Solvability of Interval Linear Equations and Data Analysis under Uncertainty*

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Abstract. Consideration was given to the problem of recognizing the solvability (nonemptyness of the solution set) of an interval systems of linear equations. A method based on the so-called recognizing functional of the solution set was proposed to solve it. A new approach to data processing under interval uncertainty based on the unconstrained maximization of the recognizing functional ("maximal consistency method") was presented as an application, and its informal interpretations were described.

DOI: 10.1134/S0005117912020099

1 Introduction

Consideration is given to the interval systems of linear algebraic equations (ISLAE) of the form

$$\begin{cases} \boldsymbol{a}_{11}x_1 & + & \boldsymbol{a}_{12}x_2 & + & \ldots & + & \boldsymbol{a}_{1n}x_n & = & \boldsymbol{b}_1, \\ \boldsymbol{a}_{21}x_1 & + & \boldsymbol{a}_{22}x_2 & + & \ldots & + & \boldsymbol{a}_{2n}x_n & = & \boldsymbol{b}_2, \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ \boldsymbol{a}_{m1}x_1 & + & \boldsymbol{a}_{m2}x_2 & + & \ldots & + & \boldsymbol{a}_{mn}x_n & = & \boldsymbol{b}_m \end{cases} \tag{1}$$

with the interval coefficients $\boldsymbol{a}_{ij}$ and right-hand sides $\boldsymbol{b}_i$,$i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, or, briefly,

$$\boldsymbol{A}x = \boldsymbol{b}, \tag{2}$$

where $\boldsymbol{A} = (\,\boldsymbol{a}_{ij})$ is the interval $(m \times n)$ matrix and $\boldsymbol{b} = (\,\boldsymbol{b}_i)$ is the interval m-vector. Systems (1)–(2) are understood as families of point linear systems $Ax = b$ of the same structure with the matrices $A \in \boldsymbol{A}$ and vectors $b \in \boldsymbol{b}$.

By the *solution set* of the interval linear equation system is meant the set

$$\varXi(\boldsymbol{A}, \boldsymbol{b}) \;=\; \bigl\{\, x \in \mathbb{R}^n \mid (\exists\, A \in \boldsymbol{A})(\exists\, b \in \boldsymbol{b})(\, Ax = b\,)\,\bigr\}, \tag{3}$$

made up by all possible solutions of the point systems $Ax = b$ with $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$ [1, 2, 3]. This object is also called the *united solution set* because the interval equations may also have other solution sets [2, 4]. They are not treated here, and therefore we briefly refer to (3) as the "solution set". We discuss below how to check the solution sets $\varXi(\boldsymbol{A}, \boldsymbol{b})$ for emptiness or nonemptyness and to determine a point in it. In the most general situation, these problems are NP-hard [4, 5].

The notation used in the paper follows the informal international standard for interval analysis [6]. In particular, the set of all closed intervals of the real axis is denoted by $\mathbb{IR}$, the intervals and interval values are denoted by bold letters $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, ..., $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, whereas the noninterval (point) values are not

*Published in AUTOMATION AND REMOTE CONTROL, February 2012, Volume 73, Issue 2, pp. 310–322.

specified in any special manner. Underlining $\underline{\boldsymbol{a}}$ and overlining $\overline{\boldsymbol{a}}$ denote the lower and upper endpoints of the interval $\boldsymbol{a}$ so that $\boldsymbol{a} = [\underline{\boldsymbol{a}}, \overline{\boldsymbol{a}}] = \{ a \in \mathbb{R} \mid \underline{\boldsymbol{a}} \leqslant a \leqslant \overline{\boldsymbol{a}} \}$. In addition, we need

$\operatorname{mid} \boldsymbol{a} = \frac{1}{2}(\overline{\boldsymbol{a}} + \underline{\boldsymbol{a}})$ is midpoint of the interval

$\operatorname{rad} \boldsymbol{a} = \frac{1}{2}(\overline{\boldsymbol{a}} - \underline{\boldsymbol{a}})$ is radius of the interval,

$\langle \boldsymbol{a} \rangle = \begin{cases} \min\{ |\overline{\boldsymbol{a}}|, |\underline{\boldsymbol{a}}| \}, & \text{if } 0 \notin \boldsymbol{a}, \\ 0, & \text{otherwise,} \end{cases}$ is mignitude of the interval, the least distance from its points to zero.

Apart from the set of real intervals, we denote by $\mathbb{IR}$ the *classical interval arithmetics*, an algebraic system made up by the intervals of the real axis with the operations of addition, subtraction, multiplication, and division defined "by representatives", that is, according to the following principle:

$$\boldsymbol{a} \star \boldsymbol{b} = \{ x \star y \mid x \in \boldsymbol{a}, \, y \in \boldsymbol{b} \} \qquad \text{for} \quad \star \in \{ +, -, \cdot, / \}.$$

In other words, the resulting interval of any arithmetic operation is a set, also an interval, made up by all possible results of this operation between the elements of the operand intervals. The expanded formulas for the interval addition, subtraction, multiplication, and division are as follows [1, 2, 3]:

$$\begin{aligned} \boldsymbol{a} + \boldsymbol{b} &= \left[\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}, \; \overline{\boldsymbol{a}} + \overline{\boldsymbol{b}} \right], \\ \boldsymbol{a} - \boldsymbol{b} &= \left[\underline{\boldsymbol{a}} - \overline{\boldsymbol{b}}, \; \overline{\boldsymbol{a}} - \underline{\boldsymbol{b}} \right], \\ \boldsymbol{a} \cdot \boldsymbol{b} &= \left[\min\{ \underline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}, \underline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}, \overline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}, \overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}} \}, \; \max\{ \underline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}, \underline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}, \overline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}, \overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}} \} \right], \\ \boldsymbol{a} / \boldsymbol{b} &= \boldsymbol{a} \cdot \left[1/\overline{\boldsymbol{b}}, \; 1/\underline{\boldsymbol{b}} \right] \qquad \text{for } \boldsymbol{b} \not\ni 0. \end{aligned}$$

The *interval vector* is defined as a vector, that is, column or row, with interval components. Its geometrical image is represented by a rectangular parallelepiped in $\mathbb{R}^n$ with edges parallel to the coordinate axes which is often called the *box*. Similarly, the interval matrix is a rectangular table of the intervals which is considered as a set of all possible point matrices with elements from the given intervals. The set-theoretical relations of membership, "$\in$", and inclusion, "$\subseteq$", are defined in a natural way for the interval vectors and matrices. The arithmetic operations between the interval vectors and matrices are analogs of the corresponding operations for the point case. In particular, the sum of two interval vectors or matrices of the same size is the interval vector (matrix) of the same size made up by the elementwise sums of operands. The same is true for the difference of the interval vectors or matrices. Multiplication of a scalar by the interval vector (matrix) amounts to the multiplication of this scalar by each element of the vector (matrix).

2 Theory

2.1 Problem Formulation and its Basic Properties

The interval equation system $\boldsymbol{A}x = \boldsymbol{b}$ of the form (1)–(2) is said to be *solvable* (*consistent* if there are $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$ such that the point equation system $Ax = b$ has a solution. This solvability quite often is called the "weak solvability" [4]. Since the present paper disregards other types of solvability of the interval equation systems, the adjective "weak" is omitted without prejudice to understanding. One can easily see that the solvability of the interval equation system is equivalent to the nonemptyness of its solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$. This formulation of the problem was first stated, although without the interval terms, in [7] where the study of solvability was reduced to the solution of a system of nonlinear inequalities which later were christened the "Oettli–Prager inequalities".

A universal method of solving the solvability recognition problem may be based on the fact that the intersections of the solution set with each orthant (coordinate angle) of the space $\mathbb{R}^n$ are convex polyhedral sets for which the equations of the boundary hyperplanes are readily put down using the matrix and the right-hand side of ISLAE (see e.g. [2]). As the result, emptiness or nonemptyness of the intersection $\Xi(\boldsymbol{A}, \boldsymbol{b})$ with each orthant $\mathbb{R}^n$ may be detected by solving a system of linear inequalities using, for example, the well-developed linear programming techniques. Overall, the employment of the above method for recognizing nonemptyness of the solution set to the interval linear system of $(m \times n)$ equations and determining its point require at most 2^n solutions of the systems of linear inequalities of size $2m+n$. This result cannot be radically improved, which, as was noticed in the Introduction, reflects intractability, that is, NP-hardness, of the considered problem [4, 5, 8]. It is also clear that this method is of passive nature and practicable only for problems of small dimensions.

One more approach to solvability of the interval system of linear equations relying on the Oettli–Prager inequalities [7] was described in [4] and [9]. However, its has the same exponential laboriousness proportional to 2^n and is also passive.

2.2 Method of Recognizing Functional

The proposed approach is based on the following proposition.

Proposition 1 *Let* $\boldsymbol{A}$ *be an interval* $(m\times n)$ *matrix and* $\boldsymbol{b}$ *be an interval* m*-vector. Then, the expression*

$$\operatorname{Uni}(x, \boldsymbol{A}, \boldsymbol{b}) = \min_{1\leqslant i\leqslant m} \left\{ \operatorname{rad} \boldsymbol{b}_i - \left\langle \operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} \boldsymbol{a}_{ij}\, x_j \right\rangle \right\} \tag{4}$$

defines the functional Uni $: \mathbb{R}^n \to \mathbb{R}$ *such that the membership of the point* $x \in \mathbb{R}^n$ *to the solution set* $\varXi(\boldsymbol{A}, \boldsymbol{b})$ *of the interval linear equation system* $\boldsymbol{A}x = \boldsymbol{b}$ *is equivalent to the nonnegativeness of the functional Uni in* x*:*

$$x \in \varXi(\boldsymbol{A}, \boldsymbol{b}) \qquad \Longleftrightarrow \qquad \operatorname{Uni}(x, \boldsymbol{A}, \boldsymbol{b}) \geqslant 0.$$

The proofs of Proposition 1 and some other results established in the present paper can be found in the Appendix.

If it is clear from the context what system is meant, then one can set down simply $\operatorname{Uni}(x)$, and not $\operatorname{Uni}(x, \boldsymbol{A}, \boldsymbol{b})$.

It is as if the functional Uni "recognizes" by the sign of its values the membership of its argument set in the set $\varXi(\boldsymbol{A}, \boldsymbol{b})$. That is why the epithet "recognizing" is applied to it. It follows from the proof of Proposition 1 (see the Appendix) that it retains its validity if the Uni is taken in the modified form

$$\operatorname{Uni}_\gamma(x, \boldsymbol{A}, \boldsymbol{b}) = \min_{1\leqslant i\leqslant m} \left\{ \gamma_i \left(\operatorname{rad} \boldsymbol{b}_i - \left\langle \operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} \boldsymbol{a}_{ij}\, x_j \right\rangle \right) \right\}, \tag{5}$$

where γ_i, $i = 1, 2, \ldots, m$, are positive numbers. As will be seen below, this form of the recognizing functional is sometimes preferable to the original form.

Proposition 2 *The functional* Uni *is concave in* x *in each orthant* $\mathbb{R}^n$. *If in the matrix* $\boldsymbol{A}$ *the columns with numbers from the set* $J = \{j_1, j_2, \ldots, j_l\}$, $l \leqslant n$ *are interval ones, the rest of them being wholly point (noninterval) ones, then the functional* $\operatorname{Uni}(x, \boldsymbol{A}, \boldsymbol{b})$ *is concave on each of the* 2^l *sets like* $\{\, x \in \mathbb{R}^n \mid x_j \gtreqless 0,\ j \in J \,\}$, *where "*$\gtreqless$*" denotes one of the relations "*$\geqslant$*" or "*$\leqslant$*".*

Proposition 3 *The functional* $\operatorname{Uni}(x, \boldsymbol{A}, \boldsymbol{b})$ *is polyhedral, that is, its graph is made up of a finite number of the hyperplane pieces.*

As an illustrating example, Figure 1 shows the graph of the recognizing functional for the interval linear equation system

$$\begin{pmatrix} [2,3] & [-1,1] \\ [-1,1] & [2,3] \\ [0,1] & [1,2] \end{pmatrix} x = \begin{pmatrix} [-2,2] \\ [0,1] \\ [-1,0] \end{pmatrix}. \tag{6}$$

Proposition 4 *The functional* $\operatorname{Uni}(x, \boldsymbol{A}, \boldsymbol{b})$ *reaches a finite maximum with respect to* x *over the entire space* $\mathbb{R}^n$.

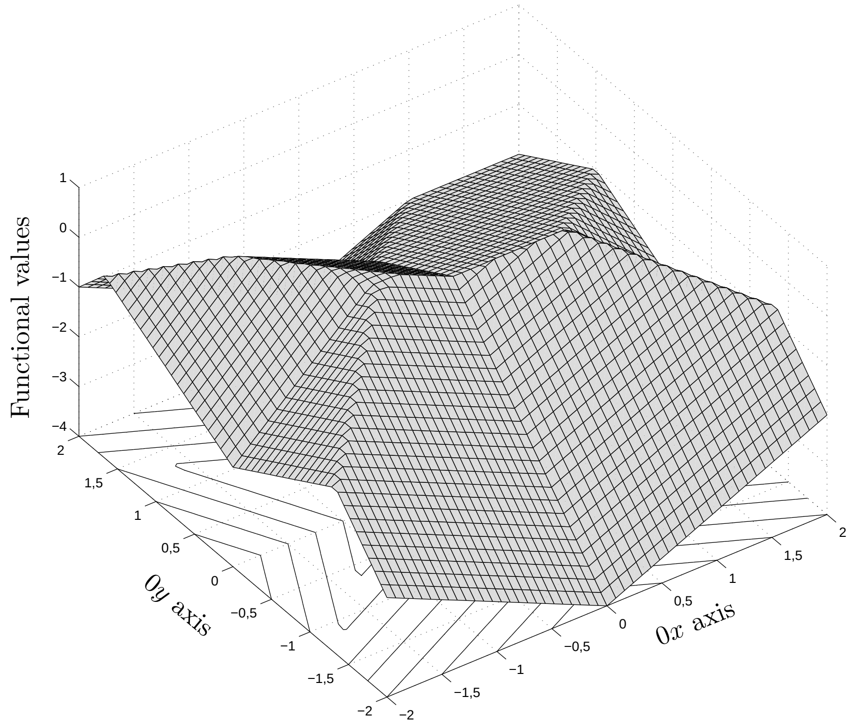

Fig. 1: Graph of the recognizing functional of the solution set to the interval system (6).

It is also possible to show that if $\mathrm{Uni}\,(\tilde{x}, \boldsymbol{A}, \boldsymbol{b}) > 0$, then $\tilde{x}$ is the point of topological interior $\mathrm{int}\ \varXi(\boldsymbol{A}, \boldsymbol{b})$ of the solution set, that is, belongs to $\varXi(\boldsymbol{A}, \boldsymbol{b})$ together with some its neighborhood. Under some additional constraints on $\boldsymbol{A}$, $\boldsymbol{b}$, and $\tilde{x}$, the inverse is true as well: $\mathrm{Uni}\,(\tilde{x}, \boldsymbol{A}, \boldsymbol{b}) > 0$ follows from the membership $\tilde{x} \in \mathrm{int}\ \varXi(\boldsymbol{A}, \boldsymbol{b})$ (see a detailed proof in [10]). The properties of the recognizing functional enable one to use it for studying the membership of the points of the solution set interior, which may be especially important at determining the bodily internal estimate of the solution set around the point-center using the procedure of [2, 11].

As the consequence of the above results, we naturally come to the following technique of studying solvability of the interval linear equation systems. We solve the problem of unconditional maximization (over the entire $\mathbb{R}^n$) of the recognizing functional Uni. If the resulting maximum of the functional is greater than or equal to zero, then, first, the system under consideration is solvable and, second, the arguments of the recognizing functional giving rise to negative values lie within the system solution set. If the maximum of the recognizing functional is negative, then the solution set of the system at hand is empty.

2.3 Correction of the Interval Equation System

The maximum of the recognizing functional $M = \max_{x\in\mathbb{R}^n} \mathrm{Uni}\,(x, \boldsymbol{A}, \boldsymbol{b})$ is an important characteristic of the interval system of linear equations enabling one to correct the system solvability.

We notice that the variables $\mathrm{rad}\ \boldsymbol{b}_i$ are included as addends in all expressions

$$\mathrm{rad}\ \boldsymbol{b}_i - \left\langle \mathrm{mid}\ \boldsymbol{b}_i - \sum_{j=1}^{n} \boldsymbol{a}_{ij}\, x_j \right\rangle$$

whose lower envelope is the recognizing functional Uni. Therefore, a simultaneous change in all $\mathrm{rad}\ \boldsymbol{b}_i$ by the same value leads to the same change in the value of the recognizing functional. In particular, if $C \geqslant 0$ and $\boldsymbol{e} = ([-1, 1], \ldots, [-1, 1])^\top$, then for the system $\boldsymbol{A}x = \boldsymbol{b} + C\boldsymbol{e}$ with the right-hand side expanded to $C\boldsymbol{e}$ we get $\mathrm{Uni}\,(x, \boldsymbol{A}, \boldsymbol{b} + C\boldsymbol{e}) = \mathrm{Uni}\,(x, \boldsymbol{A}, \boldsymbol{b}) + C$, and, consequently,

$$\max_x\ \mathrm{Uni}\,(x, \boldsymbol{A}, \boldsymbol{b} + C\boldsymbol{e}) = \max_x\ \mathrm{Uni}\,(x, \boldsymbol{A}, \boldsymbol{b}) + C.$$

If the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ is empty so that $M < 0$, then by increasing by $C \geqslant |M|$ the radii of all components in the right-hand side the set becomes nonempty because $\max_{x\in\mathbb{R}^n} \mathrm{Uni}\,(x, \boldsymbol{A}, \boldsymbol{b} + C\boldsymbol{e})$ already becomes nonnegative. If the identical expansion of the right side components is unacceptable, then a positive weight $(\varkappa_1, \varkappa_2, \dots, \varkappa_m)$, $\varkappa_i > 0$ is introduced such that an increase in the radius $\boldsymbol{b}_i$ must be proportional to $\varkappa_i$. Then, for the modified recognizing functional (5) with $\gamma_i = \varkappa_i^{-1}$, $i = 1, 2, \dots, m$, it is required to determine $M_\gamma = \max_{x\in\mathbb{R}^n} \mathrm{Uni}_\gamma(x, \boldsymbol{A}, \boldsymbol{b})$ and increase the radius of each $\boldsymbol{b}_i$ by $\varkappa_i C$, where $C \geqslant |M_\gamma|$.

The possibility of correcting the interval linear system by varying its matrix relies on the following simple property of the interval mignitude: for any real numbers p and q such that $0 \notin p + q\,[-1, 1]$,

$$\langle\, p + q\,[-1, 1]\,\rangle = |p| - |q|. \tag{7}$$

Indeed, $p + q\,[-1, 1] = [\,p - |q|, p + |q|\,]$, and if $0 \notin p + q\,[-1, 1]$, then one has to admit that $|p| > |q|$. Then, $\langle\, p + q\,[-1, 1]\,\rangle = \min\{|p - |q||, |p + |q||\,\} = |p| - |q|$.

As a consequence of (7), if $U = \mathrm{Uni}\,(x, \boldsymbol{A}, \boldsymbol{b}) < 0$, then for each $i \in \{1, 2, \dots, m\}$ such that the corresponding braced expression in (4) is negative we obtain

$$\begin{aligned}
&\mathrm{rad}\,\boldsymbol{b}_i - \left\langle \mathrm{mid}\,\boldsymbol{b}_i - \sum_{j=1}^{n} \boldsymbol{a}_{ij}\, x_j \right\rangle = \\
&\quad= \mathrm{rad}\,\boldsymbol{b}_i - \left\langle \mathrm{mid}\,\boldsymbol{b}_i - \sum_{j=1}^{n} \bigl(\,\mathrm{mid}\,\boldsymbol{a}_{ij} + [-1, 1]\cdot \mathrm{rad}\,\boldsymbol{a}_{ij}\bigr)\, x_j \right\rangle = \\
&\quad= \mathrm{rad}\,\boldsymbol{b}_i - \left\langle \mathrm{mid}\,\boldsymbol{b}_i - \sum_{j=1}^{n} (\,\mathrm{mid}\,\boldsymbol{a}_{ij})\, x_j + [-1, 1] \sum_{j=1}^{n} (\,\mathrm{rad}\,\boldsymbol{a}_{ij})\, x_j \right\rangle = \\
&\quad= \mathrm{rad}\,\boldsymbol{b}_i + \sum_{j=1}^{n} (\,\mathrm{rad}\,\boldsymbol{a}_{ij})\, |x_j| - \left| \,\mathrm{mid}\,\boldsymbol{b}_i - \sum_{j=1}^{n} (\,\mathrm{mid}\,\boldsymbol{a}_{ij})\, x_j \right|.
\end{aligned}$$

For $x \neq 0$, increasing the radii of all elements $\boldsymbol{a}_{ij}$ of the matrix $\boldsymbol{A}$ by C such that

$$C \sum_{j=1}^{n} |x_j| \;\geqslant\; |U|,$$

we get $\mathrm{Uni}\,(x, \boldsymbol{A} + C\boldsymbol{E}, \boldsymbol{b}) \geqslant 0$ for the interval $(m \times n)$ matrix $\boldsymbol{E} = ([-1, 1])$ all of whose elements are $[-1, 1]$. Therefore, the point x belongs to the nonempty solution set $\Xi(\boldsymbol{A} + C\boldsymbol{E}, \boldsymbol{b})$. It is obvious that the original system may be corrected by an unequal expansion of the elements of the matrix $\boldsymbol{A}$ at the expense of insignificant complication of this construction.

3 Application to data analysis under uncertainty

3.1 Parameter Estimation Problem

The above technique to study ISLAE for solvability may be applied to the problem of estimating the parameters of a linear dependence from inexact data. Let there be an object whose inputs and outputs are described, respectively, by the finite-dimensional vector $(\,a_1,\, a_2, \dots, a_n) \in \mathbb{R}^n$ and the scalar $b \in \mathbb{R}$, and the "input-output" dependence be linear, that is,

$$b = x_0 + a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \tag{8}$$

with some real constants x_k, $k = 0, 1, \dots, n$. It is required to determine or somehow estimate the values of x_k, that is, to identify the object parameters.

Each observation (measurement) of inputs and outputs of an object generates an expression relating the desired x_k, that is, is an equation in these x_k. If a series of such observations is "sufficiently representative", then one may try to solve the resulting equation system with resect to the unknowns x_k and determine their values. The result of the ith measurement of the object inputs is denoted for convenience by $(a_{i1}, a_{i2}, \ldots, a_{in})$, and the outputs, by b_i. The problem of parameter estimation comes to determining the solution $x = (x_0, x_1, x_2, \ldots, x_n)^\top$—ordinary or in a certain generalized sense—for the interval equation system

$$\begin{cases} x_0 + a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ x_0 + a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \quad\quad \ddots \quad\quad \vdots \quad\quad \vdots \\ x_0 + a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m, \end{cases}$$

where m is the total number of measurements (observations).

Now we turn to a more general practical situation where the object inputs and/or outputs are not known precisely, and known are only their interval estimates, that is, the memberships of a_{ij} and b_i in some intervals:

$$a_{ij} \in \boldsymbol{a}_{ij} = [\,\underline{\boldsymbol{a}}_{ij}, \overline{\boldsymbol{a}}_{ij}] \qquad \text{and} \qquad b_i \in \boldsymbol{b}_i = [\,\underline{\boldsymbol{b}}_i, \overline{\boldsymbol{b}}_i], \tag{9}$$

$i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. This uncertainty is the result not only of the inevitable measurement errors, but also can follow from the basic difficulties involved in determination of the values of some variables or their probabilistic characteristics (see, for example, a detailed discussion of these reasons in [12, 13]). It is only natural that under these new conditions we encounter the need to consider the problem of parameter estimation from the interval data (9). The tradition of formulating such problem may be traced from the pioneering paper of L.V. Kantorovich [14], and the idea of the state-of-the-art in this field can be gained from [12, 13, 15, 16, 17, 18, 19, 20, 21] and their bibliographies.

In what follows, we keep to the usual definition, saying that a set of parameters x_0, x_1, x_2, $\ldots$, x_n of the object described by (8) is *consistent* with the experimental interval data $(\boldsymbol{a}_{i1}, \boldsymbol{a}_{i2}, \ldots, \boldsymbol{a}_{in})$ and $\boldsymbol{b}_i$, $i = 1, 2, \ldots, m$ if within the limits of the measured intervals for each i, that is, each observation, there exist point representatives $a_{i1} \in \boldsymbol{a}_{i1}$, $a_{i2} \in \boldsymbol{a}_{i2}, \ldots$, $a_{in} \in \boldsymbol{a}_{in}$, and $b_i \in \boldsymbol{b}_i$ such that the relation

$$x_0 + a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i \tag{10}$$

is satisfied.

If $\boldsymbol{A} = (\,\boldsymbol{a}_{ij})$ is the $(m \times (n+1))$ interval matrix composed of m results of measurements of inputs and the first column with unit elements and $\boldsymbol{b} = (\,\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_m)^\top$ is the interval vector of m measurements of outputs, then the family of all vectors of the parameters consistent with the experimental data given in the interval form is representable as

$$\left\{\, x = (x_0, x_1, x_2, \ldots, x_n)^\top \in \mathbb{R}^{n+1} \mid \big(\exists (a_{ij}) \in (\boldsymbol{a}_{ij})\big)\big(\exists (b_i) \in (\boldsymbol{b}_i)\big)(\,Ax = b\,)\right\},$$

that is, as a set of solutions to all possible point systems $Ax = b$ with $A = (a_{ij}) \in \boldsymbol{A}$ and $b = (b_i) \in \boldsymbol{b}$. The experts in data analysis and identification call this set as the *information set*, the (*a posteriori*) *set of all possible values of parameters* [15], and so on. It is of importance below that this set is nothing else but the solution set $\varXi(\boldsymbol{A}, \boldsymbol{b})$ of the interval linear equation system $\boldsymbol{A}x = \boldsymbol{b}$ constructed from the observation (measurement) data.

As was already mentioned, this solution set is a union of at most 2^{n+1} polyhedral sets. Therefore, the length of its direct description can grow exponentially with the dimensionality $(n+1)$ of the interval system built from the measurement data. For those reason, they usually confine themselves to determining its estimates in one or another sense, that is, replace the precise description $\varXi(\boldsymbol{A}, \boldsymbol{b})$ by some approximation in compliance with the sense of the practical problem at hand. In what follows, the present author does disregards this formulation and focuses only on the determination of one ("most representative") point of the information set.

3.2 Paradox of the Interval Estimation

In the general case, the direct use of this approach leads to a paradox which was noticed first by E.Z. Demidenko [22] and may be characterized as "the worse, the better". Indeed, data uncertainty is undesirable because it distorts the true picture of reality. Therefore, reduction of uncertainty, that is, contraction of the data intervals, is a boon that is welcome for practice. On the other hand, for wider intervals of the source data, the solution set of the interval system also is wider, and, therefore, there are more possibilities to select from it the model parameters than in the case of narrow interval data where the solution set may be empty at all. So, the higher precision of the source data and the lower the interval uncertainty, the worse the parameter estimation. And vice versa, the wider the uncertainty interval and the smaller our knowledge of the precise values of the measured variables, the better the parameter estimation process and the richer the set of the results that may be obtained. This situation is depicted in Figs. 2 and 3 where the uncertainty intervals in Fig. 3 are obtained by contracting the intervals of Fig. 2. At the same time, the possibility is lost to draw a straight line passing through all uncertainty boxes, that is, coordinated with all data.

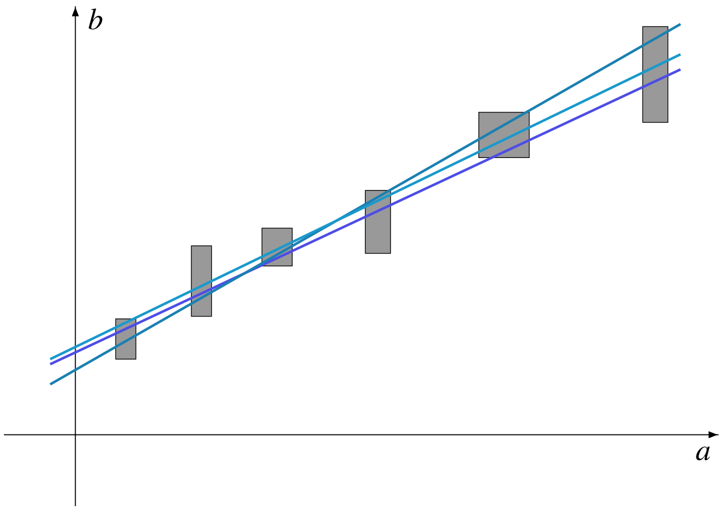

Fig. 2: Wide uncertainty intervals enable one to construct many models consistent with the data.

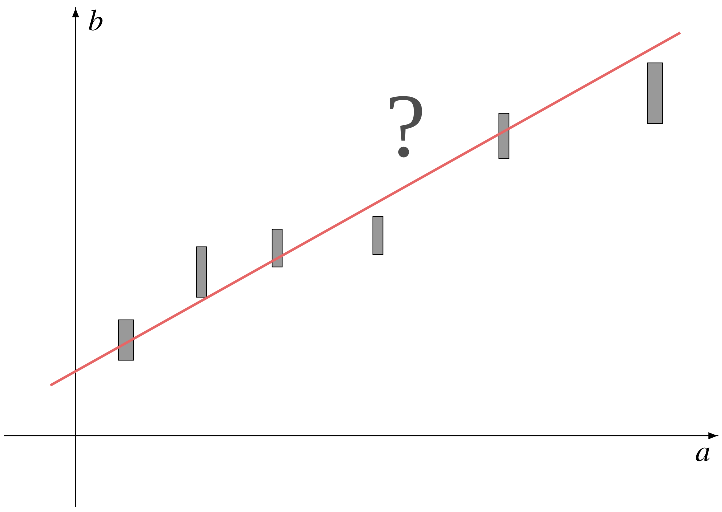

Fig. 3: For narrow uncertainty intervals, a model consistent with the data may not exist.

There are two basic ways to overcome the aforementioned paradox. The first one is based on the assumption that the intervals of the data adequately represent the boundaries of the measurement errors, so that the reduction of their width-uncertainty is positive. Hence, the impossibility of choosing model parameters consistent with these interval data (where the solution set of the interval system is empty) indicates to the inadequacy of the model used to describe the object. As the result, the model must be replaced, and the process of parameter estimation repeated using another model.

The second way assumes that the intervals of data uncertainty do not to reliably represent the set of possible values of the corresponding variables. Therefore, one does not have to obtain full consistency with the experimental data for the selected model of the object. As in the traditional case of noisy point (noninterval) data, a certain inconsistency (disagreement) is acceptable, and further one have to solve the problem of minimizing this inconsistency. Yet another situation where one has to follow this way stems from the need to retain the selected model, form of the functional dependence between the considered variables about which it is given or known a priori that "this must be the case". Following this way, one has to select a numerical "inconsistency measure" between the data and object parameters. Then, for example, a point of the parameter space where the inconsistency (disagreement) is minimal can be taken as the desired estimate.

3.3 Maximal Consistency Method

What "measure of consistency/inconsistency" between the data and model parameters can be taken? There exist natural requirements to be satisfied by such an consistency/inconsistency measure. For a nonempty information set, this measure must be positive for those points of the set, at which the "consistency" is actually attained. At that, for the points of the interior of the solution set the consistency measure, obviously, must be not smaller (or even greater) than on the boundary. Inconsistency (data contradiction) may be regarded as the "negative consistency", and on the whole the measure of consistency/inconsistency may be a functional taking values from the real axis $\mathbb{R}$.

It follows from Sec. 2.2 that the recognizing functional Uni perfectly suits the role of the consistency measure. Along this way we get an approach to parameter estimation that might be called the "method of consistency maximization" (or "maximal consistency method"): the point $\arg\max \mathrm{Uni}$ where the recognizing functional Uni of the information set has the highest value is taken as the desired parameter estimate. At that,

- if $\max \mathrm{Uni} \geqslant 0$, then this point lies in the nonempty set of the parameters consistent with the data;
- if $\max \mathrm{Uni} < 0$, then the set of parameters consistent with the data is empty, but this point minimizes the inconsistency.

Further informal interpretations of the maximal consistency method follow from Sec. 2.3. If the set of parameters consistent with the data is empty, then $\arg\max \mathrm{Uni}$ is the first point appearing in this set at uniform expansion of the right-hand side vector with respect to its midpoint. Therefore, for the empty information set this method selects as the values of the desired parameters the point where an increase in the interval uncertainty of the output variables makes this set nonempty. This principle of coordinating the contradictory data underlies the method of parameter estimation of S.I. Zhilin, *et al.* [16, 17, 18]. In the approach proposed in this paper, this principle follows mechanically as the result of the general rule for selection of the estimated parameters. We also notice that the considered procedures of processing the contradictory data are akin informally to the methods of correction of the inconsistent systems of linear inequalities developed by the Ekaterinburg scientific school of I.I. Eremin (see, for example, [23]).

3.4 Practical Implementation

Practicability of the above approach to parameter estimation depends largely on the available possibility to determine the maximum of the recognizing functional. Being a problem of optimization of the nonsmooth and multiextremal objective function, in the most general case this problem is very complex and even intractable. Appropriate global optimization methods can be used for solving it. They may be taken adaptive, that is, adjustable to the problem under solution in distinction to the passive approaches to studying the ISLAE solvability mentioned in Sec. 2.1.

The most important special case is represented by the exact values of the input variables a_{ij} and the interval uncertainty existing only at the outputs b_i. For the time being, this case is better developed by the theory of nonstatistical parameter estimation (see, in particular, [12, 13, 16, 17] and their bibliographies). We also notice that under additional statistical assumptions this special case corresponds to the prerequisites of using the traditional regression analysis, for which, in particular, the strongest results on optimality of the popular least-squares method were obtained.

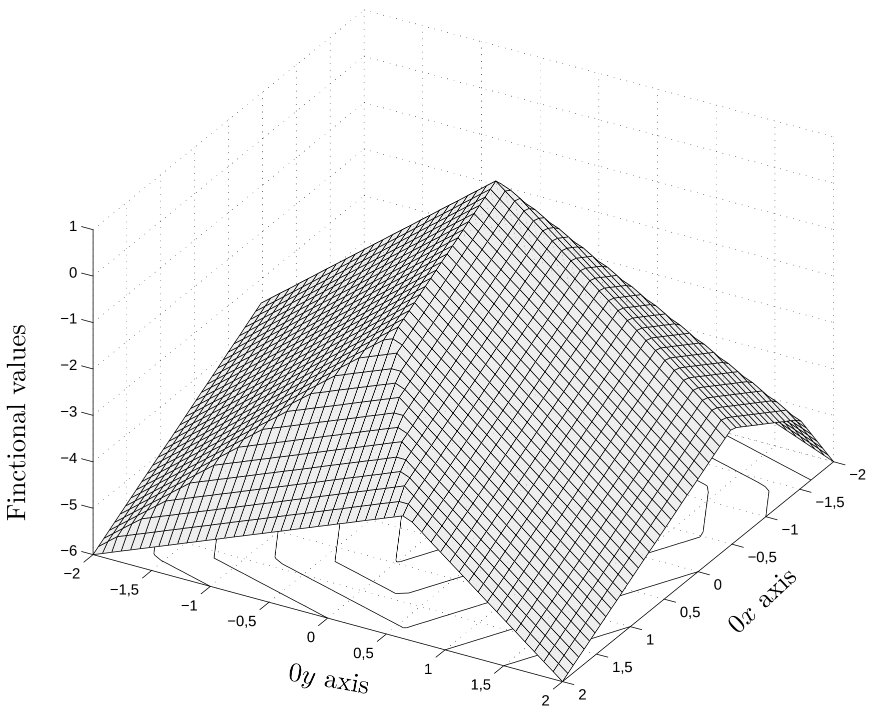

Fig. 4: Graph of the recognizing functional for the ISLAE with a point matrix.

In the case where the values of the input variables are precise, we get the interval linear system $Ax = \boldsymbol{b}$ with the point matrix $A = (a_{ij})$ where the intervality is concentrated only in the right side. As the result, the recognizing functional takes a simpler form:

$$\mathrm{Uni}\,(x, \boldsymbol{A}, \boldsymbol{b}) \;=\; \min_{1 \leqslant i \leqslant m} \left\{ \, \mathrm{rad}\, \boldsymbol{b}_i - \left| \; \mathrm{mid}\, \boldsymbol{b}_i - \sum_{j=1}^{n} a_{ij}\, x_j \; \right| \, \right\},$$

and at that it is globally concave over the entire space $\mathbb{R}^n$ in virtue of Proposition 2. Instead of the multiextremal configuration of Fig. 1, we obtain a unimodal functional looking approximately as in Fig. 4 depicting the graph of a recognizing functional of the solution set for the ISLAE

$$\begin{pmatrix} 3 & -1 \\ -1 & 2 \\ 1 & 2 \end{pmatrix} x = \begin{pmatrix} [-2, 2] \\ [0, 1] \\ [-1, 0] \end{pmatrix}.$$

Good characteristics of the functional Uni in the case of the point matrix $\boldsymbol{A}$ enable one to determine its maximum using, for example, the advanced methods of nonsmooth convex optimization (see [24, 25] and other publications). A freely distributed program implementing this version of the maximal consistency method for the computer mathematics systems Scilab and Matlab and using as basis the `ralgb5` code of P.I. Stetsyuk (Institute of Cybernetics, Ukrainian Academy of Sciences, Kiev, Ukraine) are downloadable form the server of Institute of Computational Technologies of the Siberian Branch of Russian Academy of Sciences at `http://www.nsc.ru/interval/Programing`.

Another possible way to maximize the recognizing functional in the case of the point matrix lies in solving the linear programming problem of determining the maximum of u for the pairs $(x, u) \in \mathbb{R}^{n+1}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, belonging to the convex polyhedral subgraph of the recognizing functional Uni:

$$\text{determine } \max u \text{ for} \begin{cases} u \leqslant a^{(i)}x - \underline{\boldsymbol{b}}_i, \\ u \leqslant -a^{(i)}x + \overline{\boldsymbol{b}}_i, \end{cases} \qquad i = 1, 2, \ldots, m,$$

where $a^{(i)}$ denotes the ith row of the matrix A of the considered ISLAE. On the whole, for precise input data we have an effective procedure for processing data with interval uncertainties in the output variables.

If in the general case l is the total amount of the input variables with uncertainties, then, as is shown by Proposition 2, the recognizing functional Uni has 2^l concavity domains. Therefore, in the general case of small l it is possible to determine the global maximum of the recognizing functional by enumeration of all its concavity domains and maximization of each of them with the use of the nonsmooth optimization methods of [24, 25].

4 Conclusions

By introducing the recognizing functional of the solution set, the problem of studying for solvability the interval system of the linear algebraic equations is reduced to a convenient analytical form enabling one to examine in more detail the original system and correct it. The maximal consistency method is a promising technique for processing data with interval uncertainty. It is based on maximization of the recognizing functional of the solution set (problem information set) and can serve a good alternative to the methods of regression analysis based on the probability-theoretical models of data errors.

This work was supported by the Grant of the President of Russia for Support of the Leading Scientific Schools, project no. NSh-6068.2010.9.

APPENDIX

Proof of proposition 1. The point $x \in \mathbb{R}^n$ lies, obviously, in the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ if and only if there is the matrix $\tilde{A} = (\tilde{a}_{ij}) \in \boldsymbol{A} = (\boldsymbol{a}_{ij})$ such that $\tilde{A}x \in \boldsymbol{b}$. After expansion of the matrix-vector product in the left side and representation of the intervals in the right side as "midpoint-radius", this membership is given by

$$\sum_{j=1}^{n} \tilde{a}_{ij} x_j \in \operatorname{mid} \boldsymbol{b}_i + \left[-\operatorname{rad} \boldsymbol{b}_i, \operatorname{rad} \boldsymbol{b}_i \right], \qquad i = 1, 2, \ldots, m.$$

By adding now $(-\operatorname{mid} \boldsymbol{b}_i)$ to both sides of the resulting inclusions, we get the relations

$$\sum_{j=1}^{n} \tilde{a}_{ij} x_j - \operatorname{mid} \boldsymbol{b}_i \in \left[-\operatorname{rad} \boldsymbol{b}_i, \operatorname{rad} \boldsymbol{b}_i \right], \qquad i = 1, 2, \ldots, m,$$

which in turn are equivalent to

$$\left| \sum_{j=1}^{n} \tilde{a}_{ij} x_j - \operatorname{mid} \boldsymbol{b}_i \right| \leqslant \operatorname{rad} \boldsymbol{b}_i ,$$

and therefore

$$\operatorname{rad} \boldsymbol{b}_i - \left| \operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} \tilde{a}_{ij} x_j \right| \geqslant 0, \qquad i = 1, 2, \ldots, m. \tag{11}$$

So, $x \in \Xi(\boldsymbol{A}, \boldsymbol{b})$ if and only if there are $\tilde{a}_{ij} \in \boldsymbol{a}_{ij}$, $j = 1, 2, \ldots, n$, $i = 1, 2, \ldots, m$, such that all inequalities (11) prove to be valid. This is equivalent to satisfying, for $i = 1, 2, \ldots, m$, the conditions

$$\max_{\substack{\tilde{a}_{ij} \in \boldsymbol{a}_{ij},\\ j=1,2,\ldots,n}} \left\{ \operatorname{rad} \boldsymbol{b}_i - \left| \operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} \tilde{a}_{ij} x_j \right| \right\} \geqslant 0, \tag{12}$$

and if the maximum is put within the braces, then

$$\operatorname{rad} \boldsymbol{b}_i - \min_{\substack{\tilde{a}_{ij} \in \boldsymbol{a}_{ij},\\ j=1,2,\ldots,n}} \left| \operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} \tilde{a}_{ij} x_j \right| \geqslant 0.$$

Taking into account that the natural interval extension of the expression under the sign of module coincides with its domain of values and replacing the minimum by the interval mignitude, we obtain, for $i = 1, 2, \dots, m$, from (12):

$$\operatorname{rad} \boldsymbol{b}_i \;-\; \left\langle \operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} \boldsymbol{a}_{ij} x_j \right\rangle \;\geqslant\; 0. \tag{13}$$

Finally, if by minimizing with respect to i m pieces of conditions (13) can be convoluted into one relation, then one may conclude that indeed the point x belongs to the solution set if and only if

$$\min_{1 \leqslant i \leqslant m} \left\{ \operatorname{rad} \boldsymbol{b}_i \;-\; \left\langle \operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} \boldsymbol{a}_{ij} x_j \right\rangle \right\} \;\geqslant\; 0.$$

Proof of proposition 2. It suffices to carry out it for all functions

$$\psi_i(x) \;=\; \operatorname{rad} \boldsymbol{b}_i \;-\; \left\langle \operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} \boldsymbol{a}_{ij}\, x_j \right\rangle, \qquad i = 1, 2, \dots, m,$$

because Uni is their lower envelope.

Let $\lambda \in [0, 1]$ and the points x, y belong to one orthant in $\mathbb{R}^n$ so that $x_j y_j \geqslant 0$ for all $j = 1, 2, \dots, n$. In this case, the interval multiplication is distributive in summation (see [1, 2, 3])

$$\sum_{j=1}^{n} \boldsymbol{a}_{ij} \bigl(\lambda x_j + (1-\lambda) y_j\bigr) \;=\; \lambda \sum_{j=1}^{n} \boldsymbol{a}_{ij} x_j + (1-\lambda) \sum_{j=1}^{n} \boldsymbol{a}_{ij} y_j, \tag{14}$$

thereby

$$\begin{aligned} &\psi_i\bigl(\lambda x + (1-\lambda) y\bigr) = \\ &= \operatorname{rad} \boldsymbol{b}_i - \left\langle \lambda \left(\operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} \boldsymbol{a}_{ij}\, x_j \right) + (1-\lambda) \left(\operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} \boldsymbol{a}_{ij}\, x_j \right) \right\rangle \geqslant \\ &\geqslant \operatorname{rad} \boldsymbol{b}_i - \lambda \left\langle \operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} \boldsymbol{a}_{ij}\, x_j \right\rangle + (1-\lambda) \left\langle \operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} \boldsymbol{a}_{ij}\, x_j \right\rangle = \\ &= \lambda\, \psi_i(x) + (1-\lambda)\, \psi_i(y), \end{aligned}$$

where we took advantage of the property of mignitude $\langle \boldsymbol{a} + \boldsymbol{b} \rangle \leqslant \langle \boldsymbol{a} \rangle + \langle \boldsymbol{b} \rangle$ [2, 3].

In the general case, it is impossible to do the above calculations because of the lack of distributiveness in (14). Yet if for a certain index $k \in \{1, 2, \dots, n\}$ all $\boldsymbol{a}_{ik}$ are point (noninterval) coefficients, then (14) is valid independently of the signs of x_k and y_k. Consequently, if an entire kth column in $\boldsymbol{A}$ is a point column, that is, all $\boldsymbol{a}_{ik}$, $i = 1, 2, \dots, m$, then all functions $\psi_i(x)$ (and together with them also Uni) are concave on the sets $\{\, x \in \mathbb{R}^n \mid x_1 \gtrless 0, \dots, x_{k-1} \gtrless 0, x_{k+1} \gtrless 0, \dots, x_n \gtrless 0 \,\}$ of which each is a union of two orthants in $\mathbb{R}^n$. The generalization of this reasoning to the case of more than one noninterval columns in $\boldsymbol{A}$ is evident.

Proof of proposition 3. Denoting by

$$\operatorname{hyp} \mathrm{Uni} \;=\; \bigl\{\, (x, u) \mid x \in \mathbb{R}^n,\; u \in \mathbb{R},\; \mathrm{Uni}\,(x) \geqslant u \,\bigr\}$$

the subgraph of the functional Uni, we can reformulate the main result of Proposition 2 in the following equivalent geometrical terms. For any orthant $\mathcal{O}$ in $\mathbb{R}^n$, the intersection $\operatorname{hyp} \mathrm{Uni} \cap (\mathcal{O} \times \mathbb{R})$ is convex. We demonstrate that indeed the sets $\operatorname{hyp} \mathrm{Uni} \cap (\mathcal{O} \times \mathbb{R})$ are polyhedral in $\mathbb{R}^{n+1}$, that is, they are intersections of a finite number of half-spaces in $\mathbb{R}^{n+1}$. Taking into consideration that $\langle \boldsymbol{a} \rangle = \max\{0, \underline{\boldsymbol{a}}, -\overline{\boldsymbol{a}}\}$ for any interval $\boldsymbol{a}$, we have a right to set down a chain of equalities

$$\begin{aligned}
\psi_i(x) &= \operatorname{rad} \boldsymbol{b}_i - \max \left\{ 0\,,\ \underline{\operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} \boldsymbol{a}_{ij} x_j}\ ,\ -\left(\overline{\operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} \boldsymbol{a}_{ij} x_j} \right) \right\} = \\
&= \operatorname{rad} \boldsymbol{b}_i + \min \left\{ 0\,,\ \overline{\sum_{j=1}^{n} \boldsymbol{a}_{ij} x_j} - \operatorname{mid} \boldsymbol{b}_i\,,\ \underline{\operatorname{mid} \boldsymbol{b}_i - \sum_{j=1}^{n} \boldsymbol{a}_{ij} x_j} \right\} = \\
&= \min \left\{ \operatorname{rad} \boldsymbol{b}_i\,,\ \sum_{j=1}^{n} \hat{a}_{ij} x_j - \underline{\boldsymbol{b}}_i\,,\ \overline{\boldsymbol{b}}_i - \sum_{j=1}^{n} \check{a}_{ij} x_j \right\},
\end{aligned}$$

where the corteges $(\hat{a}_{i1}, \hat{a}_{i2}, \dots, \hat{a}_{in})$ and $(\check{a}_{i1}, \check{a}_{i2}, \dots, \check{a}_{in})$ are made up of the endpoints of the components of the interval vector $(\boldsymbol{a}_{i1}, \boldsymbol{a}_{i2}, \dots, \boldsymbol{a}_{in})$, that is, the ith row of the matrix $\boldsymbol{A}$), fixed for each particular orthant $\mathcal{O}$. This fact follows from the rule of multiplication of an interval by a number. One can see from the last representation that really in $\mathcal{O}$ the functional $\operatorname{Uni} = \min_i \psi_i$ is for $i = 1, 2, \dots, m$ the lower envelope of the linear expressions under the sign of minimum. Consequently, the graph of the functional Uni is "pasted" from a finite number of pieces of the hyperplanes in $\mathbb{R}^{n+1}$, and for any orthant $\mathcal{O}$ in $\mathbb{R}^n$ the set $\operatorname{hyp}\operatorname{Uni} \cap (\mathcal{O} \times \mathbb{R})$ is polyhedral indeed.

Proof of proposition 4. We prove that the recognizing functional reaches the final maximum in each orthant $\mathcal{O} \subset \mathbb{R}^n$. The polyhedral set $\operatorname{hyp}\operatorname{Uni} \cap (\mathcal{O} \times \mathbb{R})$ in $\mathbb{R}^{n+1}$ is representable [26] as a convex combination of the extreme points (s_j, t_j), $s_j \in \mathbb{R}^n$, $t_j \in \mathbb{R}$, $j = 1, 2, \dots, p$, and extreme rays with the directing vectors (d_j, e_j), $d_i \in \mathbb{R}^n$, $e_i \in \mathbb{R}$, $j = 1, 2, \dots, q$, so that

$$\begin{aligned}
&\operatorname{hyp}\operatorname{Uni} \cap (\mathcal{O} \times \mathbb{R}) = \\
&= \left\{ \sum_{j=1}^{p} \lambda_j (s_j, t_j) + \sum_{j=1}^{q} \mu_j (d_j, e_j) \;\middle|\; \lambda_j \geqslant 0,\ \mu_j \geqslant 0, \sum_{j=1}^{p} \lambda_j = 1 \right\}.
\end{aligned} \tag{15}$$

Since $\operatorname{Uni}(x) \leqslant \min_{1 \leqslant i \leqslant m} \operatorname{rad} \boldsymbol{b}_i$, there must be $e_j \leqslant 0$ in the representation (15). Otherwise, points with arbitrarily great $(n+1)$st coordinate are in the set $\operatorname{hyp}\operatorname{Uni} \cap (\mathcal{O} \times \mathbb{R})$. Consequently,

$$\begin{aligned}
\max_{x \in \mathcal{O}} \operatorname{Uni}(x) &= \max \big\{ \ (n+1)\text{st coordinate of the points from } \operatorname{hyp}\operatorname{Uni} \cap (\mathcal{O} \times \mathbb{R}) \big\} = \\
&= \max \left\{ \sum_{j=1}^{p} \lambda_j t_j + \sum_{j=1}^{q} \mu_j e_j \;\middle|\; \lambda_j \geqslant 0,\ \mu_j \geqslant 0,\ \sum_{j=1}^{p} \lambda_j = 1 \right\} = \\
&= \max \left\{ \sum_{j=1}^{p} \lambda_j t_j \;\middle|\; \lambda_j \geqslant 0,\ \sum_{j=1}^{p} \lambda_j = 1 \right\} = \max\nolimits_{1 \leqslant j \leqslant p} t_j
\end{aligned}$$

and is reached at the extreme point of the set $\operatorname{hyp}\operatorname{Uni} \cap (\mathcal{O} \times \mathbb{R})$ having the greatest $(n+1)$st coordinate.

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This paper was recommended for publication by A.P. Kurdyukov, a member of the Editorial Board