Complexity Results on Minimal Unsatisfiable Formulas *

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A propositional formula F in conjunctive normal form is called minimal unsatisfiable if and only if F is unsatisfiable but any proper subformula of F is satisfiable. The class of minimal unsatisfiable formulas is denoted as *MU* and shown to be D^P–complete. D^P is the class of problems which can be described as the difference of two *NP*–problems. It is strongly conjectured that D^P is different from *NP* and from *coNP*.

Please note that any unsatisfiable formula in *CNF* contains a minimal unsatisfiable subformula. For proof calculi hard formulas are almost all minimal unsatisfiable. In the past decade, many breakthroughs has been made in order to have a deeper understanding of *MU*–formulas and to develop new hard formulas and new satisfiability algorithms.

In this talk, we shall report main results on the complexity concerning minimal unsatisfiability.

There are several approaches for defining natural subclasses of *MU*–formulas. For example, the deficiency, the difference between the number of clauses and the number of variables, can be restricted. It is known that any minimal unsatisfiable formula over n variables consists of at least $n + 1$ clauses.

Please note that the satisfiability problem for formulas with fixed deficiency is still *NP*-complete. However, it has been proved that for fixed k, $MU(k)$, the class of all minimal unsatisifable formulas with deficiency k, can be solved in polynomial time. This is based on the following fact: the deficiency of a formula in *MU* is greater than that of its any proper subformula.

There are some minimal unsatisfiable formulas such that removing or adding some literal to some clause will not destroy the minimal unsatisfiability. Please see the following example.

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Let

$$F = (\neg a \vee c) \wedge (b \vee a \vee c) \wedge (\neg b \vee a) \wedge \neg c.$$

It is easy to see that the resulting formula by removing c from the first clause or by adding c to the third clause is still minimal unsatisfiable. This motivates us to investigate subclasses of minimal unsatisfiable formulas to which (resp. from which) we can not add (resp. delete) any occurrence of a literal with minimal unsatisfiability still preserved.

A formula F in *MU* is called *maximal*, if for any clause $f \in F$ and any literal L which is not in f, adding L to f yields a satisfiable formula. In a certain sense maximal formulas are maximal extensions of *MU*–formulas. It has been shown that *MAX–MU*, the class of all so-called maximal minimal unsatisfiable formulas, is D^P–complete.

A *MU*–formula F is called *marginal* if, and only if removing an arbitrary occurrence of a literal from F leads to a unsatisfiable formula which is not in *MU*. The class of all marginal formulas is denoted as *MARG–MU*.

It is nor hard to see that the class *MARG–MU* or *MAX–MU* is in D^P. We will show the D^P–hardness by a reduction from the D^P–complete problem *MU*. We establish two procedures running in polynomial time and generating from a formula F two formulas $\sigma(F)$ and $\delta(F)$, respectively, such that $F \in MU$ if and only if $\sigma(F) \in MARG\text{–}MU$ ($\delta(F) \in MAX\text{–}MU$).

Another class of restrictions is based on a limited number of satisfying truth assignments. Beside the unsatisfiability, minimal unsatisfiability means that for any clause f the formula $F - \{f\}$ is satisfiable. If for any clause f, $F - \{f\}$ has exactly one satisfying truth assignment, that means $F - \{f\}$ is in *Unique–SAT*, then F is called unique minimal unsatisfiable. The class of these formulas is denoted as *Unique–MU*. At the first glance, to demand that for all clauses there is exactly one satisfying truth assignment seems to be very strong.

It has been proved that the problem *Unique–MU* is as hard as the *Unique–SAT*–problem and therefore probably not D^P–complete, because it is not known whether *Unique–SAT* is D^P–complete. It is strongly conjectured that *Unique–SAT* is neither D^P–complete nor in *NP* or *coNP*. A slight modification of *Unique–MU* is the class *Almost–Unique–MU* of almost unique minimal unsatisfiable formulas. A formula $F \in MU$ is in *Almost–Unique–MU* if for at most one clause f, $F - \{f\}$ may have more than one satisfying truth assignments. Under the assumption that *Unique–SAT* is not D^P–complete, *Almost–Unique–MU* is harder than *Unique–MU*, because we have shown the D^P–completeness of *Almost–Unique–MU*.

In order to characterize and to analyze minimal unsatisfiable formulas, we can split every formula in *MU* into two minimal unsatisfiable formulas. For a variable x we remove the clauses with literal $\neg x$ (set $\neg x = 1$) resp. x (set $x = 1$). In the remaining clauses we delete the occurrences of the literal x resp. $\neg x$. The formulas are unsatisfiable and contain therefore minimal unsatisfiable subformulas, say F_x and $F_{\neg x}$.

More precisely, given a minimal unsatisfiable formula F and a variable $x \in var(F)$, F can be represented as the following form.

$$F = \{(x \vee g_1), \cdots, (x \vee g_r)\} + B_x + C + B_{\neg x} + \{(\neg x \vee f_1), \cdots, (\neg x \vee f_q)\},$$

such that formulas $\{g_1, \cdots, g_r\} + B_x + C$, denoted as F_x, and $C + B_{\neg x} + \{f_1, \cdots, f_q\}$, denoted as $F_{\neg x}$, are minimal unsatisfiable. Where B_x, C, $B_{\neg x}$ are pairwise disjoint and contains no occurrence of x or $\neg x$. We call $(F_x, F_{\neg x})$ a splitting of F on x, and accordingly, F_x, $F_{\neg x}$ are called splitting formulas.

Generally speaking, splitting formulas F_x and $F_{\neg x}$ have common clauses, that is, C is non-empty. Whenever C is empty we call $(F_x, F_{\neg x})$ a disjunctive splitting of F on x.

A more detailed analysis of the class *Unique–SAT* leads to class *Dis–MU*. A minimal unsatisfiable formula F is in *Dis–MU* if and only if F has a disjunctive splitting on any variable. That means, for any variable x of F, F can be split into two disjoint subformulas in *MU*. *Dis–MU* is of interest, because *Dis–MU* is a proper subclass of *Unique–MU* and its close relation to tree–like decision procedures. We established a polynomial–time reduction from *Unique–SAT*, which shows that *Dis–MU* is at least as hard as *Unique–SAT*. We did not succeed in finding a reduction from a D^P–complete problem. But we conjecture that the problem *Dis–MU* is not D^P–complete.

From the above results we see that the restrictions of maximality, marginality, and disjunctive splitting, etc. can not reduce the complexity. One reason is probably that these features are not closed under splitting. Take maximality as example, suppose F is a maximal *MU* formula and $(F_x, F_{\neg x})$ a splitting of F on x, then the splitting formulas are not necessarily maximal.

For a class $K \subseteq MU$, the class K^* is the largest subclass of K closed under splitting.

It has been shown that *MAX–MU** coincides with *HIT-MU*. Here, we say a formula $F \in MU$ is in *HIT-MU*, if any two different clauses f and g of F hit each other, that is, there is some literal L with $L \in f$ and $\neg L \in g$. Please notice that any unsatisfiable hitting formula must be minimal unsatisfiable. By a result of Iwama, we know that the satisfiability problem for hitting formulas is solvable in polynomial time. Therefore, *HIT-MU*, and hence *MAX–MU**, is tractable.

There are some examples showing that the classes *MARG–MU*, *Dis–MU*, and *Unique–MU* are pairwise different. However, it has been shown that *MARG-MU** = *Unique-MU**=*Dis-MU**. The complexity *Dis-MU** remains open.

Clearly, a *CNF* formula is unsatisfiable if and only if it has a minimal unsatisfiable subformula. Thus, the problem of determining whether a formula has *MU* subformula is *coNP*-complete. However, we are interested in the problem of determining whether a formula has a simple *MU* subformula.

To decide whether a formula F has a *Horn-MU* subformula, we just consider the subformula F' which consists of all Horn clauses of F. If F' is unsatisfiable then F must contains a Horn subformula in *MU*. Therefore, the problem can be solved in linear time.

The most interesting problem is to determine whether a formula contains a subformula in $MU(1)$, since $MU(1)$ formulas also have nice structure. Unfortunately, the problem is *NP*-complete. The result remains true when replace $MU(1)$ by $MU(k)$ for any fixed $k \geq 1$.

Let H, F be formulas in *CNF* and $\phi : Lit(H) \to Lit(F)$ a map. We call ϕ a homomorphism from H to F if

(1) $\phi(\neg L) = \neg\phi(L)$ for every literal $L \in Lit(H)$, and

(2) $\phi(C) \in F$ for every clause $C \in H$

where $\phi(C) := \{\phi(L) \mid L \in C\}$. We simply write $\phi : H \to F$ if ϕ is a homomorphism from H to F.

The notion of homomorphism is of interest because homomorphisms preserve unsatisfiability. That is, if $\phi : H \to F$ is a homomorphism, and if H is unsatisfiable, then F is unsatisfiable, too.

A interesting problem is whether a tractable class M of unsatisfiable formulas is **homomorphically complete**, i.e., for any unsatisfiable formula F there is a formula H in M such that H is homomorphic to F. If M is homomorphically complete, then one can prove the unsatisfiability by establishing a homomorphism from a formula in M.

It has been proved that for any fixed k, $MU(k)$ is homomorphically complete.

However, to decide whether a formula H is homomorphic to a formula F is a hard problem even H and F are very simple.

Finally, we shall review some generalizations of minimal unsatisfiability.

The first is the notion of clause-minimal formulas. A formula F in *CNF* is said to be **clause-minimal** if for any clause f in F, $F - \{f\}$ is not equivalent to F, that is, F has no equivalent proper subformula. *CL-MIN* is the class of all clause-minimal formulas.

Please notice that a unsatisfiable formula is clause-minimal if and only if it is minimal unsatisfiable. Thus, the notion of clause-minimality is a generalization to minimal unsatisfiability.

CL-MIN is known to be *NP*-complete. Unlike $MU(k)$ which is tractable, *CL-MIN(k)*, the class of *CL-MIN* formulas with deficiency k, is still *NP*-complete. The main reason is that clause-minimal formulas may have smaller deficiency than their subformulas.

Generally, a unsatisfiable formula may have several minimal unsatisfiable subformulas, some of which are very simple and some of which are complex. Then there probably exists a subformula $F' \subseteq F$ such that the unsatisfiability decision is harder for F' than for F. Oliver

Kullmann introduced the notion of lean formulas.

A **lean formula** F is characterized by the condition that every clause of F can be used in some (tree) resolution refutation of F. For every clause-set F there is a largest lean sub-clause-set $Na(F) \subseteq F$. By reducing F to the satisfiability equivalent formula $Na(F)$ (instead of some minimally unsatisfiable formula) we have overcome the above problem by eliminating only absolutely superfluous clauses).

Please notice that every minimal unsatisfiable formula is lean.

The problem of deciding whether a formula is lean is *coNP*–complete.

Any *QCNF* -formula Φ has the form $\Phi = Q_1 x_1 \cdots Q_n x_n \phi$, where $Q \in \{\exists, \forall\}$ and ϕ is a *CNF* –formula. Sometimes we use an abbreviation and write $\Phi = Q\phi$.

A quantified boolean formula $\Phi \in$ *QCNF* is termed **minimal false**, if Φ is false and after removing an arbitrary clause the resulting formula is true. The set of minimal false formulas is denoted as *MF* .

Given a formula $\Phi \in$ *QCNF* , the deficiency of Φ, denoted as $d(\Phi)$, is not the difference between the number of clauses and the number of variables, but the difference between the number of clauses and the number of existential variables. *MF* (k) is the class of minimal false formulas with deficiency k.

If $\Phi \in MF$, then for any proper subformula Φ' of Φ, $d(\Phi') < d(\Phi)$.

However, we do not know whether *MF* (k) is solvable in polynomial time for fixed $k \geq 1$?