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MODEL-THEORETIC PROPERTIES OF MULTI-VALUED FIELDS

I. Valued Fields ([1, 2]).

Let F be a field of characteristic 0 and $R \leq F$ be a valuation ring of F, i.e. a subring of F such that for any $a \in F^{\times} = F \setminus \{0\}$ either $a \in R$ or $a^{-1} \in R$. A pair $\langle F, R\rangle$ is called a *valued field* ([1]).

For a valuation ring R we denote by $\mathfrak{m}(R)$ the maximal ideal of R; $F_R \rightleftharpoons R/\mathfrak{m}(R)$ is the *residue field*; $U(R) \rightleftharpoons R \setminus \mathfrak{m}(R)$ is the *unit group* of the ring R, $\Gamma_R \rightleftharpoons F^{\times}/U(R)$ is a linearly ordered group (with the order defined via the cone $R^{\times}/U(R) \leq \Gamma_R$), which is called the *valuation group* (with additive notation); a natural homomorphism $v_R : F^{\times} \to \Gamma_R$ is the *valuation*, induced by the ring R.

A valuation ring R (valued field $\langle F, R\rangle$) is called *a Henselian valuation ring (valued field)* if for any unitary polynomial $f \in R[x]$ the following holds: if the image $\bar{f} \in F_R[x]$ has a prime root α then there exists a root $a \in R$ of the polynomial f such that $\alpha = a + \mathfrak{m}/R$.

For any valued field $\mathbb{F} = \langle F, R\rangle$ there exists the least extension $\mathbb{H}_R(F) = \langle H_R(F), H(R)\rangle \geq \mathbb{F}$ which is a Henselian valued field — the *Henselization* of $\mathbb{F}$.

The absolute ramification index $a_*(R)$ of a ring R is equal to 1 in case when F_R is of characteristic 0, and equal to $n \in \omega$ in case when F_R is of characteristic $p > 0$ and the set $[0, v_R(p)) \rightleftharpoons \{\gamma | \gamma \in \Gamma_R,\ 0 \leq \gamma < v_R(p)\}$ has cardinality n ($n = |[0, v_R(p))|$); $a_*(R) = \omega$ in case when F_R is of characteristic $p > 0$ and the set $[0, v_R(p)]$ is infinite.

THEOREM 1. *Let $\langle F, R\rangle \leq \langle F', R'\rangle$ be an extension of Henselian valued fields. If $a_*(R) < \omega$ then this extension is elementary if and only if the induced extensions $F_R \leq F_{R'}$ and $\Gamma_R \leq \Gamma_{R'}$ of the residue fields and valuation groups, respectively, are elementary.*

THEOREM 2. *Suppose that $\langle F_0, R_0\rangle, \langle F_1, R_1\rangle$ are Henselian valued fields; $a_*(R_0) < \omega$; $F \leq F_0, F_1$ is a common subfield; $R \leq F, R_0, R_1$ is a common subring such that*

$$F = q(R),\ \ R_0 \cap F = R_{\mathfrak{m}(R_0)\cap R},\ R_1 \cap F = R_{\mathfrak{m}(R_1)\cap R};$$

F_{R_0} is a separable extension of $F_{R_0 \cap F}$; $\Gamma_{R_0 \cap F}$ is a pure subgroup of Γ_{R_0}. Then $\langle F_0, R_0\rangle$ is elementary equivalent to $\langle F_1, R_1\rangle$ over R, $\langle F_0, R_0\rangle \equiv_R \langle F_1, R_1\rangle$, if and only if $F_{R_0} \equiv_R F_{R_1}$ and $\Gamma_{R_0} \equiv_{R^{\times}} \Gamma_{R_1}$.

II. Multi-Valued Fields (Boolean Families)([1, 2]).

A multi-valued field is a pair $\langle F, H\rangle$, where H is a Prüfer subring of the field F (of characteristic 0) such that $F = q(H)$. Recall that a ring H is called *Prüfer* if for any maximal ideal $\mathfrak{m}$ of H ($\mathfrak{m} \in mSpec\ H$) the residue ring $H_{\mathfrak{m}}$ is a valuation ring. We denote by $W(H)$ the family $\{H_{\mathfrak{m}} | \mathfrak{m} \in mSpec\ R\}$ of all valuation rings of the field F. The *Zariski topology* (Z-topology) on $W(H)$ is defined via the base $U_a \rightleftharpoons \{H_{\mathfrak{m}} | a \notin \mathfrak{m}\},\ a \in H^{\times} = H \setminus \{0\}$. Family $W(H)$ is called *Boolean* (*near Boolean)* if all of the sets $U_a,\ a \in H^{\times}$, are closed (compact) with respect to Zariski topology. The *C-topology* on $W(H)$ is defined via the subbase $W_n \rightleftharpoons W(H) \setminus U_a,\ a \in H^{\times}$. If $W(H)$ is Boolean then Z-topology and C-topology are equivalent.

A family $W(H)$ is said to have *C-continuous elementary properties* if, for any formula $\varphi(\underline{x})$ of the signature of valued fields and for any tuple $\bar{a}$, the set $\{R | R \in W(H),\ \ \mathbb{H}_R(F) \models \varphi(\bar{a})\}$ is C-open.

A ring H (family $W(H)$, multi-valued field $\langle F, H \rangle$) is said to satisfy the *local-global arithmetical* principle LG_A if the following holds:

any affine (absolutely irreducible) variety V, defined over F, has a simple H-rational point in case then V has a simple $H(R)$-rational point for any $R \in W(H)$.

A ring H (family $W(H)$, multi-valued field $\langle F, H \rangle$) is said to satisfy the principle of *maximality M* if the following holds:

if $f \in F[x]$ is an irreducible polynomial over F and f has a root in $H_R(F)$ for any $R \in W(H)$, then f is linear.

It is natural to consider a multi-valued field $\langle F, H \rangle$ with Boolean family $W(H)$ in the signature expanded by the predicate J distinguishing the Jacobson radical of the ring H.

Let $\langle F, H, J(H) \rangle \leq \langle F', H', J(H') \rangle$ be an extension of multi-valued fields; $W(H),\ W(H')$ are Boolean families; $a_*(R) < \omega$ $(a_*(R') < \omega)$ for any $R \in W(H)$ $(R' \in W(H'))$; $H(H')$ satisfies principles LG_A and M, and the local elementary properties of the family $W(H)$ $(W(H'))$ are C-continuous. Under these assumptions we have

THEOREM 3. *An embedding $\langle F, H, J(H) \rangle\rangle \leq \langle F', H', J(H') \rangle$ is elementary if and only if the induced embeddings of rings $H/J(H) \leq H'/J(H')$ and groups $F^*/U(H) \leq F'^{\times}/U(H')$ are elementary.*

Note that the ring $H/(J(H)$ is a subring of the direct product

$$\prod_{R \in W(H)} F_R$$

of the residue fields, and the group $F^{\times}/U(H)$ is a subgroup of the direct product

$$\prod_{R \in W(H)} \Gamma_R$$

of the valuation groups; from C-continuity of the local elementary properties it follows that the embeddings above are *elementary products* (see [1]).

III. Multi-valued fields

(near Boolean families)([1, 2]).

In this section we consider multi-valued fields $\langle F, H\rangle$ for which $W(H)$ is a near Boolean family. A natural expansion of signature in this case is obtained by adding a symbol for the preorder $\sqsubseteq_H$, defined on F in the following way: for $a, b \in F$,

$$a \sqsubseteq_H b \rightleftharpoons \forall \mathfrak{m} \in mSpec\ H(a \in \mathfrak{m} \Rightarrow b \in m).$$

Under assumption that H is a Bézout ring (i.e. any finitely generated ideal of H is principal), the partial order induced on $E_H(F) \rightleftharpoons F/\equiv_H$ by the preorder $\sqsubseteq_H$ defines on $E_H(F)$ a structure of distributive lattice with the least element ($[1]_H$) and relative complements.

If the local elementary properties of $W(H)$ are C-continuous then, for any formula $\varphi(\bar{x})$ and tuple $\bar{a} \in F$, we can consider the following ideal of the lattice $E_H(F)$:

$$T^F_{\varphi,\bar{a}} \rightleftharpoons \{[d]_H | d \in H^\times,\ \forall R \in W_d(\mathbf{H}_R(F) \models \varphi(\bar{a})\}.$$

Suppose that $\langle F, H, \sqsubseteq_H\rangle \leq \langle F', H', \sqsubseteq_{H'}\rangle$; the local elementary properties of the family $W(H)$ ($W(H')$) are C-continuous; $H(H')$ satisfies principles LG_A and M (from LG_A it follows that $H(H')$ is a Bézout ring); for any prime p there exists a natural number n_p such that, for any $R \in W(H)(R' \in W(H'))$, if $F_R(F_{R'})$ is of characteristic p, then $a_*(R) \leq n_p$ ($a_*(R') \leq n_p$). Under these assumptions we have

THEOREM 4. *An embedding $\langle F, H, \sqsubseteq_H\rangle \leq \langle F', H', \sqsubseteq'\rangle$ is elementary if and only if the induced embedding of lattices $E_H(F) \leq E_{H'}(F')$ is elementary after expansion of these lattices by ideals $T^F_{\varphi,\bar{a}}$ ($T^{F'}_{\varphi,\bar{a}}$), $\bar{a} \in F$.*

We now state the condition of elementary equivalence: let

$$\langle F, H, \sqsubseteq_H\rangle,\quad \langle F', H', \sqsubseteq_{H'}\rangle$$

satisfy assumptions stated for $\langle F, H, \sqsubseteq_H\rangle$ before Theorem 4.

THEOREM 5. *Let $F \leq F_0, F_1$ be a common subfield, $H \leq F, H_0, H_1$ is a common subring, such that $F = q(H)$ and for any $R_0 \in W(H_0)$ we have $R \hookrightarrow R_0 \cap F = H_{\mathfrak{m}(R_0)\cap H}$, F_{R_0} is a separable extension F_R, and Γ_R is a pure subgroup of Γ_{R_0}. If*

$$\langle E_{H_0}(F_0),\ T^{F_0}_{\varphi,\bar{a}} | \bar{a} \in H\rangle \equiv \langle E_{H_1}(F_1), T^{F_1}_{\varphi,\bar{a}} | \bar{a} \in H\rangle$$

then $\langle F_0, H_0, \sqsubseteq_{H_0}\rangle \equiv \langle F_1, H_1, \sqsubseteq_{H_1}\rangle$.

IV. Some applications

The theorems obtained above allow us to introduce some new classes of multi-valued fields with decidable theory [3, 4]; to define an analog of Henselization [6] for multi-valued fields; in particular, for fields of algebraic numbers (wonderful extensions [5]) and to use them for effectivization of the global theory of fields of classes [7].

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