \operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} a_{ij} t_j \right| \\
&=\; \operatorname{rad} \boldsymbol{b}_i - \big|\, \operatorname{mid} \boldsymbol{b}_i - (At)_i \,\big|.
\end{aligned}
$$

As far as $Ay = Ax - At$, we get

$$
(At)_i - \operatorname{rad} \boldsymbol{b}_i + \big|\, \operatorname{mid} \boldsymbol{b}_i - (At)_i \,\big| \;\leq\; (Ax)_i \;\leq\; (At)_i + \operatorname{rad} \boldsymbol{b}_i - \big|\, \operatorname{mid} \boldsymbol{b}_i - (At)_i \,\big|
$$

or, which is equivalent,

$$
\begin{gathered}
\underline{\boldsymbol{b}}_i - (\, \operatorname{mid} \boldsymbol{b}_i - (At)_i) + |\, \operatorname{mid} \boldsymbol{b}_i - (At)_i \,| \\
\leq (Ax)_i \leq \\
\overline{\boldsymbol{b}}_i - (\, \operatorname{mid} \boldsymbol{b}_i - (At)_i) - |\, \operatorname{mid} \boldsymbol{b}_i - (At)_i \,|.
\end{gathered}
\tag{2.12}
$$

Taking into account that

$$
-z + |z| \geq 0 \quad \text{and} \quad -z - |z| \leq 0
$$

for any real z, the inequality (2.12) implies for every $i = 1, 2, \ldots, m$

$$
\underline{\boldsymbol{b}}_i \leq (Ax)_i \leq \overline{\boldsymbol{b}}_i,
$$

i.e. $Ax \in \boldsymbol{b}$. This means that the point x is a member of the solution set to the interval linear system $Ax = \boldsymbol{b}$. So, the formula (2.10) is proved for the systems (2.1)–(2.2) with only the right-hand side being interval, not the matrix.

We suppose now that the matrix $\boldsymbol{A}$ in the interval linear system (2.1)–(2.2) is essentially interval, i.e. has nonzero width, the corresponding solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ is nonempty and $t \in \Xi(\boldsymbol{A}, \boldsymbol{b})$. We consider the totality of all the systems $Ax = \boldsymbol{b}$ with point matrices $A \in \boldsymbol{A}$ and inner estimates $\boldsymbol{U}_A$ of their solution sets $\Xi(A, \boldsymbol{b})$. By virtue of the fact that

$$
\Xi(\boldsymbol{A}, \boldsymbol{b}) = \bigcup_{A \in \boldsymbol{A}} \Xi(A, \boldsymbol{b}),
$$

the union of all or some of the inner estimates of the sets $\Xi(A, \boldsymbol{b})$ for $A \in \boldsymbol{A}$ is an inner estimate of $\Xi(\boldsymbol{A}, \boldsymbol{b})$ too.

Let $\boldsymbol{U}_A$ be a cube, with the fixed center t, included in the solution set of $Ax = \boldsymbol{b}$. Clearly, such inner estimates exist not for every solution set $\Xi(A, \boldsymbol{b})$ with $A \in \boldsymbol{A}$, but only for those that contain the point t. However, the union of the inner cubes $\boldsymbol{U}_A \subseteq \Xi(A, \boldsymbol{b})$ that still exist for the given t can be found in an especially simple way: it is a cube with the same center t, its size being equal to the maximum of sizes of the cubes to be united (see Fig. 2.2). In particular, if the sizes of the cubes are

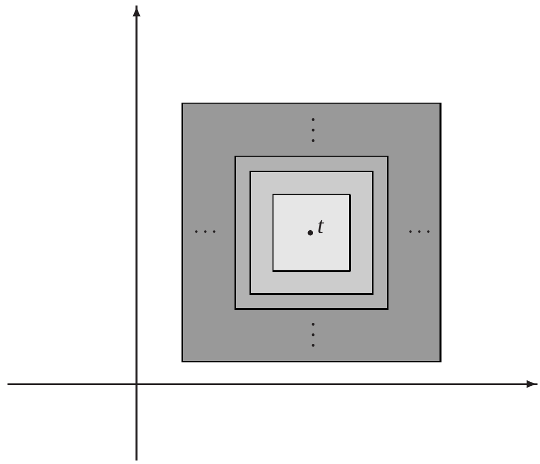

Fig. 2.2: Union of cubes with a common center is also a cube with the same center

defined by the formula (2.11), then the box

$$\boldsymbol{U} = t + \rho \boldsymbol{e}$$

is also entirely included into the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ for

$$\rho \;=\; \max_{A \in \boldsymbol{A}} \rho_A \;=\; \max_{A \in \boldsymbol{A}} \min_{1 \leq i \leq m} \left\{ \frac{\operatorname{rad} \boldsymbol{b}_i - \left| \operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} a_{ij} t_j \right|}{\sum_{j=1}^{n} |a_{ij}|} \right\}. \tag{2.13}$$

In this expression, we have the right to take the maximum with respect to A over the whole interval matrix $\boldsymbol{A}$, no matter whether $t \in \Xi(A, \boldsymbol{b})$ or not for specific $A \in \boldsymbol{A}$. The point is that $\rho_A < 0$ in case of $t \notin \Xi(A, \boldsymbol{b})$, and such negative values of the inner minimum in the expression (2.13) in no way affect the overall nonnegative maximum of (2.13).

Finally, we can rearrange the minimum and maximum in (2.13), since, for different indices i, the expressions in the curly braces have *nonintersecting sets of arguments*, namely, they are taken over different rows of the matrix $\boldsymbol{A}$. Finally,

$$\rho \;=\; \min_{1 \leq i \leq m} \max_{A \in \boldsymbol{A}} \left\{ \frac{\operatorname{rad} \boldsymbol{b}_i - \left| \operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} a_{ij} t_j \right|}{\sum_{j=1}^{n} |a_{ij}|} \right\}.$$

This completes the proof of the theorem.

One cannot but notice a beautiful duality of the above result with the formula derived in [4, 16] for the size of inner estimate of the *tolerable solution set* to the interval linear system (2.1)–(2.2). The tolerable solution set is defined as

$$\begin{aligned} \Xi_{tol}(\boldsymbol{A},\boldsymbol{b}) &= \{\, x\in\mathbb{R}^n \mid (\forall A\in\boldsymbol{A})(\exists b\in\boldsymbol{b})(Ax=b)\,\} \\ &= \{\, x\in\mathbb{R}^n \mid (\forall A\in\boldsymbol{A})(Ax\in\boldsymbol{b})\,\} \\ &= \{\, x\in\mathbb{R}^n \mid \boldsymbol{A}x\subseteq\boldsymbol{b}\,\} \end{aligned}$$

and has many interesting practical applications (see e.g. [13, 19]). It turns out that, if $t\in\Xi_{tol}(\boldsymbol{A},\boldsymbol{b})\neq\varnothing$, then

$$\sigma \;=\; \min_{1\le i\le m}\;\min_{A\in\boldsymbol{A}} \left\{ \frac{\operatorname{rad}\boldsymbol{b}_i - \left|\,\operatorname{mid}\boldsymbol{b}_i - \sum_{j=1}^{n} a_{ij}t_j\,\right|}{\sum_{j=1}^{n}|\,a_{ij}|} \right\} \;\ge\; 0 \tag{2.14}$$

and the interval vector $(t+\sigma\boldsymbol{e})$, $\boldsymbol{e}=([-1,1],\dots,[-1,1])^\top$, is included into the tolerable solution set $\Xi_{tol}(\boldsymbol{A},\boldsymbol{b})$. Changing the logical quantifier that stands at the matrix in the definition of the solution set — from "$\exists$" to "$\forall$" — leads to changing the sense of the internal extremum in the expression (2.10) for the size of the inner box: we get minimum over $A\in\boldsymbol{A}$ instead of maximum.

An unpleasant feature of the formula (2.10) is that it produces zero, if the radius of a right-hand side component is zero. This can be partially corrected after substituting the coordinates of the center into the interval system (2.1) and transferring any interval column into the right-hand side, which acquires nonzero radius as the result.

In the expression (2.10), taking the minimum over $i\in\{1,2,\dots,m\}$ involves no difficulties, so that the main problem in the computation of ρ is to find, for each i, the internal maximums

$$\max_{(a_{i1},\dots,a_{in})\in(\boldsymbol{a}_{i1},\dots,\boldsymbol{a}_{in})} \left\{ \frac{\operatorname{rad}\boldsymbol{b}_i - \left|\,\operatorname{mid}\boldsymbol{b}_i - \sum_{j=1}^{n} a_{ij}t_j\,\right|}{\sum_{j=1}^{n}|\,a_{ij}|} \right\}$$

or to estimate them from below.

For further convenience, we denote the box $(\boldsymbol{a}_{i1},\boldsymbol{a}_{i2},\dots,\boldsymbol{a}_{in})$ through

$$(\boldsymbol{X}_1,\boldsymbol{X}_2,\dots,\boldsymbol{X}_n)=\boldsymbol{X},$$

regardless of the index $i \in \{1, 2, \dots, m\}$, while the objective function $\mathbb{R}^n \to \mathbb{R}$, defined by the expression inside the curly braces in (2.10) and (2.14), will be denoted as

$$\Phi(x) \;=\; \frac{R - \left| M - \sum_{j=1}^{n} x_j t_j \right|}{\sum_{j=1}^{n} |x_j|}, \tag{2.15}$$

where $R = \operatorname{rad} \boldsymbol{b}_i$, $M = \operatorname{mid} \boldsymbol{b}_i$ are real constants. As the result, constructing inner interval estimate of the set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ around the known center point reduces to the solution of the following optimization problem

$$\boxed{\text{Find } \max_{x \in \boldsymbol{X}} \Phi(x) \text{ or, at least, its nonnegative estimate from below.}} \tag{2.16}$$

Nonnegativity constraint is evidently implied by the practical sense of the required estimate as a radius of the inner box.

2.6 Computing Size of Inner Estimate

It is obvious that, in (2.16), the estimate of the sought-for $\max_{x \in \boldsymbol{X}} \Phi(x)$ from below may be the value the objective function $\Phi(x)$ takes at any point of the box $\boldsymbol{X}$. Therefore, if we are not going to get involved into laborious computations, then the simplest way to solve the problem (2.16) is to take maximum of the values of the objective function in several special points of its domain $\boldsymbol{X}$.

Let us denote

$$G(x) = R - \left| M - \sum_{j=1}^{n} x_j t_j \right|, \qquad H(x) = \sum_{j=1}^{n} |x_j|,$$

so that

$$\Phi(x) \;=\; \frac{G(x)}{H(x)}.$$

$G(x)$ and $H(x)$ are quite simple expressions that have only one occurrence of every variable x_j, so that their extrema over $\boldsymbol{X}$ can be easily computed as the lower and upper endpoints of the natural interval extensions $G(\boldsymbol{X})$ and $H(\boldsymbol{X})$ for the respective expressions. In particular,

$$\max_{x \in \boldsymbol{X}} G(x) \;=\; \overline{G(\boldsymbol{X})} \;=\; R - \left\langle M - \sum_{j=1}^{n} \boldsymbol{X}_j t_j \right\rangle$$

and

$$\min_{x\in\boldsymbol{X}} H(x) \;=\; \underline{H(\boldsymbol{X})} \;=\; \sum_{j=1}^{n} \langle \boldsymbol{X}_j \rangle.$$

Further, along with the values of these extrema, we can find the arguments that they deliver, tracing which of the endpoints of the intervals $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$ produce the endpoints of the interval extensions $G(\boldsymbol{X})$ and $H(\boldsymbol{X})$ as the result of the operations with them, i.e. addition, subtraction, multiplication and taking the modulus. Overall, the simplest estimate of the solution to the problem (2.16) can be taken, for instance, as maximum of the values of the objective function $\Phi(x)$

in the center ("most representative point") of the box $\boldsymbol{X}$,

in the point where the denominator $H(x)$ attains its minimum,

in the point where the numerator $G(x)$ attains its maximum.

If the center t of the inner box lies in the solution set $\Xi(\boldsymbol{A},\boldsymbol{b})$, then we have seen that $\max_{x\in\boldsymbol{X}} G(x) \geq 0$. So, the overall maximum of the values of $\Phi(x)$ in the above three points is greater or equal to zero, thus satisfying the nonnegativity requirement in the formulation (16).

We turn now to more developed techniques for the solution of the optimization problem (2.16). Recall

Definition [2]. Let D be a convex set in $\mathbb{R}^n$. The function $f : D \to \mathbb{R}$ is referred to as *quasiconcave*, if for every $x, y \in D$ and $0 \leq \lambda \leq 1$ there holds

$$f\big(\lambda x + (1-\lambda)y\big) \geq \min\{\, f(x), f(y)\,\}.$$

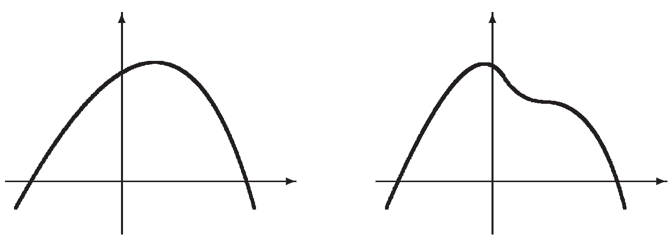

Fig. 2.3: Graphs of concave and quasiconcave functions

The function $f : D \to \mathbb{R}$ is known to be quasiconcave [2] if and only if its Lebesgue sets

$$\{\, x \in D \mid f(x) \geq \alpha \,\}$$

are convex for every $\alpha \in \mathbb{R}$ (see Fig. 2.3). In particular, a quasiconcave function cannot have several local maxima that differ in value from each other. Computing one local maximum of such functions is, at the same time, the solution of global maximization problem.

Theorem 2.3. *Let $0 \notin \boldsymbol{X} \subseteq \mathbb{R}^n$. The set $\mathfrak{D}$ of all the points from $\boldsymbol{X}$ for which the function $\Phi(x)$ defined by (2.15) takes nonnegative values is convex, and $\Phi(x)$ is quasiconcave on $\mathfrak{D}$.*

Proof. For a given level $\alpha \geq 0$, we denote through

$$S_\alpha = \{\, x \in \boldsymbol{X} \subset \mathbb{R}^n \mid \Phi(x) \geq \alpha \,\}$$

the Lebesgue set of the function $\Phi(x)$. In particular, $S_0 = \mathfrak{D}$.

If S_α is empty, there is nothing to talk about. If $S_\alpha \neq \varnothing$, then let the points x, y (not necessarily different) belong to the set S_α, so that $\Phi(x) \geq \alpha$, $\Phi(y) \geq \alpha$. Therefore,

$$R - \left| M - \sum_{j=1}^{n} x_j t_j \right| \;\geq\; \alpha \sum_{j=1}^{n} |x_j|,$$

$$R - \left| M - \sum_{j=1}^{n} y_j t_j \right| \;\geq\; \alpha \sum_{j=1}^{n} |y_j|.$$

Taking any $\lambda \in [0,1]$ and summing the above inequalities with the nonnegative weights λ and $(1-\lambda)$, we come up with the inequality of the same sense:

$$\begin{aligned} R - \lambda \left| M - \sum_{j=1}^{n} x_j t_j \right| - (1-\lambda) \left| M - \sum_{j=1}^{n} y_j t_j \right| \\ \geq \alpha \left(\lambda \sum_{j=1}^{n} |x_j| + (1-\lambda) \sum_{j=1}^{n} |y_j| \right). \end{aligned} \tag{2.17}$$

Further, applying the triangle inequality for the absolute values of intervals, we can change the left-hand side of the inequality (2.17) to a greater or equal quantity

$$R - \left| \lambda \left(M - \sum_{j=1}^{n} x_j t_j \right) + (1-\lambda) \left(M - \sum_{j=1}^{n} y_j t_j \right) \right|,$$

while the right-hand side (2.17) can be changed (due to $\alpha \geq 0$) to a smaller or equal quantity

$$\alpha \left(\sum_{j=1}^{n} |\lambda x_j + (1-\lambda) y_j| \right).$$

Finally, we have

$$R - \left| M - \sum_{j=1}^{n} (\lambda x_j + (1-\lambda) y_j) t_j \right| \;\geq\; \alpha \left(\sum_{j=1}^{n} |\lambda x_j + (1-\lambda) y_j| \right),$$

which is equivalent to

$$\Phi\big(\lambda x + (1-\lambda)y\big) \geq \alpha.$$

The point $\lambda x + (1-\lambda)y$ thus lies within the set S_α too, i.e. S_α is convex. This completes the proof of the theorem.

It is worth noting that the condition of nonnegativity on $\Phi(x)$ is not so burdensome for applications of the above result, since negativity of $\Phi(x)$ for all $x \in \boldsymbol{X}$ is only possible for uninteresting cases when the center point t does not lie within the solution set. This follows from that the negativity of $\Phi(x)$ is equivalent to negativity of the numerator in the fraction (2.15) and, hence, of the "recognizing" functional Uni in the point t (see Section 2.4). Then we have to take care of a better choice for the center point t.

The presence of moduli in the expression (2.15) makes the objective function $\Phi(x)$ nonsmooth, although it is continuous. The function is still differentiable almost everywhere over its domain of definition. Therefore, the quasiconcavity of $\Phi(x)$ may result in gradient-type methods for the solution of the problem (2.16). For instance, if $\mathrm{Pr}_{\boldsymbol{X}}$ means projection onto the box $\boldsymbol{X}$, we can apply the simplest gradient projection method

$$x^{(k+1)} := x^{(k)} + \gamma^{(k)}\,\mathrm{Pr}_{\boldsymbol{X}}\Big(\nabla\Phi\big(x^{(k)}\big)\Big), \qquad k = 0,1,2,\ldots, \tag{2.18}$$

with the appropriate choice of the step size $\gamma^{(k)} \in \mathbb{R}_+$ (see e.g. [2]). The components of the gradient $\nabla\Phi(x)$ are easily seen to have the form

$$\Big(\nabla\Phi(x)\Big)_i =$$

$$\frac{t_i\cdot \mathrm{sgn}\left(M - \sum_{j=1}^n x_j t_j\right)\cdot\left(\sum_{j=1}^n |x_j|\right) - \left(R - \left|M - \sum_{j=1}^n x_j t_j\right|\right)\cdot \mathrm{sgn}\, x_i}{\left(\sum_{j=1}^n |x_j|\right)^2},$$

$$i = 1,2,\ldots,n,$$

where "sgn" means the usual sign function.

A good choice of the initial approximation $x^{(0)}$ for the process (2.18) will be a point where the objective function $\Phi(x)$ is already nonnegative. How can we find this?

As follows from the results of Section 2.4, the membership of a point t in the solution set $\Xi(\boldsymbol{A},\boldsymbol{b})$ is equivalent to

$$\mathrm{Uni}(t,\boldsymbol{A},\boldsymbol{b}) = \min_{1\leq i\leq m}\left\{ \mathrm{rad}\,\boldsymbol{b}_i - \left\langle \mathrm{mid}\,\boldsymbol{b}_i - \sum_{j=1}^n \boldsymbol{a}_{ij}t_j \right\rangle \right\} \;\geq\; 0,$$

which, in its turn, holds true if and only the same inequality is valid for the separate i-th row of the matrix $\boldsymbol{A}$, $i = 1, 2, \dots, m$. In terms of the function Φ defined by (2.15), this means that

$$R - \left\langle M - \sum_{j=1}^{n} \boldsymbol{X}_j t_j \right\rangle \geq 0, \tag{2.19}$$

where $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2, \dots, \boldsymbol{X}_n) = (\boldsymbol{a}_{i1}, \boldsymbol{a}_{i2}, \dots, \boldsymbol{a}_{in})$, $R = \operatorname{rad} \boldsymbol{b}_i$, $M = \operatorname{mid} \boldsymbol{b}_i$ for a fixed index i. Therefore, to find nonnegativity points for the objective function $\Phi(x)$, we have to trace the endpoints of the intervals $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, ..., $\boldsymbol{X}_n$, at which the value of the expression

$$\left\langle M - \sum_{j=1}^{n} \boldsymbol{X}_j t_j \right\rangle$$

is attained, similar to what has been recommended in the beginning of the section. The numbers thus obtained constitute components of the sought-for starting approximation $x^{(0)}$ for the gradient ascending method (2.18).

2.7 Numerical Examples

Let us consider a numerical example with the interval linear system

$$\begin{pmatrix} [2,3] & [0,1] \\ [1,2] & [2,3] \end{pmatrix} x = \begin{pmatrix} [0,120] \\ [60,240] \end{pmatrix}, \tag{2.20}$$

proposed by E. Hansen (see [5] and earlier works). Its solution set is shown at Fig. 2.4.

In formal-algebraic approach to inner estimation of the solution set, we have to carry our considerations into Kaucher complete interval arithmetic and organize the so-called dualization equation

$$\begin{pmatrix} [3,2] & [1,0] \\ [2,1] & [3,2] \end{pmatrix} x = \begin{pmatrix} [0,120] \\ [60,240] \end{pmatrix},$$

having the matrix dualized and the right-hand side vector unchanged, and then compute its formal (algebraic) solution [8, 15, 17, 18]. It can be computed by several ways, and the most efficient subdifferential Newton method[1] in 2 iterations finds the vector

$$\begin{pmatrix} [-12,60] \\ [24,90] \end{pmatrix}. \tag{2.21}$$

[1] C-sources and executable files of its implementation for Windows are downloadable from `http://www.nsc.ru/interval/shary/Codes/progr.html`

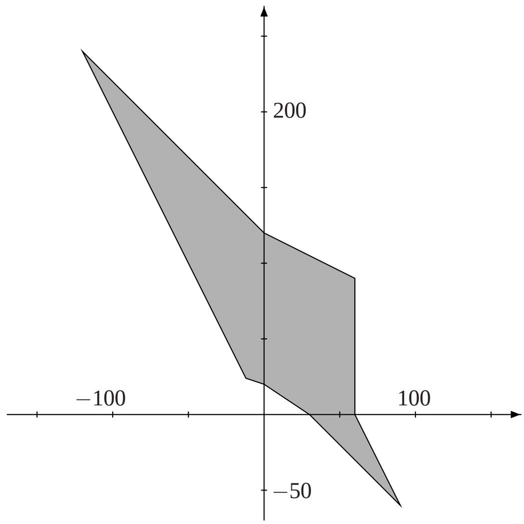

Fig. 2.4: Solution set of Hansen system (2.20)

One can make sure that this is an inclusion maximal inner estimate of the solution set for Hansen system.

Inner interval estimation with the use of our "center approach" starts from solving the midpoint system

$$\begin{pmatrix} 2.5 & 0.5 \\ 1.5 & 2.5 \end{pmatrix} x = \begin{pmatrix} 60 \\ 150 \end{pmatrix}. \tag{2.22}$$

Its solution is $(13.6364, 51.8182)^\top$ and, due to regularity of the matrix in (2.22), this vector is within the estimated solution set and can be taken as the center t of the inner box.[2]

When solving the optimization problem (2.15)–(2.16) for the first equation of the system (2.20), we have to take

$$R = 60, \qquad M = 60, \qquad \boldsymbol{X} = \big([2,3],[0,1]\big).$$

Then

$$\left\langle M - \sum_{j=1}^{2} \boldsymbol{X}_j t_j \right\rangle = 0,$$

and this value is attained at $(2.5, 0.5) \in \boldsymbol{X}$ which can serve as a starting point $x^{(0)}$ for the method (2.18).

Launched from this $x^{(0)}$, with $\Phi(x^{(0)}) = 20$, the gradient ascending (2.18) reaches the boundary of the box $\boldsymbol{X}$ at the point $\tilde{x} = (2.0, 0.631581)$ (the exact number of

[2] We keep no more than six digits in the numerical data of this section.

steps depends on the specific choice of the step size $\gamma^{(k)}$). The point $\tilde{x}$ turns out to be maximum of $\boldsymbol{\Phi}$ in $\boldsymbol{X}$ with $\boldsymbol{\Phi}(\tilde{x}) = 22.8$.

For the second equation of (2.20), the optimization problem (2.15)–(2.16) corresponds to

$$R = 90, \qquad M = 150, \qquad \boldsymbol{X} = \big([1,2],[2,3]\big).$$

We have

$$\left\langle M - \sum_{j=1}^{2} \boldsymbol{X}_j t_j \right\rangle = 0,$$

which is attained at $(1.5, 2.5)$. It is taken as the starting point $x^{(0)}$ for the method (2.18), while $\boldsymbol{\Phi}(x^{(0)}) = 22.5$. The gradient ascending (2.18) reaches the boundary of the domain box $\boldsymbol{X}$ at the point $\tilde{x} = (1.0, 2.63158)$ that delivers maximal value $\boldsymbol{\Phi}(\tilde{x}) = 24.7826$ to the objective function.

According to Theorem 2 (Section 5) and formula (2.10), we get an inner interval estimate for the solution set of Hansen system in the form

$$\begin{pmatrix} 13.6364 \\ 51.8182 \end{pmatrix} + \min\big\{22.8, 24.7826\big\} \cdot \begin{pmatrix} [-1,1] \\ [-1,1] \end{pmatrix},$$

that is,

$$\begin{pmatrix} [-9.16364, 36.4364] \\ [29.0182, 74.6182] \end{pmatrix}.$$

This is slightly worse than (2.21), but no so bad at all!

Next, we consider the interval linear system

$$\begin{pmatrix} 3.5 & [0,2] & [0,2] \\ [0,2] & 3.5 & [0,2] \\ [0,2] & [0,2] & 3.5 \end{pmatrix} x = \begin{pmatrix} [-1,1] \\ [-1,1] \\ [-1,1] \end{pmatrix}, \tag{2.23}$$

with the solution set as in Fig. 2.5 (it is shown at the jacket of the book [10], but in another projection).

Since the middle of the right-hand side vector is $(0,0,0)^\top$, the solution to the midpoint system is the zero vector too, and we can take the center of the inner box as $t = 0$. This crucially simplifies our technique, since then the numerator of the expression (2.15) does not depend on x any more. We have

$$\max_{x \in \boldsymbol{X}} \boldsymbol{\Phi}(x) \;=\; \max_{x \in \boldsymbol{X}} \left(\frac{R - |M|}{\sum_j |x_j|} \right) \;=\; \frac{R - |M|}{\min_{x \in \boldsymbol{X}} \big(\sum_j |x_j| \big)} = \frac{R - |M|}{\sum_j \langle \boldsymbol{X}_j \rangle}, \tag{2.24}$$

which is easily computable.

For the system (2.23), the expressions (2.24) taken over all three rows of the matrix coincide and equal

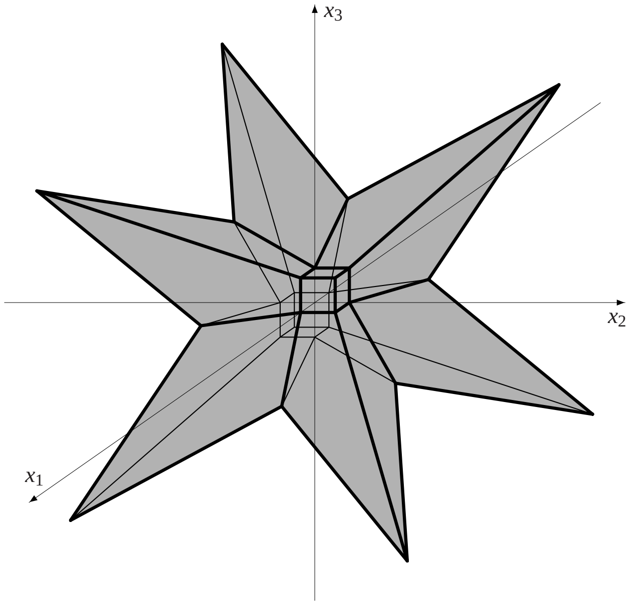

Fig. 2.5: Solution set for Neumaier system (2.23)

$$\frac{1-0}{\langle 3.5\rangle + \langle [0,2]\rangle + \langle [0,2]\rangle} = \frac{1}{3.5} = 0.285714.$$

Therefore, the inner interval box for the solution set of (2.23) should be

$$\begin{pmatrix} [-0.285714, 0.285714] \\ [-0.285714, 0.285714] \\ [-0.285714, 0.285714] \end{pmatrix}. \tag{2.25}$$

It coincides with the inner estimate obtained by formal-algebraic approach, as a proper formal solution to the interval linear system in Kaucher arithmetic

$$\begin{pmatrix} 3.5 & [2,0] & [2,0] \\ [2,0] & 3.5 & [2,0] \\ [2,0] & [2,0] & 3.5 \end{pmatrix} x = \begin{pmatrix} [-1,1] \\ [-1,1] \\ [-1,1] \end{pmatrix}.$$

The cube (2.25) is actually an inclusion maximal inner interval estimates of the solution set to (2.23) that "exhaust" its central part adjacent to the origin of coordinates.

2.8 Conclusions

The work presents a new method ("center approach") for inner interval estimation of the solution sets to interval linear systems, which is a good supplement to the earlier developed techniques.

For interval linear systems with square matrices, the quality of the results produced by the new method is slightly worse in comparison to those of formal-algebraic approach. But the new method is conceptually simpler and has wider applicability scope, being able to compute inner estimates for the solution sets to interval linear systems with general rectangular matrices. A notable feature of the "center approach" is the possibility to easily control the location of the inner box within the solution set, through changing the position of its center. Additionally, the new approach can be adapted to interval linear systems with dependencies between the entries of the matrix.

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References

1. Alefeld, G., Herzberger, J.: Introduction to Interval Computations. Academic Press, New York (1983)
2. Bazaraa, M.S., Shetti, C.M.: Nonlinear Programming. Theory and Algorithms. John Wiley and Sons, New York (1979).
3. Cope, J., Rust, B.: Bounds on solutions of linear systems with inaccurate data. SIAM J. Numer. Anal. **16**, 950–963 (1979)
4. Dobronets, B.S., Shaidurov, V.V.: Two-sided Numerical Methods. Nauka, Novosibirsk (1990) (in Russian)
5. Hansen, E.R., Walster, G.W.: Global Optimization Using Interval Analysis. Marcel Dekker, New York (2003)
6. Kearfott, R.B., Nakao, M.T., Neumaier, A., Rump, S.M., Shary, S.P., van Hentenryck, P.: Standardized notation in interval analysis. Comput. Technol. **15**, No. 1, 7–13 (2010) (an earlier electronic version of the paper is downloadable from URL `http://www.nsc.ru/interval/INotation.pdf`)
7. Kreinovich, V., Lakeyev, A., Rohn, J., Kahl, P.: Computational Complexity and Feasibility of Data Processing and Interval Computations. Kluwer, Dordrecht (1997).
8. Kupriyanova, L.: Inner estimation of the united solution set of interval linear algebraic system. Reliab. Comput. **1**, No. 1, 15–31 (1995)
9. Moore, R.E., Kearfott, R.B., Cloud, M.J.: Introduction to Interval Analysis. SIAM, Philadelphia (2009)
10. Neumaier, A.: Interval Methods for Systems of Equations. Cambridge Univ. Press, Cambridge (1990)
11. Oettli, W.: On the solution set of a linear system with inaccurate coefficients. SIAM J. Numer. Anal. **2**, No. 1, 115–118 (1965)
12. Rex, G., Rohn, J.: Sufficient conditions for regularity and singularity of interval matrices. SIAM J. Matr. Anal. Appls. **20**, 437–445 (1999)
13. Rohn, J.: Input-output planning with inexact data. Freiburger Intervall-Berichte 78/9, 1–16 (1978)

14. Shary, S.P.: On characterization of the united solution set to interval linear algebraic systems. Krasnoyarsk, 1990. 20 p. Deposited in VINITI, No. 726-B91. (in Russian)
15. Shary, S.P.: Linear static systems under interval uncertainty: algorithms to solve control and stabilization problems. In: Kreinovich, V. (ed.) Int. J. of Reliab. Comput. Supplement. Extended Abstracts of APIC'95, Int. Workshop on Applications of Interval Computations, El Paso, TX, Feb. 23-25, 1995, pp. 181–184. El Paso, University of Texas at El Paso, 1995, (an electronic version of the paper is downloadable from URL `http://www.nsc.ru/interval/shary/Papers/ElPaso.pdf`
16. Shary, S.P.: Solving the linear interval tolerance problem. Math. Comput. Simul. **39**, 53–85 (1995)
17. Shary, S.P.: Algebraic approach to the interval linear static identification, tolerance and control problems, or One more application of Kaucher arithmetic. Reliab. Comput. **2**, No. 1, 3–33 (1996)
18. Shary, S.P.: A new technique in systems analysis under interval uncertainty and ambiguity. Reliab. Comput. **8**, No. 5, 321–418 (2002)
19. Smagina, Ye., Brewer, I.: Using interval arithmetic for robust state feedback design. Syst. & Control Lett. **46**, 187–194 (2002)