THE DEGREES OF UNSOLVABILITY OF CONTINUOUS FUNCTIONS

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Computable analysis provides a standard notion of computability for continuous functions on the real numbers. This notion was first explicitly formulated and studied by Lacombe and Grzegorczyk in the 1950's, although it can be traced back to Turing and beyond that to Brouwer. However, a satisfactory notion of the degrees of unsolvability of continuous functions has only recently been introduced. While the Turing degrees measure the effective content of sets and functions over $\mathbb{N}$, we have proved that they are not sufficient to capture the complexity of continuous functions on the real numbers. For this, we need the *continuous degrees*, which are a proper extension of the Turing degrees and a proper substructure of the enumeration degrees. The study of this new degree structure was initiated by the author in [3]; this lecture is mainly an exposition of that work.

In 1936, Alan Turing [9] defined the *computable real numbers* as those with computable decimal expansions. The next year, he suggested an alternate definition, "modifying the manner in which computable numbers are associated with computable sequences, the totality of computable numbers being left unaltered" [10]. He noted that the non-uniformity of decimal representation at rational numbers with finite decimal expansions made it unsuitable for the study of computable functions on the real numbers. In particular, the function $x \mapsto 3x$ is not induced by any computable functional on decimal (or binary) expansions. To fix this problem, Turing suggested an alternative representation of the real numbers. This representation, for which he credited Brouwer, is suitable for studying computable functions on the real numbers, although he did not do so.

Our choice of representation differs from Turing's, but it is *equivalent* in the sense of Kreitz and Weihrauch [2]; in particular, they induce the same computable structure on $\mathbb{R}$. We take a *representation* of a real number $x \in \mathbb{R}$ to be any sequence of rational intervals $\{I_n\}_{n\in\mathbb{N}}$ such that $\bigcap_{n\in\mathbb{N}} I_n = \{x\}$. By coding rational intervals with natural numbers, we can view a representation as an element of $\mathbb{N}^{\mathbb{N}}$, hence it has a Turing degree. It is well known that every real $x \in \mathbb{R}$ has a representation of least Turing degree. Furthermore, this degree is exactly the Turing degree of the binary (or decimal) expansion of x. Hence, the Turing degrees are quite sufficient to measure the effective content of real numbers.

The same analysis might be attempted for continuous functions, or more generally, for the members of any computably presented metric space. The notion of representation can be generalized to an arbitrary computable metric space by replacing the rational intervals with a suitable family of metric balls. As before, representations can be viewed as elements of $\mathbb{N}^{\mathbb{N}}$ and thus have Turing degree. These degrees are meaningful; a continuous function $f \in \mathcal{C}[0,1]$ has a representation of Turing degree $\mathbf{a}$ iff there is an $\mathbf{a}$-computable functional that, for all $x \in [0,1]$, maps every representation of x to a representation of $f(x)$. Every continuous function has infinitely many representations. But one might ask: is there a least complicated representation—one which captures exactly the *intrinsic* difficulty of computing the function? Formally, does a continuous function $f\colon [0,1] \to \mathbb{R}$ necessarily have a representation of least Turing degree among all representations of f? This question was raised by Pour-El and Lempp (10.2 in Slaman's list of Questions in Recursion Theory [8]).

One motivation for this question was the work of L. J. Richter [5] in computable model theory, where an analogous situation arises. A countable group $\mathcal{G}$ can be *presented* as a subset of $\mathbb{N}$ with a binary relation representing multiplication. Other countable structures, such as linear orders and graphs, can be presented similarly. Just as a function $f \in \mathcal{C}[0,1]$ has infinitely many representations, $\mathcal{G}$ will have infinitely many presentations. Say that $\mathcal{G}$ computes $A \subseteq \mathbb{N}$ if every one of its presentations computes A, and that $\mathcal{G}$ has Turing degree $\mathbf{a}$ if this is the least degree of any presentation. Richter proved that there are groups of every Turing degree, but also that there are groups that have no Turing degree. There are even non-computable groups that compute no non-computable subsets of $\mathbb{N}$. The situation for linear orders is more restrictive; no linear order can compute a non-computable subset of $\mathbb{N}$, so in fact, no non-computable linear order has a Turing degree.

We not only answer Pour-El and Lempp's question in the negative, but in doing so, we introduce a natural degree structure that captures the complexity of the continuous functions. Our methods are very different from those used by Richter. The outcome is also different; we not only show that there are continuous functions with no Turing degree, but also that every non-computable $f \in \mathcal{C}[0,1]$ computes a non-computable subset of $\mathbb{N}$, distinguishing the effective content of continuous functions from that of groups and linear orders and from the various other classes of discrete structures that have been studied.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be computable metric spaces. We say that $m_1 \in \mathcal{M}_1$ is *representation reducible* to $m_2 \in \mathcal{M}_2$ (or less formally, that m_2 *computes* m_1) if every representation of m_2 computes a representation of m_1. The equivalence classes induced by this relation are called the *continuous degrees*. Representation reducibility agrees with Turing reducibility on (the computable metric spaces) $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$, so the continuous degrees extend the Turing degrees. On the other hand, the continuous degrees embed into the enumeration degrees, a degree structure from classical computability theory that captures the difficulty of enumerating sets of natural numbers [6]. A function $f \in \mathcal{C}[0,1]$ has total degree iff it has a least Turing degree representation. It is not hard to show that every continuous degree contains an element of $\mathcal{C}[0,1]$ (hence the name). Call a continuous degree that corresponds to a Turing degree *total*. Therefore, proving the existence of a non-total continuous degree gives a negative answer to Pour-El and Lempp's question.

Sequences of reals play an important role in understanding the continuous degrees. As with continuous functions, it can be shown that every continuous degree contains an element of $[0,1]^{\mathbb{N}}$. We say that a sequence $\alpha \in [0,1]^{\mathbb{N}}$ is *not computably diagonalizable* if it lists every real $x \in [0,1]$ that it computes. The non-total continuous degrees are exactly the degrees of sequences that are not computably diagonalizable. Therefore, the existence of such sequences implies that the continuous degrees properly extend the Turing degrees. This existence is proved by constructing a mutivalued operator $\Psi\colon [0,1]^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ with the property that no fixed point of Ψ is computably diagonalizable. A classical topological fixed point theorem of Eilenberg and Montgomery [1] is used to prove that Ψ has a fixed point. This answers Pour-El and Lempp's question and proves that the Turing degrees fail to adequately measure the effective content of continuous functions.

Although the motivating question is now answered, many other questions present themselves. For example, what distinguishes the metric spaces whose members all have total degree, such as $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{R}$, from the metric spaces that have members in every continuous degree, such as $\mathcal{C}[0,1]$ and $[0,1]^{\mathbb{N}}$? Vasco Brattka conjectured a connection between the dimension of a computable metric space and the degrees of its members. In unpublished work with the author, he effectivized a result from dimension theory to show that no finite dimensional computable metric space has elements in all continuous degrees. In fact, there are continuous degrees that contain only elements from infinite dimensional computable metric spaces.

Another basic problem is to distinguish the continuous degrees from the enumeration degrees (and also contrast our results from those of Richter). Moving from classical topology to Russian constructive analysis, a modification of Orevkov's [4] constructive retraction of (the constructive points of) the unit square onto its boundary is used to show that every sequence of computable reals is computably diagonalizable. This implies that every non-computable continuous function computes a non-computable subset of $\mathbb{N}$; hence the continuous degrees properly embed into the enumeration degrees (where this property fails).

Having shown that the continuous degrees are a new degree structure, it is natural to study their relationship to the substructure of the Turing degrees. This is closely connected to two concepts from the classical study of complete extensions of Peano arithmetic: PA degrees and Scott systems. If $\mathbf{a}$ and $\mathbf{b}$ are Turing degrees, we say that $\mathbf{a}$ is a *PA degree relative to* $\mathbf{b}$ ($\mathbf{b} \ll \mathbf{a}$) if every infinite $\mathbf{b}$-computable subtree of $2^{<\mathbb{N}}$ has an infinite path computable from $\mathbf{a}$. A *Scott ideal* is a countable ideal in the Turing degrees such that for every $\mathbf{b} \in \mathcal{I}$ there is an $\mathbf{a} \in \mathcal{I}$ with $\mathbf{b} \ll \mathbf{a}$. If $\mathcal{I}$ is a Scott ideal, then the collection of subsets of $\mathbb{N}$ with degree in $\mathcal{I}$ is called a *Scott system*. Alternately, $\mathcal{S} \subseteq 2^{\mathbb{N}}$ to be a Scott system iff it is the field of (standard initial segments of) sets arithmetically definable in some complete extension of Peano arithmetic [7].

These notions allow us to pinpoint exactly where non-total continuous degrees appear relative to the Turing degrees. There is a non-total degree between total degrees $\mathbf{a} < \mathbf{b}$ iff $\mathbf{b}$ is a PA degree relative to $\mathbf{a}$. Furthermore, the collection $\mathcal{I}_{\mathbf{v}}$ of Turing degrees below a non-total continuous degree $\mathbf{v}$ is a Scott ideal and every Scott ideal is represented in this way. In fact, if $\mathcal{I}$ is a Scott ideal, then there are $2^{\aleph_0}$ pairwise incomparable continuous degrees $\mathbf{v}$ such that $\mathcal{I}_{\mathbf{v}} = \mathcal{I}$. Translating back to continuous functions, there are computably incomparable functions $f, g \in \mathcal{C}[0,1]$—in other words, some representation of f computes no representation of g and vice versa—such that f and g compute exactly the same subsets of $\mathbb{N}$.

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