An uncountably categorical theory whose only computably presentable model is saturated

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Abstract

An important theme in computable model theory is the study of computable models of complete first-order theories. More precisely, given a complete first-order theory T, one would like to know which models of T have computable copies and which do not. A special case of interest is when T is an $\aleph_1$-categorical theory. In this paper we are interested in computable models of $\aleph_1$-categorical theories.

A complete theory T in a language L is $\aleph_1$*-categorical* if any two models of T of power $\aleph_1$ are isomorphic. We say that a model $\mathcal{A}$ of T is *computable* if its domain and its atomic diagram are computable. A model $\mathcal{A}$ is *computably presentable* if it is isomorphic to a computable model, which is called a *computable presentation* of $\mathcal{A}$.

In [1], Baldwin and Lachlan developed the theory of $\aleph_1$-categoricity in terms of strongly minimal sets. They showed that the countable models of an $\aleph_1$-categorical theory T can be listed in an $\omega + 1$ chain

$$\mathcal{A}_0 \preccurlyeq \mathcal{A}_1 \preccurlyeq \cdots \preccurlyeq \mathcal{A}_\omega,$$

where the embeddings are elementary, $\mathcal{A}_0$ is the prime model of T, and $\mathcal{A}_\omega$ is the saturated model of T. The following definition is given in [5]:

Definition. Let T be an $\aleph_1$-categorical theory and let $\mathcal{A}_0 \preccurlyeq \mathcal{A}_1 \preccurlyeq \cdots \preccurlyeq \mathcal{A}_\omega$ be the countable models of T. The *spectrum of computable models* of T is the set $\{i : \mathcal{A}_i \text{ has a computable presentation}\}$.

If $X \subseteq \omega + 1$ is the spectrum of computable models of some $\aleph_1$-categorical theory, then we say that X is *realized as a spectrum.*

There has been some previous work on the possible spectra of computable models of $\aleph_1$-categorical theories. For example, Nies [7] gave an upper bound of $\Sigma^0_3(\emptyset^\omega)$ for the complexity of the sets realized as spectra. Interestingly, the following are the only subsets of $\omega + 1$ known to be realizable as spectra: the empty set, $\omega + 1$ itself ([3], [4]), the initial segments $\{0, \ldots, n\}$, where $n \in \omega$ ([2], [6]), the sets $(\omega + 1) \setminus \{0\}$ and ω ([5]), and the intervals $\{1, \ldots, n\}$, where $n \in \omega$ ([7]). Our main result is the following

Theorem. *There exists an $\aleph_1$-categorical but not $\aleph_0$-categorical theory whose only computably presentable model is the saturated one.*

Thus $\{\omega\}$ is also realized as a spectrum.

References

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