INTERPOLATION PROPERTY AND PRINCIPLE OF VARIABLE SEPARATION IN SUBSTRUCTURAL LOGICS

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We will discuss Craig's interpolation property (CIP), deductive interpolation property (DIP), pseudo-relevance property (PRP), principle of variable separation (PVS) and Halldén completeness (HC) of substructural logics, and give algebraic characterizations of these properties. These characterizations have been studied for modal and superintuitionistic logics, e.g. in Maksimova (1977), [2] etc., Wroński [3] and so on. Our main aim is to show relations among these properties in substructural logics and to clarify how structural rules affect algebraic characterizations of them. This study comes out of my joint work with N. Galatos [1] and H. Kihara (in preparation).

1 Interpolation Property and Pseudo-Relevance Property

In the following, by a *substructural logic* we mean a substructural logic over **FL**, i.e. a set of formulas containing all formulas provable in **FL**, which is closed under modus ponens, adjunction, conjugates and substitution. We assume that our language consists of logical connectives $\wedge, \vee, \cdot, \backslash, /$ and constants $\top, \bot, 1, 0$. When it is commutative, i.e. it is closed under exchange rule, formulas $\varphi\backslash\psi$ and ψ/φ become equivalent and are expressed usually as $\varphi \to \psi$. The constant 0 is used for defining negation(s). The least commutative substructural logic is denoted by $\mathbf{FL_e}$. Also, the least substructural logic which is closed under weakening rule (closed under both exchange and weakening rules) is denoted by $\mathbf{FL_w}$ ($\mathbf{FL_{ew}}$, respectively). For the precise definition, see [1].

Algebraic structures for substructural logics are called **FL**-*algebras*, which are defined as follows. Let us first define a *residuated lattice* to be an algebra of the form $\mathbf{A} = \langle A; \vee, \wedge, \cdot, \backslash, /, 1\rangle$ such that

1. $\langle A; \vee, \wedge\rangle$ is a lattice,
2. $\langle A; \cdot, 1\rangle$ is a monoid with the unit 1,
3. $xy \leq z \iff y \leq x\backslash z \iff x \leq z/y$ for all $x, y, z \in A$.

An **FL**-*algebra* is a RL $\mathbf{A}$ with an (*arbitrary*) element $0 \in A$. (In the present paper, we assume always that each RL is a bounded lattice with the greatest element $\top$ and the least $\bot$.) An $\mathbf{FL_e}$-algebra ($\mathbf{FL_w}$-algebra) is an **FL**-algebra with a commutative monoid operation (and an **FL**-algebra such that $1 = \top$ and $0 = \bot$, respectively.) An $\mathbf{FL_{ew}}$-algebra is an $\mathbf{FL_e}$-algebra which is at the same time an $\mathbf{FL_w}$-algebra.

It is shown that there is a one-to-one correspondence between the class of substructural logics and the class of subvarieties of the variety $\mathcal{FL}$, which consists of all **FL**-*algebras.* More precisely, for each substructural logic $\mathbf{L}$, define a class $\mathcal{V}(\mathbf{L})$ of $\mathbf{FL}$-algebras as follows: An algebra $\mathbf{A}$ is in $\mathcal{V}(\mathbf{L})$ iff $\mathbf{A} \models \varphi$ for every $\varphi \in \mathbf{L}$. Here, $\mathbf{A} \models \varphi$ means that $w(\varphi) \geq 1$ in $\mathbf{A}$ for any valuation w on $\mathbf{A}$. Then, $\mathcal{V}(\mathbf{L})$ forms a variety. Conversely, for each class $\mathcal{V}$ of $\mathcal{FL}$, define a set $\mathbf{L}(\mathcal{V})$ of formulas as follows: $\varphi \in \mathbf{L}(\mathcal{V})$ iff $\mathbf{A} \models \varphi$ for every $\mathbf{A} \in V$. Then $\mathbf{L}(\mathcal{V})$ is shown to be a substructural logic. In particular, when $\mathcal{V}$ consists of a single algebra $\mathbf{A}$, we write the logic $\mathbf{L}(\mathcal{V})$ simply as $\mathbf{L}(\mathbf{A})$. These correspondences $\mathcal{V}(*)$ and $\mathbf{L}(*)$ form mutually dual lattice isomorphisms between the lattice of all substructural logics and the lattice of all subvarieties of $\mathcal{FL}$. For the details, see [1].

For each substructural logic $\mathbf{L}$, we can introduce the *deducibility relation* $\vdash_{\mathbf{L}}$ of $\mathbf{L}$ as follows: $\Gamma \vdash_{\mathbf{L}} \varphi$ iff $w(\alpha) \geq 1$ for all $\alpha \in \Gamma$ implies $w(\varphi) \geq 1$, for each $\mathbf{A} \in \mathcal{V}(\mathbf{L})$ and for each valuation w on $\mathbf{A}$. We can show the *algebraization* and the *(parametrized) local deduction theorem* for the deducibility relation $\vdash_{\mathbf{L}}$ for every substructural logic $\mathbf{L}$ ([1]).

A substructural logic $\mathbf{L}$ has the *Craig interpolation property* (CIP), if for all formulas φ, ψ, if $\varphi\backslash\psi \in \mathbf{L}$, there exists a formula σ such that (1) both $\varphi\backslash\sigma$ and $\sigma\backslash\psi$ belong to $\mathbf{L}$, and (2) $Var(\sigma) \subseteq Var(\varphi) \cap Var(\psi)$, where $Var(\phi)$ denotes the set of propositional variables in a formula ϕ. Also, $\mathbf{L}$ has the *strong deductive interpolation property* (SDIP), if for all sets of formulas $\Gamma \cup \Sigma \cup \{\psi\}$, if $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$, then there exists a formula δ such that (1) $\Gamma \vdash_{\mathbf{L}} \delta$ and $\delta, \Sigma \vdash_{\mathbf{L}} \psi$, and (2) $Var(\delta) \subseteq Var(\Gamma) \cap Var(\Sigma \cup \{\psi\})$. When Γ is restricted to a singleton set of a formula and Σ is to the empty set in the above, we say that $\mathbf{L}$ has the *deductive interpolation property* (DIP).

A subvariety $\mathcal{V}$ of $\mathcal{FL}$ has the *congruence extension property* (CEP), if for every $\mathbf{A} \in \mathcal{V}$, every subalgebra $\mathbf{B}$ of $\mathbf{A}$ and for every congruence θ on $\mathbf{B}$, there exists a congruence θ' on $\mathbf{A}$ such that $\theta' \cap B^2 = \theta$. It is well-known that $\mathcal{V}(\mathbf{L})$ has the CEP if and only if a *local deduction theorem* holds for $\mathbf{L}$. Thus, $\mathcal{V}(\mathbf{L})$ has the CEP for every commutative substructural logic $\mathbf{L}$. Also, a variety $\mathcal{V}$ has the *amalgamation property* (AP), if for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathcal{V}$ and for all embeddings $f : \mathbf{A} \to \mathbf{B}$, $g : \mathbf{A} \to \mathbf{C}$, there exists an algebra $\mathbf{D}$ in $\mathcal{V}$ and embeddings $f' : \mathbf{B} \to \mathbf{D}$, $g' : \mathbf{C} \to \mathbf{D}$ such that $f' \circ f = g' \circ g$. Then the following can be shown (see [1]).

Theorem 1 *1. CIP implies SDIP for commutative substructural logics.*
2. $\mathbf{L}$ *has the SDIP iff* $\mathcal{V}(\mathbf{L})$ *has both AP and CEP.*
3. SDIP implies DIP for every $\mathbf{L}$*, while the converse holds whenever* $\mathcal{V}(\mathbf{L})$ *has the CEP.*

A substructural logic $\mathbf{L}$ has the *strong pseudo-relevance property* (SPRP), if for all sets of formulas $\Gamma \cup \Sigma \cup \{\psi\}$ with $Var(\Gamma) \cap Var(\Sigma \cup \{\psi\}) = \emptyset$, $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$ implies either $\Gamma \vdash_{\mathbf{L}} \bot$ or $\Sigma \vdash_{\mathbf{L}} \psi$. When Γ is restricted to a singleton set and Σ is to the empty set in this definition, $\mathbf{L}$ is said to have the *pseudo-relevance property* (PRP). Algebras $\mathbf{B}, \mathbf{C}$ are *jointly embeddable* into an algebra $\mathbf{D}$, if there exists embeddings $h : \mathbf{B} \to \mathbf{D}$ and $j : \mathbf{C} \to \mathbf{D}$. Then, we have the following (cf. Maksimova [2]).

Theorem 2 *1.* $\mathbf{L}$ *has the SPRP iff every pair of subdirectly irreducible (s.i.) algebras in* $\mathcal{V}(\mathbf{L})$ *are jointly embeddable into an algebra in* $\mathcal{V}(\mathbf{L})$*.*
2. SPRP implies PRP for every $\mathbf{L}$*, while the converse holds when* $\mathcal{V}(\mathbf{L})$ *has the CEP.*
3. If a subvariety $\mathcal{V}$ *of the variety of all* $\mathbf{FL_w}$*-algebras has the AP, then all pairs of s.i. algebras in* $\mathcal{V}$ *are jointly embeddable into an algebra in* $\mathcal{V}$*. Thus, the DIP implies the PRP for every substructural logic over* $\mathbf{FL_w}$*.*

The implication in the above 3 doesn't hold always, if we drop either of conditions $1 = \top$ and $0 = \bot$. For instance, $\mathbf{FL_e}$ has the CIP and hence the DIP, but it doesn't have the PRP. We can show the following in the same way as a result by Komori (1978), by using an extension of Glivenko's theorem obtained by Galatos-Ono (2005).

Theorem 3 *Every extension of the logic* $\mathbf{FL_{ew}}$ *with the axiom* $\neg(\alpha \wedge \neg\alpha)$ *has the SPRP.*

2 Halldén Completeness and Principle of Variable Separation

In this section we consider only commutative substructural logics and $\mathbf{FL_e}$-algebras. We note first that for any $\mathbf{FL_e}$-algebra $\mathbf{A}$, $\mathbf{A}$ is s.i. if and only if there exists an element $a \in A$ such that $a \not\geq 1$ and for any $x \not\geq 1$ there exists a natural number m such that $(x \wedge 1)^m \leq a$. Such an element a in a s.i. algebra $\mathbf{A}$ is called an *opremum* of $\mathbf{A}$.

A substructural logic $\mathbf{L}$ is *Halldén complete* (HC), if for all formulas φ and ψ which have no variables in common, $\vdash_{\mathbf{L}} \varphi \vee \psi$ implies $\vdash_{\mathbf{L}} \varphi$ or $\vdash_{\mathbf{L}} \psi$. Obviously the disjunction property implies the HC. As the following theorem shows, we can extend results by both Lemmon (1966) and Wroński [3], for substructural logics over $\mathbf{FL_{ew}}$.

Theorem 4 *The following conditions are equivalent for every substructural logic* $\mathbf{L}$ *over* $\mathbf{FL_{ew}}$*.*
1. $\mathbf{L}$ *is Halldén complete,*
2. $\mathbf{L}$ *is meet irreducible in the lattice of all substructural logics over* $\mathbf{FL_{ew}}$*, i.e,* $\mathbf{L}$ *cannot be represented as the intersection of two incomparable logics,*
3. $\mathbf{L} = \mathbf{L}(\mathbf{A})$ *for some well-connected* $\mathbf{FL_{ew}}$*-algebra* $\mathbf{A}$*, i.e, an* $\mathbf{FL_{ew}}$*-algebra such that* $x \vee y = 1$ *implies* $x = 1$ *or* $y = 1$*.*

On the other hand, if we try to show the similar result for substructural logics over $\mathbf{FL_e}$, a certain modification becomes necessary since the unit 1 is not always equal to the greatest $\top$. If we keep the condition of the meet irreducibility of a logic $\mathbf{L}$ and compare it with other conditions, then we have the following.

Theorem 5 *The following conditions are equivalent for every substructural logic* $\mathbf{L}$ *over* $\mathbf{FL_e}$.
1. $\mathbf{L}$ *is weakly Halldén complete, i.e. for all formulas* φ *and* ψ *which have no variables in common,* $\vdash_{\mathbf{L}} (\varphi \wedge 1) \vee (\psi \wedge 1)$ *implies* $\vdash_{\mathbf{L}} \varphi$ *or* $\vdash_{\mathbf{L}} \psi$*,*
2. $\mathbf{L}$ *is meet irreducible in the lattice of all substructural logics over* $\mathbf{FL_e}$*,*
3. $\mathbf{L} = \mathbf{L}(\mathbf{A})$ *for some weakly well-connected* $\mathbf{FL_e}$*-algebra* $\mathbf{A}$*, i.e, an* $\mathbf{FL_e}$*-algebra such that* $x \vee y = 1$ *implies* $x = 1$ *or* $y = 1$ *for all* $x, y \leq 1$.

In [3], it is shown that for superintuitionistic logics the third condition (and hence also other conditions) in Theorem 4 is equivalent to the following.

$\mathbf{L} = \mathbf{L}(\mathbf{A})$ for some s.i. Heyting-algebra $\mathbf{A}$.

It seems that the similar equivalence doesn't hold in general even for substructural logics over $\mathbf{FL_{ew}}$. On the other hand, if we assume that the axiom of n-potency, i.e, $\alpha^n \to \alpha^{n+1}$, holds for some n in $\mathbf{L}$ over $\mathbf{FL_e}$, the following condition is also equivalent to any of conditions in Theorem 5.

$\mathbf{L} = \mathbf{L}(\mathbf{A})$ for some s.i. $\mathbf{FL_e}$-algebra $\mathbf{A}$.

Let us consider next *principle of variable separation*(PVS). A substructural logic $\mathbf{L}$ has the PVS if for all sets of formulas $\Gamma \cup \{\varphi\}$ and $\Sigma \cup \{\psi\}$ such that $\Gamma \cup \{\varphi\}$ and $\Sigma \cup \{\psi\}$ have no variables in common, $\Gamma, \Sigma \vdash_{\mathbf{L}} \varphi \vee \psi$ implies $\Gamma \vdash_{\mathbf{L}} \varphi$ or $\Sigma \vdash_{\mathbf{L}} \psi$. It is easy to see that we can always assume that both Γ and Σ are singleton sets of formulas, and also that the condition $\gamma, \sigma \vdash_{\mathbf{L}} \varphi \vee \psi$ is equivalent to the condition $\gamma \wedge \sigma \vdash_{\mathbf{L}} \varphi \vee \psi$. The PVS for basic substructural logics is studied syntactically by Naruse-Bayu Surarso-Ono (1998).

Obviously, both Halldén completeness and the SPRP of a given logic follow from the PVS by either taking the empty set for both Γ and Σ, or taking $\perp$ for φ. In the same way as Theorem 4.3 in Maksimova [2] we can show the following.

Theorem 6 *The following conditions are equivalent for every substructural logic* $\mathbf{L}$ *over* $\mathbf{FL_{ew}}$.
1. $\mathbf{L}$ *has the PVS,*
2. all pairs of s.i. algebras in $V(\mathbf{L})$ *are jointly embeddable into a well-connected (or even a s.i.) algebra in* $V(\mathbf{L})$.

By comparing this with Theorem 2, we can see how the PVS relates to the HC semantically. On the other hand, it is not so obvious how the second condition of Theorem 6 implies the third condition of Theorem 4, while the PVS implies trivially the HC. To show this, we need an argument using ultraproduct construction. The proof, though we omit here, show why we have some difficulties in replacing the well-connectedness by the subdirect irreducibility in the third condition of Theorem 4, while we can do so in the second condition of Theorem 6. The reason is that the well-connectedness of $\mathbf{FL_{ew}}$-algebras is a first-order property, while the subdirect irreducibility is not. Therefore, the latter is not preserved under ultraproduct. On the other hand, if we assume the axiom of n-potency for some n, the latter becomes also a first-order property.

Also to substructural logics over $\mathbf{FL_e}$, a similar result to Theorem 6 holds by modifying definitions of the PVS just as we have done in the case of the HC, and taking weak well-connected algebras instead of well-connected algebras. Also, we can give also an algebraic characterization of the PVS (in the original form) for these logics, as shown below.

Theorem 7 *The following conditions are equivalent for every substructural logic* $\mathbf{L}$ *over* $\mathbf{FL_e}$.
1. $\mathbf{L}$ *has the PVS,*
2. for all s.i. algebras $\mathbf{B}, \mathbf{C}$ *in* $V(\mathbf{L})$ *and for all oprema* $x \in \mathbf{B}, y \in \mathbf{C}$*, there exist a s.i. algebra* $\mathbf{D}$ *in* $V(\mathbf{L})$*, and embeddings* $h : \mathbf{B} \to \mathbf{D}$ *and* $j : \mathbf{C} \to \mathbf{D}$ *such that* $h(x) \vee j(y) \not\geq 1$.

It is interesting to compare the above theorem with Theorem 2.1 of Maksimova [2]. In the theorem, our second condition is given for characterizing the HC of normal modal logics. This happened since the HC is equivalent to the PVS for them. On the other hand, as shown in Chagrov-Zakharyaschev (1993) there exist uncountably many superintuitionistic logics with the HC but without the PVS.

References

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