CONTINGENCY IN S4

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0. In every modal system expression Qp ("it's contingent, that p") can be introduced as the abbreviation for such the expressions as $Pp \wedge P\neg p$, or $Pp \wedge \neg Np$, or $\neg Np \wedge \neg N\neg p$, or $\neg(Pp \Rightarrow Np)$, where Pp and Np means "it's possible, that p" and "it's necessary, that p" respectively. Conversely, if operator Q is considered as primitive, then expressions Np and Pp will be the abbreviations for $p \wedge \neg Qp$ and $p \vee Qp$ respectively. According to these definitions one can discover some "natural" properties of the contingency operator. For instance, it's obvious, that $Qp \equiv Q\neg p$.

However, the explication of the "purely mathematical" properties (i. e., of such the properties, which are used in a daily language occasionally) of this operator depends on the choice of some definite modal system. Here we consider S4, partly because of its extreme closeness to "the natural language".

1. Does Q distributes over the connectors $\wedge$, $\vee$, $\Rightarrow$, and $\equiv$? Only *one* of the eight implications, which express these distributivities, is derivable (note that in the case of N *five* of these implications are derivable):

$$S4 \vdash Q(p \vee q) \Rightarrow (Qp \vee Qq).$$

Different "deontic-like" modifications of the property of distributivity are underivable as well, but:

$$S4 \vdash \neg Q(p \equiv q) \Rightarrow (Qp \equiv Qq).$$

2. Iterated Q demonstrates the behavior, which is analogous to the behavior of the necessity operator N in $S3$. Namely,

$$S4 \vdash QQQp \equiv QQp,$$

but only

$$S4 \vdash QQp \Rightarrow Qp.$$

(Recall that $S4 \vdash NNp \equiv Np$.)

Let's define as a *modality* any expression of a kind $X_1X_2...X_np$, where p is a propositional variable, $X_i \in \{N, P, Q, \neg\}$. Then, the number of the irreducible *modalities* is finite. The following theorems of reduction are useful:

$$S4 \vdash PQp \equiv Qp;$$

$$S4 \vdash NQQp \equiv \bot$$

(i.e. $S4 \vdash \neg NQQp$. Formula $\neg QQp$ is underiveble in S4, though it is derivable in S5).

Some implications are also interesting:

$$S4 \vdash QN \Rightarrow \neg Np,$$

$$S4 \vdash QPp \Rightarrow Qp,$$

$$S4 \vdash PQp \Rightarrow Pp.$$

3. The main result here is that derivable S4 formula with Q as a main sign *doesn't exists.* Moreover, if $X_1X_2...X_nA$ is a derivable substitution into any *modality*, and there exists m $(1 < m \leq n)$ such that X_m is Q, then there exists k $(1 \leq k < m)$ such that X_k is $\neg$.