On Π^0_1-presentations of algebras

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Abstract

Definition 1. Say that an algebra $\mathcal{A}$ is computable, Σ^0_1- or Π^0_1-algebra if its atomic diagram is computable, computably enumerable (c.e.) or co-c.e. respectevely.

There has been some research on Σ^0_1-algebras (see for example [1], [2], [3]) but not much is known about Π^0_1-algebras and their properties. The question that we are going to study here is which computable algebras admit non-computable Π^0_1-presentation.

Examples of typical computable algebras are the arithmetic $(\omega, S, +, \times)$, finitely generated term algebras and fields. We would like to know whether or not the isomorphism types of these typical computable structures contain non-computable but Π^0_1-algebras. In regard to this, it is worth to note that all these mentioned structures fail to be isomorphic to non-computable Σ_1-algebras (see [3]). To settle this question we introduce the following notion:

Definition 2. Let $\mathcal{A}$ be an algebra. We say that $\mathcal{A}$ is **term-separable** if for every finite set of terms $\{t_1(x,y), \dots, t_n(x,y)\}$ with parameters from A, every $J \subseteq \{1, \dots, n\}^2$ and every $a \in A$ the following holds:

$$\mathcal{A} \models \bigwedge_{\langle k,l\rangle \in J} t_k(a,a) \neq t_l(a,a) \longrightarrow$$
$$\exists b_1 \, \exists b_2 \, (b_1 \neq b_2) \wedge \bigwedge_{\langle k,l\rangle \in J} t_k(b_1,b_2) \neq t_l(b_1,b_2).$$

The following algebras are term-separable: the arithmetic $(\omega, S, +, \times)$, the term algebra with any generator set X, any infinite field, any torsion-free abelian group, any infinite vector space over the field of rational numbers.

The main theorem of this paper is

Theorem 1. *Any computable term-separable algebra $\mathcal{A}$ possesses a non-computable Π^0_1-presentation.*

As a corollory we have that the following structures have non-conputable Π^0_1-presentation: the arithmetic $(\omega, S, +, \times)$, term algebras with at most countable generator set, any infinite computable field $(F, +, \times, 0, 1)$, any computable torsion-free abelian group, infinite computable vector spaces over the field of rational numbers.

References

[1] L. Feiner. Hierarchies of Boolean Algebras. *Journal of Symbolic Logic*, 35, p. 365-374, 1970.

[2] N. Kasymov. Positive Algebras with Congruences of Finite Index. *Algebra and Logic*, 30, no. 3, p. 293-305, 1991.

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