MATHEMATICAL
RESEARCH

Volume 90

Scientific Computing and Validated Numerics

edited by
Götz Alefeld
Andreas Frommer
Bruno Lang

Akademie Verlag

Scientific Computing and Validated Numerics

Proceedings of the International Symposium on Scientific Computing, Computer Arithmetic and Validated Numerics SCAN-95 held in Wuppertal, Germany, September 26–29, 1995

edited by
Götz Alefeld
Andreas Frommer
Bruno Lang

Akademie Verlag

Editors:

Prof. Dr. Götz Alefeld, University of Karlsruhe;
Prof. Dr. Andreas Frommer, Dr. Bruno Lang, University of Wuppertal

With 31 figures and 20 tables

1st edition
Editorial Director: Dipl.-Math. Gesine Reiher

Library of Congress Card Number pending

Die Deutsche Bibliothek – CIP-Einheitsaufnahme

Scientific computing and validated numerics : proceedings of
the international Symposium on Scientific Computing,
Computer Arithmetic and Validated Numerics, SCAN-95 ;
held in Wuppertal, Germany, September 26–29, 1995 / ed. by
Götz Alefeld ... – 1. ed. – Berlin : Akad. Verl., 1996
(Mathematical research ; Vol. 90)
ISBN 3-05-501737-4
NE: Alefeld, Götz [Hrsg.]; International Symposium on Scientific
Computing, Computer Arithmetic and Validated Numerics <4, 1995,
Wuppertal>; GT

ISSN 0138-3019

Akademie Verlag is a member of the VCH Publishing Group.

Printed on non-acid paper.
The paper used corresponds to both the U.S. standard ANSI Z.39.48 – 1984
and the European standard ISO TC 46.

Printing: GAM Media GmbH, Berlin
Bookbinding: Verlagsbuchbinderei Mikolai GmbH, Berlin

Printed in the Federal Republic of Germany

Akademie Verlag GmbH
Postfach · D-13162 Berlin
Federal Republic of Germany

VCH Publishers, Inc.
220 East 23rd Street
New York, NY 10010-4606

A new approach to the analysis of static systems under interval uncertainty

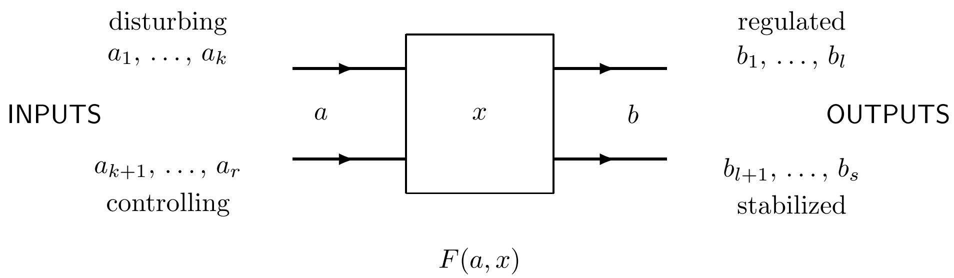
Sergey P. Shary

0 Problem statement

The subject of this work is some mathematical and computational aspects of modeling of static systems under interval uncertainty, and our main practical example will be the *inverse problem* of the systems analysis:

$$\boxed{\begin{array}{c}\textit{Given the input and output of the system,}\\ \textit{find (or somehow estimate) its states.}\end{array}}$$

The peculiarity of the situation we deal with is that the input and output of the system are only supposed to be within some bounds, lower and upper, or, which is equivalent, we are given merely intervals of their possible variations. We shall denote intervals and interval objects (vectors, matrices) by boldface letters (for instance, $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{x}, \mathbf{y}, \mathbf{z}$). Also, $\underline{\mathbf{x}}$ and $\overline{\mathbf{x}}$ designate the lower and upper endpoints of the interval $\mathbf{x}$.



Let the system input, state and output be described by the vectors $a \in \mathbb{R}^r$, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^s$ respectively. In the set of all inputs, we distinguish *perturbations* $a_1, \ldots, a_k$, which are independent of our will, acting within the intervals $\mathbf{a}_1, \ldots, \mathbf{a}_k$, as well as *controls* $a_{k+1}, \ldots, a_r$, which we can choose from the intervals $\mathbf{a}_{k+1}, \ldots, \mathbf{a}_r$. Similarly, the set of all system outputs is divided into the components $b_1, b_2, \ldots, b_l$ (*regulated outputs*) that we must be able to transform to any values from the prescribed attainability intervals $\mathbf{b}_1, \ldots, \mathbf{b}_l$ and the components $b_{l+1}, \ldots, b_s$ (*stabilized outputs*) that must fall into some intervals $\mathbf{b}_{l+1}, \ldots, \mathbf{b}_s$ with guarantee. The input-output relationship is assumed to be of the form

$$F(a, x) = b, \tag{1}$$

where the components $F_i(a, x)$, $i = 1, 2, \ldots, s$, are *rational expressions* that we will understand from now on as finite combinations of variables a, x and some constants with elementary arithmetical operations (cf. [9, 10]). Also, we assume all F_i to be continuous over their domains. The question that inspired the mathematical constructions of our work is as follows:

> *For what system states x can we choose, for any perturbations $a_1, \ldots,$ a_k which are inside the limits of $\mathbf{a}_1, \ldots, \mathbf{a}_k$ respectively and for any* a priori *given output values $b_1, \ldots, b_l$ from the respective attainability intervals $\mathbf{b}_1, \ldots, \mathbf{b}_l$, the corresponding controls $a_{k+1} \in \mathbf{a}_{k+1}, \ldots,$ $a_r \in \mathbf{a}_r$ such that the output response of the system $F(x, a)$ would be exactly equal to $b_1, \ldots, b_l$ in the regulated outputs and would be inside $\mathbf{b}_{l+1}, \ldots, \mathbf{b}_s$ in the stabilized outputs?* (2)

Some people mind using the terms "control", "controlling", etc. in the situations like the above described. They argue that what we mean is something different from the respective notions of the classical automatic control theory. That is so indeed. However, the automatic control theory is not the only discipline that has to do with "controls". We would like to remind that in the operations research the following definition is generally adopted [1]: an operation is a purposeful action that can be characterized as

$$U = f(X_i, Y_j),$$

where U is a utility or a value of the criterion that gives a quality of system functioning, X_i are the variables that we can *control* and Y_j are the variables that can not *be controlled* (i.e., they are uncontrolled, or, in other words, *disturbing*). Thus, the sense in which we use the term "control" (and related terms) is rather the sense in which these words are used in the operations research. Anyway, they are quite rightful.

In the special case of all inputs and outputs of the system being determined precisely, the solution of the problem (2) reduces to the solution of the equation (1) with respect to x. If the input and output values have interval uncertainty, then, according to the terminology tradition of the interval analysis, we shall also refer to the solution process of the problem (2) as the *solution of the interval equations*

$$F(\mathbf{a}, x) = \mathbf{b} \tag{3}$$

with $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r)^\top$ and $\mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_s)^\top$, but we must first specify what is meant by the "solution set" to (3).

Formally, the set of all states x that satisfy the requirement (2) is described by the following definition:

$$\begin{array}{l} \{\ x \in \mathbb{R}^n \mid \\ \qquad (\ \ \forall a_1 \in \mathbf{a}_1\ \ ) \cdots (\forall a_k \in \mathbf{a}_k\,)(\ \ \forall b_1 \in \mathbf{b}_1\ \ ) \cdots (\ \forall b_l \in \mathbf{b}_l\ ) \\ \qquad (\,\exists a_{k+1} \in \mathbf{a}_{k+1}\,) \cdots (\ \exists a_r \in \mathbf{a}_r\,)(\,\exists b_{l+1} \in \mathbf{b}_{l+1}\,) \cdots (\ \exists b_s \in \mathbf{b}_s\ ) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\ F(a,x) = b\ )\ \}. \end{array} \tag{4}$$

The property which is written out as a predicate behind the vertical line in the record (4) selects some values of x that constitute the solution set and therefore, we call this property *selecting predicate* of the set (4). We emphasize that apart from setting the function F and interval vectors $\mathbf{a}$, $\mathbf{b}$, the key point in the definition (4) is indication of quantifiers at various elements of the system (3), or, to put it differently, determination of the r-vector $\alpha = (\,\alpha_i)$ and s-vector $\beta = (\beta_i)$ made up of quantifiers and such that

$$\begin{cases} \alpha_i = & \text{the quantifier standing at } a_i \text{ in the record (4),} \\ \beta_i = & \text{the quantifier standing at } b_i \text{ in the record (4).} \end{cases} \tag{5}$$

One could arrive at introducing the general definition of the solution sets (4) from the abstract standpoint as well. Notice that the interval uncertainty of a parameter may be interpreted in two ways in accordance with the dual understanding of what the interval is. In some cases, when we say that a real-life quantity is described by an interval, we mean that *all* values from this interval are possible; in some other cases, the interval means the bound on the (unknown) value, i.e., that *some* values are possible, and all these possible values belong to the given interval. Mathematically, this difference is expressed by using the universal quantifier $\forall$ and existential quantifier $\exists$: in the first case we write $\forall x \in \mathbf{x}$, while in the second $\exists x \in \mathbf{x}$. When strictly defining the solution set to an interval problem, one must clearly demarcate these two types of uncertainty.

Overall, the mathematical object defined by (4) has a significance of its own, and it makes sense to single it out as a separate notion. But, before doing this, it should be recognized that the definition (4) is still not the most general. Since different quantifiers do not commute with each other, we may form the other solution sets to interval equations through combining $\forall$ and $\exists$ with the parameters of the equation and changing their order. Generally, these solution sets can be practically interpreted as solutions to some games or multistep decision-making processes under interval uncertainty. In this work, we shall consider only the solution sets of the form (4), or, in other words, the solution sets with the selecting predicate in which *all occurrences of the universal quantifier* $\forall$ *precede all occurrences of the existential quantifier* $\exists$.

Definition 1. *Let for the interval equation* $F(\mathbf{a}, x) = \mathbf{b}$ *the distribution of various uncertainty types of its interval elements be represented by the quantifier vectors* α *and* β *defined by (5). We will call the set (4)* $\alpha\beta$-solution set *to the interval equation* $F(\mathbf{a}, x) = \mathbf{b}$ *and denote it by* $\Sigma_{\alpha\beta}(F, \mathbf{a}, \mathbf{b})$.

The particular cases of the above definition are the following three solution sets which have been the subject of (more or less) active research in the modern interval analysis:

♡ the *united solution set*, formed by solutions of all point equations $F(a, x) = b$ with $a \in \mathbf{a}$ and $b \in \mathbf{b}$, i.e., the set

$$\Sigma_{\exists\exists}(F, \mathbf{a}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \mid (\exists a \in \mathbf{a})(\exists\, b \in \mathbf{b})(F(a, x) = b)\,\}, \tag{6}$$

historically first and undoubtedly the most popular of the solution sets; sometimes it is called simply as *solution set*;

♡ the *tolerable solution set*, formed by all point vectors x such that the image $F(a, x) \in \mathbf{b}$ for any $a \in \mathbf{a}$, i.e., the set

$$\Sigma_{\forall\exists}(F, \mathbf{a}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \mid (\forall a \in \mathbf{a})(\exists\, b \in \mathbf{b})(F(a, x) = b)\,\} \tag{7}$$

(see, e.g., [10, 14]);

♡ the *controllable solution set*

$$\Sigma_{\exists\forall}(F, \mathbf{a}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \mid (\forall\, b \in \mathbf{b})(\exists\, a \in \mathbf{a})(F(a, x) = b)\,\}, \tag{8}$$

formed by all point vectors $x \in \mathbb{R}^n$, such that for any desired $b \in \mathbf{b}$ we can find a corresponding $a \in \mathbf{a}$ satisfying $F(a, x) = b$ (see [13]).

Even in the simple practical situations, a direct computation and description of $\alpha\beta$-solution sets prove, as a rule, arduous and sometimes almost impossible. For instance, the length of the direct description of $\Sigma_{\alpha\beta}(F, \mathbf{a}, \mathbf{b})$ grows exponentially with n in the linear case, when $F(a, x) = Ax$ with some $s \times n$-matrix A. It makes sense to confine ourselves to the problem of an approximate description of the solution sets by simpler sets, and, taking into account our practical interpretation, we shall consider an *inner* estimation, i.e., by *subsets*.

Why subsets? The point is that other estimating sets may contain points which have nothing to do with the solutions to the main problem (2). The other relevant explanation is that only for subsets $\Pi \subseteq \Sigma_{\alpha\beta}(F, \mathbf{a}, \mathbf{b})$ the answer to the question (2) remains positive for all points $x \in \Pi$.

Taking the simpler subsets in the form of the axis-aligned boxes (interval vectors), we thus arrive at the problem of inner interval estimation of the solution sets (4):

$$\boxed{\begin{array}{c}\textit{Find an interval vector that is included in the solution}\\ \textit{set } \Sigma_{\alpha\beta}(F, \mathbf{a}, \mathbf{b}) \textit{ of the interval equation } F(\mathbf{a}, x) = \mathbf{b}.\end{array}} \tag{9}$$

So far, these problems have been solved only by minimax methods of mathematical programming (see, e.g., [3]). We propose a new interval approach to the problem, which we call the *algebraic approach*, and, on its basis, we develop (for some specific cases) a number of efficient numerical algorithms that enables us to compute an interval solution to (9) fast and with high quality.

1 Analytical characterization of $\alpha\beta$-solution sets

To describe the distribution of various types of uncertainty in interval elements of the system $F(\mathbf{a}, x) = \mathbf{b}$, it is reasonable to introduce, along with the quantifier vectors α and β, the following interval vectors $\mathbf{a}^{\forall} = (\,\mathbf{a}_i^{\forall})$ and $\mathbf{a}^{\exists} = (\,\mathbf{a}_i^{\exists})$ and interval vectors $\mathbf{b}^{\forall} = (\,\mathbf{b}_i^{\forall})$ and $\mathbf{b}^{\exists} = (\,\mathbf{b}_i^{\exists})$ of the same size as $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{a}_i^{\forall} = \begin{cases} \mathbf{a}_i, & \text{if } \alpha_i = \forall, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathbf{a}_i^{\exists} = \begin{cases} \mathbf{a}_i, & \text{if } \alpha_i = \exists, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathbf{b}_i^{\forall} = \begin{cases} \mathbf{b}_i, & \text{if } \beta_i = \forall, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathbf{b}_i^{\exists} = \begin{cases} \mathbf{b}_i, & \text{if } \beta_i = \exists, \\ 0, & \text{otherwise}. \end{cases}$$

In particular,

$$\begin{aligned} & \mathbf{a} = \mathbf{a}^{\forall} + \mathbf{a}^{\exists}, && \mathbf{b} = \mathbf{b}^{\forall} + \mathbf{b}^{\exists}, & \\ \text{and} \quad & \mathbf{a}_i^{\forall}\,\mathbf{a}_i^{\exists} = 0, && \mathbf{b}_i^{\forall}\,\mathbf{b}_i^{\exists} = 0 & \text{for all } i. \end{aligned}$$

Proposition 1. *Let the mapping F be be such that each controlling parameter $a_{k+1}, \ldots, a_r$, which correspond to $\exists$-type of uncertainty, occurs in at most one of the components $F_i(a, x)$. Then the membership $x \in \Sigma_{\alpha\beta}(F, \mathbf{a}, \mathbf{b})$ is equivalent to the following system of inequalities:*

$$\left\{ \begin{array}{l} \left\{ \begin{array}{lcl} \displaystyle\min_{a' \in \mathbf{a}^{\forall}} \; \max_{a'' \in \mathbf{a}^{\exists}} \; F_i(a, x) & \geq & \overline{\mathbf{b}}_i, \\ \displaystyle\max_{a' \in \mathbf{a}^{\forall}} \; \min_{a'' \in \mathbf{a}^{\exists}} \; F_i(a, x) & \leq & \underline{\mathbf{b}}_i, \end{array} \right. \\ \qquad\qquad \text{— \textit{for the regulated outputs}, } i = 1, \ldots, l, \\ \\ \left\{ \begin{array}{lcl} \displaystyle\min_{a' \in \mathbf{a}^{\forall}} \; \max_{a'' \in \mathbf{a}^{\exists}} \; F_i(a, x) & \geq & \underline{\mathbf{b}}_i, \\ \displaystyle\max_{a' \in \mathbf{a}^{\forall}} \; \min_{a'' \in \mathbf{a}^{\exists}} \; F_i(a, x) & \leq & \overline{\mathbf{b}}_i, \end{array} \right. \\ \qquad\qquad \text{— \textit{for the stabilized outputs}, } i = l+1, \ldots, s, \end{array} \right. \tag{10}$$

where we denoted $a = a' + a''$, $a', a'' \in \mathbb{R}^r$.

Proof. Let $b = (\,b_1, b_2, \ldots, b_s) = b' + b''$, $b', b'' \in \mathbb{R}^s$. We perform the following equivalent transformations with the selecting predicate of the solution set to the interval equation:

$$\begin{aligned} & \Sigma_{\alpha\beta}(F, \mathbf{a}, \mathbf{b}) \\ & \quad = \{\; x \in \mathbb{R}^n \mid (\forall a' \in \mathbf{a}^{\forall})(\forall b' \in \mathbf{b}^{\forall})(\exists a'' \in \mathbf{a}^{\exists})(\exists b'' \in \mathbf{b}^{\exists})(\, F(a, x) = b\,)\;\} \end{aligned}$$

$$
\begin{aligned}
= \{\ x \in \mathbb{R}^n \mid & (\forall a' \in \mathbf{a}^\forall)(\forall b' \in \mathbf{b}^\forall)(\exists a'' \in \mathbf{a}^\exists) \\
& (\ F_1(a,x) = b_1 \ \& \\
& \quad \cdots \quad \& \\
& \ F_l(a,x) = b_l \ \& \\
& \ F_{l+1}(a,x) \in \mathbf{b}_{l+1} \ \& \\
& \quad \cdots \quad \& \\
& \ F_s(a,x) \in \mathbf{b}_s \ )\ \} \\
= \{\ x \in \mathbb{R}^n \mid & (\forall a' \in \mathbf{a}^\forall)(\forall b' \in \mathbf{b}^\forall)(\exists a'' \in \mathbf{a}^\exists) \\
& (\ F_1(a,x) \geq b_1 \ \&\ F_1(a,x) \leq b_1 \ \& \\
& \quad \cdots \quad \& \\
& \ F_l(a,x) \geq b_l \ \&\ F_l(a,x) \leq b_l \ \& \\
& \ F_{l+1}(a,x) \geq \underline{\mathbf{b}}_{l+1} \ \&\ F_{l+1}(a,x) \leq \overline{\mathbf{b}}_{l+1} \ \& \\
& \quad \cdots \quad \& \\
& \ F_s(a,x) \geq \underline{\mathbf{b}}_s \ \&\ F_s(a,x) \leq \overline{\mathbf{b}}_s \ )\ \} \\
= \{\ x \in \mathbb{R}^n \mid & (\forall a' \in \mathbf{a}^\forall)(\forall b' \in \mathbf{b}^\forall) \\
& (\ (\exists a'' \in \mathbf{a}^\exists)(F_1(a,x) \geq b_1) \ \&\ (\exists a'' \in \mathbf{a}^\exists)(F_1(a,x) \leq b_1) \ \& \\
& \quad \cdots \quad \& \\
& \ (\exists a'' \in \mathbf{a}^\exists)(F_l(a,x) \geq b_l) \ \&\ (\exists a'' \in \mathbf{a}^\exists)(F_l(a,x) \leq b_l) \ \& \\
& \ (\exists a'' \in \mathbf{a}^\exists)(F_{l+1}(a,x) \geq \underline{\mathbf{b}}_{l+1}) \ \&\ (\exists a'' \in \mathbf{a}^\exists)(F_{l+1}(a,x) \leq \overline{\mathbf{b}}_{l+1}) \ \& \\
& \quad \cdots \quad \& \\
& \ (\exists a'' \in \mathbf{a}^\exists)(F_s(a,x) \geq \underline{\mathbf{b}}_s) \ \&\ (\exists a'' \in \mathbf{a}^\exists)(F_s(a,x) \leq \overline{\mathbf{b}}_s) \ )\ \},
\end{aligned}
$$

the last equality being true due to the restriction we impose upon F: the sets of variables from non-zero components of $\mathbf{a}^\exists$ that occur in different components of F simply do not intersect with each other.

Notice that for functions f which are continuous over $\mathbf{a}$ we have the following equivalences:

$$
\begin{aligned}
(\exists a \in \mathbf{a})(\ f(a) \geq b\ ) \qquad &\Longleftrightarrow \qquad \max_{a \in \mathbf{a}} f(a) \geq b, \\
(\exists a \in \mathbf{a})(\ f(a) \leq b\ ) \qquad &\Longleftrightarrow \qquad \min_{a \in \mathbf{a}} f(a) \leq b.
\end{aligned}
$$

Hence, we may continue our transformations:

$$
\begin{aligned}
&\Sigma_{\alpha\beta}(F, \mathbf{a}, \mathbf{b}) \\
&\quad = \{\ x \in \mathbb{R}^n \mid (\forall a' \in \mathbf{a}^\forall)(\forall b' \in \mathbf{b}^\forall) \\
&\qquad (\ (\max\nolimits_{a'' \in \mathbf{a}^\exists} F_1(a,x) \geq b_1) \ \&\ (\min\nolimits_{a'' \in \mathbf{a}^\exists} F_1(a,x) \leq b_1) \ \& \\
&\qquad\quad \cdots \quad \& \\
&\qquad\ (\max\nolimits_{a'' \in \mathbf{a}^\exists} F_l(a,x) \geq b_l) \ \&\ (\min\nolimits_{a'' \in \mathbf{a}^\exists} F_l(a,x) \leq b_l) \ \& \\
&\qquad\ (\max\nolimits_{a'' \in \mathbf{a}^\exists} F_{l+1}(a,x) \geq \underline{\mathbf{b}}_{l+1}) \ \&\ (\min\nolimits_{a'' \in \mathbf{a}^\exists} F_{l+1}(a,x) \leq \overline{\mathbf{b}}_{l+1}) \ \& \\
&\qquad\quad \cdots \quad \& \\
&\qquad\ (\max\nolimits_{a'' \in \mathbf{a}^\exists} F_s(a,x) \geq \underline{\mathbf{b}}_s) \ \&\ (\min\nolimits_{a'' \in \mathbf{a}^\exists} F_s(a,x) \leq \overline{\mathbf{b}}_s) \ )\ \}
\end{aligned}
$$

Further,

$$
\begin{aligned}
(\forall\, b \in \mathbf{b})(\, f(a) \geq b\,) \qquad &\Longleftrightarrow \qquad f(a) \geq \overline{\mathbf{b}},\\
(\forall\, b \in \mathbf{b})(\, f(a) \leq b\,) \qquad &\Longleftrightarrow \qquad f(a) \leq \underline{\mathbf{b}},
\end{aligned}
$$

so we have

$$
\begin{aligned}
&\Sigma_{\alpha\beta}(F, \mathbf{a}, \mathbf{b})\\
&\quad = \{\ x \in \mathbb{R}^n \mid (\forall a' \in \mathbf{a}^{\forall})\\
&\qquad\qquad (\ (\, \max\nolimits_{a'' \in \mathbf{a}^{\exists}} F_1(a, x) \geq \overline{\mathbf{b}}_1) \ \&\ (\, \min\nolimits_{a'' \in \mathbf{a}^{\exists}} F_1(a, x) \leq \underline{\mathbf{b}}_1) \ \&\\
&\qquad\qquad\quad \cdots \qquad \&\\
&\qquad\qquad\quad (\, \max\nolimits_{a'' \in \mathbf{a}^{\exists}} F_l(a, x) \geq \overline{\mathbf{b}}_l) \ \&\ (\, \min\nolimits_{a'' \in \mathbf{a}^{\exists}} F_l(a, x) \leq \underline{\mathbf{b}}_l) \ \&\\
&\qquad\qquad\quad (\, \max\nolimits_{a'' \in \mathbf{a}^{\exists}} F_{l+1}(a, x) \geq \underline{\mathbf{b}}_{l+1}) \ \&\ (\, \min\nolimits_{a'' \in \mathbf{a}^{\exists}} F_{l+1}(a, x) \leq \overline{\mathbf{b}}_{l+1}) \ \&\\
&\qquad\qquad\quad \cdots \qquad \&\\
&\qquad\qquad\quad (\, \max\nolimits_{a'' \in \mathbf{a}^{\exists}} F_s(a, x) \geq \underline{\mathbf{b}}_s) \ \&\ (\, \min\nolimits_{a'' \in \mathbf{a}^{\exists}} F_s(a, x) \leq \overline{\mathbf{b}}_s)\ )\ \}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
(\forall\, a \in \mathbf{a})(\, f(a) \geq b\,) \qquad &\Longleftrightarrow \qquad \min_{a \in \mathbf{a}} f(a) \geq b,\\
(\forall\, a \in \mathbf{a})(\, f(a) \leq b\,) \qquad &\Longleftrightarrow \qquad \max_{a \in \mathbf{a}} f(a) \leq b,
\end{aligned}
$$

and we get

$$
\begin{aligned}
&\Sigma_{\alpha\beta}(F, \mathbf{a}, \mathbf{b})\\
&\quad = \{\ x \in \mathbb{R}^n \mid (\ (\, \min_{a' \in \mathbf{a}^{\forall}} \max_{a'' \in \mathbf{a}^{\exists}} F_1(a, x) \geq \overline{\mathbf{b}}_1) \ \&\ (\, \max_{a' \in \mathbf{a}^{\forall}} \min_{a'' \in \mathbf{a}^{\exists}} F_1(a, x) \leq \underline{\mathbf{b}}_1) \ \&\\
&\qquad\qquad\qquad \cdots \qquad \&\\
&\qquad\qquad\quad (\, \min_{a' \in \mathbf{a}^{\forall}} \max_{a'' \in \mathbf{a}^{\exists}} F_l(a, x) \geq \overline{\mathbf{b}}_l) \ \&\ (\, \max_{a' \in \mathbf{a}^{\forall}} \min_{a'' \in \mathbf{a}^{\exists}} F_l(a, x) \leq \underline{\mathbf{b}}_l) \ \&\\
&\qquad\qquad\quad (\, \min_{a' \in \mathbf{a}^{\forall}} \max_{a'' \in \mathbf{a}^{\exists}} F_{l+1}(a, x) \geq \underline{\mathbf{b}}_{l+1}) \ \&\ (\, \max_{a' \in \mathbf{a}^{\forall}} \min_{a'' \in \mathbf{a}^{\exists}} F_{l+1}(a, x) \leq \overline{\mathbf{b}}_{l+1}) \ \&\\
&\qquad\qquad\qquad \cdots \qquad \&\\
&\qquad\qquad\quad (\, \min_{a' \in \mathbf{a}^{\forall}} \max_{a'' \in \mathbf{a}^{\exists}} F_s(a, x) \geq \underline{\mathbf{b}}_s) \ \&\ (\, \max_{a' \in \mathbf{a}^{\forall}} \min_{a'' \in \mathbf{a}^{\exists}} F_s(a, x) \leq \overline{\mathbf{b}}_s)\ )\ \},
\end{aligned}
$$

which coincides with the system (10). $\square$

2 Kaucher interval arithmetic

One can readily see from the characterization (10) that the problems (2), (9) are minimax by their nature. Then, to solve minimax problems we need a special "minimax" interval arithmetic, that is, an interval arithmetic which implements computation of minimax at each elementary arithmetical operation — addition, subtraction, multiplication and division. Classical interval arithmetic and its well-known generalizations — Kahan arithmetic, Hansen arithmetic and some others — are designed for evaluation of the range of operations and can not be directly used for our purposes. Luckily, such a minimax arithmetic does exist and we do not need to construct it by ourselves on a bare place. It is *Kaucher interval arithmetic.*

The classical interval arithmetic is known to be the algebraic system $\langle \mathbb{IR}, +, -, \cdot, / \rangle$, where $\mathbb{IR}$ is the set of all real intervals $[\underline{x}, \overline{x}]$, $\underline{x} \le \overline{x}$, while the binary operations — addition, subtraction, multiplication and division — are defined according to the following fundamental principle:

$$\mathbf{x} \star \mathbf{y} = \{\, x \star y \mid x \in \mathbf{x},\ y \in \mathbf{y} \,\} \tag{11}$$

for all intervals $\mathbf{x}$, $\mathbf{y}$ such that $(x \star y)$, $\star \in \{\, +, -, \cdot, / \,\}$ makes sense for any $x \in \mathbf{x}$, $y \in \mathbf{y}$ [2, 9, 10]. The explicit formulas of the interval arithmetical operations are

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= \left[\, \underline{\mathbf{x}} + \underline{\mathbf{y}},\ \overline{\mathbf{x}} + \overline{\mathbf{y}} \,\right], \\
\mathbf{x} - \mathbf{y} &= \left[\, \underline{\mathbf{x}} - \overline{\mathbf{y}},\ \overline{\mathbf{x}} - \underline{\mathbf{y}} \,\right], \\
\mathbf{x} \cdot \mathbf{y} &= \left[\, \min\{\underline{\mathbf{x}}\,\underline{\mathbf{y}}, \underline{\mathbf{x}}\,\overline{\mathbf{y}}, \overline{\mathbf{x}}\,\underline{\mathbf{y}}, \overline{\mathbf{x}}\,\overline{\mathbf{y}}\},\ \max\{\underline{\mathbf{x}}\,\underline{\mathbf{y}}, \underline{\mathbf{x}}\,\overline{\mathbf{y}}, \overline{\mathbf{x}}\,\underline{\mathbf{y}}, \overline{\mathbf{x}}\,\overline{\mathbf{y}}\} \,\right], \\
\mathbf{x}/\mathbf{y} &= \mathbf{x} \cdot \left[\, 1/\overline{\mathbf{y}},\ 1/\underline{\mathbf{y}} \,\right] \qquad \text{for } \ \mathbf{y} \not\ni 0.
\end{aligned}$$

Algebraic properties of the classical interval arithmetic are meager. It is not even a group both with respect to addition and multiplication [4]: intervals with nonzero width, that is, the majority of elements of $\mathbb{IR}$, do not have algebraic opposite and inverse ones (in the group sense). Besides, $\mathbb{IR}$ is not a lattice [4] with respect to the natural inclusion ordering. The first of the operations

$$\begin{aligned}
\mathbf{x} \wedge \mathbf{y} &= \left[\, \max\{\underline{\mathbf{x}}, \underline{\mathbf{y}}\},\ \min\{\overline{\mathbf{x}}, \overline{\mathbf{y}}\} \,\right], \ \text{—}\textit{join}, \\
\mathbf{x} \vee \mathbf{y} &= \left[\, \min\{\underline{\mathbf{x}}, \underline{\mathbf{y}}\},\ \max\{\overline{\mathbf{x}}, \overline{\mathbf{y}}\} \,\right], \ \text{—}\textit{meet},
\end{aligned}$$

is not always applicable in the classical interval arithmetic.

“Incompleteness” both of the algebraic and of the order structures of $\mathbb{IR}$ naturally stimulated attempts to create a “more convenient” interval arithmetic based on it. The joint order-algebraic completion of $\mathbb{IR}$ carried out in the works by Kaucher [6, 7] resulted in the algebraic system called “the extended interval arithmetic $\mathbb{I}\mathbb{IR}$”. We shall also use this term as well as “Kaucher interval arithmetic”. Afterward, Gardeñes and Trepat studied this arithmetic and established some its helpful properties and important applications [5].

Elements of $\mathbb{I}\mathbb{IR}$ are pairs of real numbers $[\,\underline{x},\ \overline{x}\,]$, that are not necessarily related by the condition $\underline{x} \le \overline{x}$. Thus, $\mathbb{I}\mathbb{IR}$ is obtained by adjoining *improper* intervals $[\underline{x}, \overline{x}]$, $\underline{x} > \overline{x}$, to the set $\mathbb{IR} = \{\, [\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in \mathbb{R},\ \underline{x} \le \overline{x} \,\}$ of *proper* intervals as well as real numbers, which are identified with the corresponding degenerate intervals. Elements of the Kaucher extended interval arithmetic and other objects formed of these elements shall be denoted by boldface letters, like the common intervals.

The proper and improper intervals, the two "halves" of $\mathbb{IIR}$, change places as the result of *dualization* mapping dual $: \mathbb{IIR} \to \mathbb{IIR}$, such that

$$\operatorname{dual} \mathbf{x} = [\,\overline{\mathbf{x}}, \underline{\mathbf{x}}\,].$$

As in classical interval arithmetic, we can define the inclusion as

$$\mathbf{x} \subseteq \mathbf{y} \quad \Longleftrightarrow \quad \underline{\mathbf{x}} \geq \underline{\mathbf{y}} \;\&\; \overline{\mathbf{x}} \leq \overline{\mathbf{y}}. \tag{12}$$

This definition makes Kaucher arithmetic $\mathbb{IIR}$ a lattice [4] with respect to the inclusion order relation, in contrast to $\mathbb{IR}$.

Addition and multiplication by real numbers are defined on $\mathbb{IIR}$ by

$$\mathbf{x} + \mathbf{y} := \left[\, \underline{\mathbf{x}} + \underline{\mathbf{y}}, \overline{\mathbf{x}} + \overline{\mathbf{y}} \,\right],$$

$$\lambda \cdot \mathbf{x} := \begin{cases} [\,\lambda\underline{\mathbf{x}}, \lambda\overline{\mathbf{x}}\,], & \text{if } \lambda \in \mathbb{R}^+, \\ [\,\lambda\overline{\mathbf{x}}, \lambda\underline{\mathbf{x}}\,], & \text{otherwise.} \end{cases}$$

Thus, each element $\mathbf{x}$ from $\mathbb{IIR}$ has a unique algebraic opposite element $[\,-\underline{\mathbf{x}}, -\overline{\mathbf{x}}\,]$, and with respect to addition, Kaucher interval arithmetic $\mathbb{IIR}$ is a commutative group, which is isomorphic to the additive group of the standard linear space $\mathbb{R}^2$.

To write out explicit formulas for the multiplication, we separate in $\mathbb{IIR}$ the following subsets:

$$\mathcal{G} := \{\, \mathbf{x} \in \mathbb{IIR} \mid (\underline{\mathbf{x}} > 0) \;\&\; (\overline{\mathbf{x}} > 0) \,\}, \qquad \mathcal{H} := \{\, \mathbf{x} \in \mathbb{IIR} \mid -\mathbf{x} \in \mathcal{G} \,\},$$

$$\mathcal{U} := \{\, \mathbf{x} \in \mathbb{IIR} \mid \underline{\mathbf{x}} \leq 0 \leq \overline{\mathbf{x}} \,\}, \qquad \mathcal{V} := \{\, \mathbf{x} \in \mathbb{IIR} \mid \operatorname{dual} \mathbf{x} \in \mathcal{U} \,\},$$

so that $\mathbb{IIR} = \mathcal{G} \cup \mathcal{H} \cup \mathcal{U} \cup \mathcal{V}$. Then the multiplication in Kaucher arithmetic is described by the following table [7]:

Multiplication in Kaucher interval arithmetic

$\cdot$	$\mathbf{y} \in \mathcal{G}$	$\mathbf{y} \in \mathcal{U}$	$\mathbf{y} \in \mathcal{H}$	$\mathbf{y} \in \mathcal{V}$
$\mathbf{x} \in \mathcal{G}$	$[\,\underline{\mathbf{x}}\,\underline{\mathbf{y}}, \overline{\mathbf{x}}\,\overline{\mathbf{y}}\,]$	$[\,\overline{\mathbf{x}}\,\underline{\mathbf{y}}, \overline{\mathbf{x}}\,\overline{\mathbf{y}}\,]$	$[\,\overline{\mathbf{x}}\,\underline{\mathbf{y}}, \underline{\mathbf{x}}\,\overline{\mathbf{y}}\,]$	$[\,\underline{\mathbf{x}}\,\underline{\mathbf{y}}, \underline{\mathbf{x}}\,\overline{\mathbf{y}}\,]$
$\mathbf{x} \in \mathcal{U}$	$[\,\underline{\mathbf{x}}\,\overline{\mathbf{y}}, \overline{\mathbf{x}}\,\overline{\mathbf{y}}\,]$	$[\,\min\{\underline{\mathbf{x}}\,\overline{\mathbf{y}}, \overline{\mathbf{x}}\,\underline{\mathbf{y}}\},$ $\max\{\underline{\mathbf{x}}\,\underline{\mathbf{y}}, \overline{\mathbf{x}}\,\overline{\mathbf{y}}\}\,]$	$[\,\overline{\mathbf{x}}\,\underline{\mathbf{y}}, \underline{\mathbf{x}}\,\underline{\mathbf{y}}\,]$	0
$\mathbf{x} \in \mathcal{H}$	$[\,\underline{\mathbf{x}}\,\overline{\mathbf{y}}, \overline{\mathbf{x}}\,\underline{\mathbf{y}}\,]$	$[\,\underline{\mathbf{x}}\,\overline{\mathbf{y}}, \underline{\mathbf{x}}\,\underline{\mathbf{y}}\,]$	$[\,\overline{\mathbf{x}}\,\overline{\mathbf{y}}, \underline{\mathbf{x}}\,\underline{\mathbf{y}}\,]$	$[\,\overline{\mathbf{x}}\,\overline{\mathbf{y}}, \overline{\mathbf{x}}\,\underline{\mathbf{y}}\,]$
$\mathbf{x} \in \mathcal{V}$	$[\,\underline{\mathbf{x}}\,\underline{\mathbf{y}}, \overline{\mathbf{x}}\,\underline{\mathbf{y}}\,]$	0	$[\,\overline{\mathbf{x}}\,\overline{\mathbf{y}}, \underline{\mathbf{x}}\,\overline{\mathbf{y}}\,]$	$[\,\max\{\underline{\mathbf{x}}\,\underline{\mathbf{y}}, \overline{\mathbf{x}}\,\overline{\mathbf{y}}\},$ $\min\{\underline{\mathbf{x}}\,\overline{\mathbf{y}}, \overline{\mathbf{x}}\,\underline{\mathbf{y}}\}\,]$

As one can see, multiplication in Kaucher arithmetic allows nontrivial zero divisors. For instance, $[-1, 2] \cdot [5, -3] = 0$. The extended interval multiplication is both commutative and associative, like its predecessor in $\mathbb{IR}$, but the multiplicative group of $\mathbb{KR}$ is formed only by intervals $\mathbf{x}$ with $\underline{\mathbf{x}}\,\overline{\mathbf{x}} > 0$ (i.e. $\mathcal{G} \cup \mathcal{H}$), since the cancellation law does not hold on any wider subset of $\mathbb{KR}$ [6].

Definitions of the interval subtraction and division in Kaucher arithmetic are similar to those in the traditional interval arithmetic $\mathbb{IR}$:

$$\mathbf{x} - \mathbf{y} = \mathbf{x} + (-1) \cdot \mathbf{y},$$
$$\mathbf{x} \,/\, \mathbf{y} = \mathbf{x} \cdot \left[\,1/\overline{\mathbf{y}}, 1/\underline{\mathbf{y}}\,\right] \qquad \text{for} \;\; \underline{\mathbf{y}}\,\overline{\mathbf{y}} > 0.$$

It is important that the inclusion monotonicity holds in Kaucher interval arithmetic too:

$$\mathbf{x} \subseteq \mathbf{x}', \;\; \mathbf{y} \subseteq \mathbf{y}' \quad \Longrightarrow \quad \mathbf{x} \star \mathbf{y} \subseteq \mathbf{x}' \star \mathbf{y}'$$

for $\star \in \{\, +\, , \,-\, , \,\cdot\, , \,/\, \}$ and any $\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}' \in \mathbb{KR}$. It should be mentioned that in general the distributivity property of the multiplication with respect to addition is not valid in Kaucher interval arithmetic [5, 7]. Besides, there is no distributivity of the multiplication with respect to the lattice operations $\vee$ and $\wedge$.

Vector and matrix operations in Kaucher interval arithmetic are defined similarly to the same operations on $\mathbb{IR}$ (see, e.g., [2, 9, 10]) and inclusion orderings on the sets of interval vectors and matrices are the direct product [4] of inclusion orders on separate components, that is,

$$\mathbf{x} \subseteq \mathbf{y} \quad \Longleftrightarrow \quad \mathbf{x}_i \subseteq \mathbf{y}_i \;\; \text{for all } i.$$

Similarly, the action of the dualization operation "dual" upon interval vectors and matrices will be understood componentwise.

The most wonderful fact with Kaucher arithmetic is that the following representation holds that generalizes the formula (11):

$$\mathbf{x} \star \mathbf{y} = \bigvee\!\!\!\!\bigwedge^{\mathbf{x}}_{x \in \mathrm{pro}\, \mathbf{x}} \; \bigvee\!\!\!\!\bigwedge^{\mathbf{y}}_{y \in \mathrm{pro}\, \mathbf{y}} (x \star y), \tag{13}$$

where

$$\bigvee\!\!\!\!\bigwedge^{\mathbf{x}} := \begin{cases} \bigvee, & \text{if } \mathbf{x} \text{ is proper}, \\ \bigwedge, & \text{otherwise}, \end{cases} \qquad \text{— conditional lattice operation,}$$

$$\mathrm{pro}\, \mathbf{x} := \begin{cases} \mathbf{x}, & \text{if } \mathbf{x} \text{ is proper}, \\ \mathrm{dual}\, \mathbf{x}, & \text{otherwise}, \end{cases} \qquad \text{— proper projection of the interval.}$$

This representation expresses the connection between the interval operation $\mathbf{x} \star \mathbf{y}$ and the results of the point operations $x \star y$ for $x \in \mathrm{pro}\, \mathbf{x}$ and $y \in \mathrm{pro}\, \mathbf{y}$.

Notice that, as follows from (13), Kaucher interval arithmetic is the desired minimax interval arithmetic! Indeed, in Kaucher arithmetic, endpoints of the resulting interval are minimax and maximin of the results of the arithmetical operation, if the intervals under operation have different directions. It is fairly simple to conclude from (13), using induction, that, if a rational expression $f(x, y) = f(x_1, \ldots, x_p, y_1, \ldots, y_q)$ has only one occurrence of each variable (if at all) and to the first power only, then for any proper interval vectors $\mathbf{x}$, $\mathbf{y}$, we have

$$\left[\, \min_{x\in\mathbf{x}} \max_{y\in\mathbf{y}} f(x,y),\ \max_{x\in\mathbf{x}} \min_{y\in\mathbf{y}} f(x,y) \,\right] = f(\mathbf{x}, \mathrm{dual}\,\mathbf{y}). \tag{14}$$

The more complex case which can also be proved by induction: given a rational expression $f(x, y) = f(x_1, \ldots, x_p, y_1, \ldots, y_q)$ that has only one occurrence of each variable y_i (if at all) and to the first power only, for any proper interval vectors $\mathbf{x}$, $\mathbf{y}$, we have

$$\left[\, \min_{x\in\mathbf{x}} \max_{y\in\mathbf{y}} f(x,y),\ \max_{x\in\mathbf{x}} \min_{y\in\mathbf{y}} f(x,y) \,\right] \subseteq f(\mathbf{x}, \mathrm{dual}\,\mathbf{y}). \tag{15}$$

3 Algebraic approach

The basis of the algebraic approach to the solution of the problem (9) (i.e., a procedure for the analysis of static systems under interval uncertainty) is the following results:

Proposition 2. *Let the mapping F be such that each of the variables $a_{k+1}, \ldots, a_r$ that represent $\exists$-type of uncertainty occurs only once and to the first power in at most one of the component expressions $F_1, F_2, \ldots, F_s$. If for the vector $x \in \mathbb{R}^n$ the inclusion*

$$F(\,\mathbf{a}^{\forall} + \mathrm{dual}\,\mathbf{a}^{\exists}, x\,) \subseteq \mathrm{dual}\,\mathbf{b}^{\forall} + \mathbf{b}^{\exists} \tag{16}$$

holds, then $x \in \Sigma_{\alpha\beta}(F, \mathbf{a}, \mathbf{b})$.

If each of the variables $a_1, a_2, \ldots, a_r$ occurs only once and to the first power in only one of the component expressions $F_1, F_2, \ldots, F_s$, then the inverse assertion is also true, that is, $x \in \Sigma_{\alpha\beta}$ is equivalent to the inclusion (16).

Proof. Relying on the definition (12) we rewrite conditions (10) in terms of Kaucher interval arithmetic. They are equivalent to

$$\left[\, \min_{a'\in\mathbf{a}^{\forall}} \max_{a''\in\mathbf{a}^{\exists}} F_i(a,x),\ \max_{a'\in\mathbf{a}^{\forall}} \min_{a''\in\mathbf{a}^{\exists}} F_i(a,x) \,\right] \subseteq \mathrm{dual}\,\mathbf{b}_i, \qquad i = 1, 2, \ldots, l,$$

$$\left[\, \min_{a'\in\mathbf{a}^{\forall}} \max_{a''\in\mathbf{a}^{\exists}} F_i(a,x),\ \max_{a'\in\mathbf{a}^{\forall}} \min_{a''\in\mathbf{a}^{\exists}} F_i(a,x) \,\right] \subseteq \mathbf{b}_i, \qquad i = l+1, \ldots, s,$$

or, uniformly,

$$\left[\, \min_{a'\in\mathbf{a}^{\forall}} \max_{a''\in\mathbf{a}^{\exists}} F_i(a,x),\ \max_{a'\in\mathbf{a}^{\forall}} \min_{a''\in\mathbf{a}^{\exists}} F_i(a,x) \,\right] \subseteq (\,\mathrm{dual}\,\mathbf{b}^{\forall} + \mathbf{b}^{\exists})_i, \qquad i = 1, \ldots, s. \tag{17}$$

If for some point x (16) holds, then combining it with (15) results in (17), that is, $x \in \Sigma_{\alpha\beta}(F, \mathbf{a}, \mathbf{b})$.

The second assertion follows directly from the equality (14). $\square$

The key concept in our consideration is that of *algebraic solution* to the interval equation first considered in [11, 12]:

Definition 2. *An interval vector is called an* algebraic solution *to the interval equation if substituting this vector into the equation and executing all interval operations according to the rules of the interval arithmetic result in the equality.*

Thus, the notion of algebraic solution corresponds to the usual concept of a solution to an equation. The essence of our algebraic approach is to change the problem (2) to the problem of finding an algebraic solution to a special equation in Kaucher interval arithmetic $\mathbb{IR}$, thus reducing the initial problem to a traditional problem of the numerical analysis. This reduction is a very attractive feature, notwithstanding that the algebraic solution to the auxiliary interval equation does not need to exist even if the corresponding original problem (9) has solutions.

Proposition 3. *Let the mapping F be such that each of the variables $a_{k+1}, \dots, a_r$ that represent $\exists$-type of uncertainty occurs only once and to the first power in at most one of the component expressions $F_1, F_2, \dots, F_s$. If the proper interval vector $\mathbf{x}$ is an algebraic solution to the equation*

$$F(\,\mathbf{a}^{\forall} + \mathrm{dual}\,\mathbf{a}^{\exists}, x\,) = \mathrm{dual}\,\mathbf{b}^{\forall} + \mathbf{b}^{\exists} \tag{18}$$

then $\mathbf{x} \subseteq \Sigma_{\alpha\beta}(F, \mathbf{a}, \mathbf{b})$, that is, $\mathbf{x}$ is a solution to the problem (9).

Definition 3. *For the interval system $F(\mathbf{a}, x) = \mathbf{b}$, we will call the equation (18) the* dualization equation *that corresponds to its $\alpha\beta$-solution set.*

Proof. Let a proper interval vector $\mathbf{x}$ be the algebraic solution to the dualization equation (18) and $\tilde{x} \in \mathbf{x}$. So, $[\tilde{x}, \tilde{x}] \subseteq \mathbf{x}$ and, in view of inclusion monotonicity of the arithmetical operations in $\mathbb{IR}$, we have

$$F(\,\mathbf{a}^{\forall} + \mathrm{dual}\,\mathbf{a}^{\exists}, \tilde{x}\,) \subseteq F(\,\mathbf{a}^{\forall} + \mathrm{dual}\,\mathbf{a}^{\exists}, \mathbf{x}\,) = \mathrm{dual}\,\mathbf{b}^{\forall} + \mathbf{b}^{\exists}.$$

Thus, $\tilde{x} \in \Sigma_{\alpha\beta}(F, \mathbf{a}, \mathbf{b})$ by Proposition 2. Since it is true for any $\tilde{x} \in \mathbf{x}$, we get $\mathbf{x} \subseteq \Sigma_{\alpha\beta}$ as required. $\square$

Let us now describe important particular cases of the above general statements that relate to the inner interval estimation of the united, tolerable and controlled solution sets (6)–(8):

- Let the mapping F be such that each of the variables $a_1, a_2, \dots, a_r$ occurs only once (if at all) and to the first power in at most one of the component expressions $F_1, F_2, \dots, F_s$. If the proper interval vector $\mathbf{x}$ is an algebraic solution to the equation

$$F(\mathrm{dual}\,\mathbf{a}, x) = \mathbf{b},$$

then $\mathbf{x} \subseteq \Sigma_{\exists\exists}(F, \mathbf{a}, \mathbf{b})$, that is, $\mathbf{x}$ is an inner interval estimate of the united solution set to the equation $F(\mathbf{a}, x) = \mathbf{b}$. This result is applicable, in particular, to interval linear systems, i.e., when $F(a, x) = Ax$ with some matrix $A = (\, a_{ij})$ (see [17]).

- If the proper interval vector $\mathbf{x}$ is an algebraic solution to the equation

$$F(\mathbf{a}, x) = \mathbf{b},$$

then $\mathbf{x} \subseteq \Sigma_{\forall\exists}(F, \mathbf{a}, \mathbf{b})$, that is, $\mathbf{x}$ is an inner interval estimate for the tolerable solution set to the equation $F(\mathbf{a}, x) = \mathbf{b}$ (or, in other words, a solution to the *tolerance problem* [10, 14]).

- Let the mapping F be such that each of the variables $a_1, a_2, \ldots, a_r$ occurs only once and to the first power in at most one of the component expressions $F_1, F_2, \ldots, F_s$. If the proper interval vector $\mathbf{x}$ is an algebraic solution to the equation

$$F(\mathrm{dual}\,\mathbf{a}, x) = \mathrm{dual}\,\mathbf{b},$$

then $\mathbf{x} \subseteq \Sigma_{\exists\forall}(F, \mathbf{a}, \mathbf{b})$, that is, $\mathbf{x}$ is an inner interval estimate of the controllable solution set to the equation $F(\mathbf{a}, x) = \mathbf{b}$.

Finally, the last question we are going to study in this part of our work is that of how "good" the inner interval estimate of $\Sigma_{\alpha\beta}(F, \mathbf{a}, \mathbf{b})$ obtained by using the algebraic approach is.

Proposition 4. *If all the components $F_i(a, x)$ are bilinear in a, x (that is, $F(a, x) = Ax$ with some $s \times n$-matrix A) and each a_i occurs only in one of the component expressions F_i, then the inclusion-maximal algebraic solution to the dualization equation (18) is an inclusion-maximal inner estimate for the solution set $\Sigma_{\alpha\beta}$, that is, an inclusion-maximal solution to the problem (9).*

Proof can be found in [17].

4 What is next?

The proposed algebraic approach enables us to reduce the problem of inner interval estimation of the generalized solution set to solving one non-interval equation — dualization equation, — i.e., to a traditional numerical analysis problem. One would naturally like to have this reduction available for the widest possible class of mappings F and not only for those with simple occurrences of the control variables as specified in Proposition 3. It turns out that, using fine Gardeñes-Trepat theorem about reduction of the dependency widening [5] and some other results, we are able to get free considerable enlargement of the set of mappings F for which the algebraic approach can be applied. The corresponding results will be presented in the expanded and elaborated version of this paper.

Overall, the practicality and efficiency of the algebraic approach crucially depend on efficiency of the numerical algorithms solving the dualization equation (18). It is important to note that, when solving the dualization equation, in very few cases, we could use symbolic and algebraic manipulation (computer algebra) algorithms, etc. The explanation is that algebraic properties of $\mathbb{IR}$ are still poor. Though they are better than those of the classical interval arithmetic, the lack of distributivity makes it impossible even such simplest operation as the reduction of similar terms. This is the reason why the algorithms which implement our algebraic approach should be essentially *numerical*.

Recently, Lakeyev managed to prove that finding algebraic solutions even to the interval linear equations in Kaucher interval arithmetic is NP-hard [8]. Nonetheless, in spite of this unfavorable fact, there has been constructed a number of efficient numerical methods, which work well providing the input intervals are not "too wide". These are subdifferential Newton method [17] (which turns into quasidifferential Newton method in the general case) and various versions of single-step iteration methods based on splitting of the interval matrix of the equation [16]. The construction of numerical procedures for the solution of the dualization equation in the general nonlinear case is an interesting important open problem and, in solving it, the decisive role should play impetus and needs from practice and industry.

5 Acknowledgements

This work was partially supported by Krasnoyarsk Regional Science Foundation, Krasnoyarsk, Russia, under grant 4F307 and by Deutsche Forschungsgemeinschaft. The author is also grateful to Hoon Hong for unforgettable discussions of the above results.

References

[1] R. L. Ackoff and M. W. Sasieni, *Fundamentals of Operations Research*, John Wiley, New York, 1968.

[2] G. Alefeld and J. Herzberger, *Introduction to Interval Computations*, New York, Academic Press, 1983.

[3] L. T. Ashchepkov, *Towards the problem of increasing the survivability of controlled systems*, in Models and Methods of Operations Research, B. A. Bel'tyukov and V. P. Bulatov, eds., Novosibirsk, Nauka, 1988, pp. 69–85. (in Russian)

[4] G. Birkhoff and T. C. Bartee, *Modern Applied Algebra*, McGrow-Hill, 1968.

[5] E. Gardeñes and A. Trepat, *Fundamentals of SIGLA, an interval computing system over the completed set of intervals*, Computing, 24 (1980), pp. 161–179.

[6] E. Kaucher, *Algebraische Erweiterungen der Intervallrechnung unter Erhaltung Ordnungs- und Verbandsstrukturen*, Computing Suppl., 1 (1977), pp. 65–79.

[7] E. Kaucher, *Interval analysis in the extended interval space* $\mathbb{IR}$, Computing Suppl., 2 (1980), pp. 33–49.

[8] A. V. Lakeyev, *Linear algebraic equation in Kaucher arithmetic*, in Reliable Computing, 1995, Supplement, Extended Abstracts of APIC'95: International Workshop on Applications of Interval Computations, El Paso, TX, February 23–25, 1995, pp. 130–133.

[9] R. E. Moore, *Interval Analysis*, Englewood Cliffs, Prentice Hall, 1966.

[10] A. Neumaier, *Interval Methods for Systems of Equations*, Cambridge University Press, Cambridge, 1990.

[11] K. Nickel, *Die Auflösbarkeit linearer Kreissheiben- und Intervall- Gleichungssysteme*, Linear Algebra Appl., 44 (1982), pp. 19–40.

[12] H. Ratschek and W. Sauer, *Linear interval equations*, Computing, 28 (1982), pp. 105–115.

[13] S. P. Shary, *On controlled solution set of interval algebraic systems*, Interval Computations, 6 (1992), 6, pp. 66–75.

[14] S. P. Shary, *Solving the linear interval tolerance problem*, Mathematics and Computers in Simulation, 39 (1995), pp. 53–85.

[15] S. P. Shary, *Algebraic approach to the interval linear static identification, tolerance and control problems, or One more application of Kaucher arithmetic*, Reliable Computing, 2 (1996), 1, to appear.

[16] S. P. Shary, *Numerical computation of algebraic solution to interval linear systems*, in Discrete Mathematics, Krasnoyarsk State Technical University, Krasnoyarsk, 1996, pp. 129–145. (in Russian)

[17] S. P. Shary, *Algebraic solutions to interval linear equations and their applications*, in Numerical Methods and Error Bounds, G. Alefeld and J. Herzberger, eds., Akademie-Verlag, Berlin, 1996, to appear.

Address:

S.P. Shary, Institute of Computational Technologies,
6 Lavrentiev avenue, 630090 Novosibirsk, Russia.
E-mail: `shary@ict.nsk.su`