On a problem of L. D. Beklemishev

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4 апреля 2005 г.

Among subsystems of PA, $I\Delta_n$ is one of the less well understood, the main problem concerning it being the following, which was set by J. Paris and appeared as Problem 34 in the list of open problems compiled by P. Clote and J. Krajiček (see [2]).

Problem 1 *Does $I\Delta_n$ imply $B\Sigma_n$ $(n \geq 1)$?*

Motivated by this problem, several authors have recently studied the strength of $I\Delta_n$ and its parameter free counterpart, $I\Delta_n^-$, especially for the case $n = 1$ (see, e.g., [1] and [7]).

One of the first results concerning $B\Sigma_n$, proved by C. Parsons (see [6]), is that it is not implied by the set of Π_{n+1} sentences true in the standard model.

Theorem 1 *For $n \geq 1$, $\Pi_{n+1}(\mathbf{N}) \nvdash B\Sigma_n$.*

Relationships among $B\Sigma_n$ and other fragments of PA were extensively studied in [5]; some are summarized as follows.

Theorem 2 *For all $n \in \mathbf{N}$, $I\Sigma_{n+1} \Rightarrow B\Sigma_{n+1} \Rightarrow I\Sigma_n \Leftrightarrow L\Sigma_n$ (the implications are strict).*

By using (easy) modifications of some arguments in [5] and an argument due to R. Gandy (unpublished, see p. 66 of [4]), one obtains

Theorem 3 *For all $n \in \mathbf{N}$, $B\Sigma_{n+1} \Leftrightarrow L\Delta_{n+1} \Rightarrow I\Delta_{n+1}$.*

Motivated by the following picture,

$$\begin{array}{ccc} I\Sigma_1 & \Rightarrow & I\Delta_0 + exp \\ \Downarrow & & \Uparrow \\ B\Sigma_1 & \not\Leftarrow & \Pi_2(\mathbf{N}) \\ \Downarrow & & \\ I\Delta_1 & & \end{array}$$

which emerged as a synthesis of earlier results for $n = 1$, Beklemishev considered the problem whether $\Pi_2(\mathbf{N})$ implies $I\Delta_1$ or not and he solved it by proving

Theorem 4 $\Pi_2(\mathbf{N}) \not\Rightarrow I\Delta_1$.

At the end of his paper, Beklemishev noted that Theorem 4 (and other results of his paper) can be easily generalized for any $n > 1$ and stated some questions motivated by Problem 1. Two of his questions were the following.

Problem 2 *Does $I\Delta_1^-$ follow from EA? From any r.e. set of true Π_1 sentences?*

($I\Delta_1^-$ denotes parameter free $I\Delta_1$, while EA is a theory he called "Elementary Arithmetic" and is easily seen to be equivalent to $I\Delta_0 + exp$).

Problem 3 *Does $B\Sigma_1$ follow from $I\Delta_1$ together with all true Π_2 sentences?*

Slaman considered a variant of Problem 1, namely the same question but taking as base theory $I\Delta_0 + exp$ instead of $I\Delta_0$, and showed that the answer is "yes", i.e. he proved

Theorem 5 *For all $n \geq 1$, $I\Delta_n + exp \Rightarrow B\Sigma_n$.*

As a consequence, he obtained positive solutions to (i) Problem 1 for $n > 1$, since $I\Delta_n \Rightarrow I\Sigma_{n-1} \Rightarrow exp$, for such n, (ii) Problem 3, since exp is a particular case of a true Π_2 sentence.

By modifying an argument from [5], we obtained an alternative proof of Theorem 4 and we proved in [3] the following

Theorem 6 *(a) $EA \nvdash I\Delta_1^-$.*

(b) $I\Sigma_{n-1} \nvdash I\Delta_n^-$, for any $n > 1$.

Список литературы

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