ON LIMIT EXISTENCE PRINCIPLES IN FORMAL ARITHMETIC

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First order Peano arithmetic PA can be axiomatized over a basic elementary arithmetic by one of the two main principles: the induction schema or the collection (or boundedness) schema. Although these schemata are equivalent, they lead to different hierarchies of fragments of PA when restricted to some subclasses of the class of arithmetical formulas such as Σ^0_n or Π^0_n. A good deal of research in formal arithmetic concerns the study of such hierarchies of fragments [2]. The most important of such fragments is $I\Sigma_1$, the theory obtained by restricting the induction schema to Σ^0_1-formulas in the standard axiomatization of PA.

We study the arithmetical schema asserting that every eventually decreasing elementary recursive function has a limit, that is, the principle

$$\exists m \forall n \geq m\, h(n+1) \leq h(n) \rightarrow \exists m \forall n \geq m\, h(n) = h(m), \qquad (Lim)$$

for each elementary function h, and the corresponding inference rule

$$\frac{\exists m \forall n \geq m\, h(n+1) \leq h(n)}{\exists m \forall n \geq m\, h(n) = h(m)}. \qquad (LimR)$$

Some other related principles are also formulated.

Our interest in the study of such principles is twofold. Firstly, they naturally appear in the proofs of some notable theorems such as Solovay's arithmetical completeness theorem for provability logic and in its extensions to the logics of interpretability and conservativity. Secondly, these principles capture different levels of logical complexity from those of the standard fragments of PA and yield new interesting series of such fragments.

Theorem. Over the elementary arithmetic, the rule $(LimR)$ provides an axiomatization of the Σ^0_2-consequences of $I\Sigma_1$.

We also establish the relationship of the limit existence principles with restricted parameter-free induction schemata. Using these results we are able to improve the arithmetical completeness theorems for conservativity logics due to P. Hajek and F. Montagna [1].

Further, we obtain a general characterization of the Σ^0_n-consequences of the standard fragments of Peano arithmetic in terms of iterated reflection principles. This is important in view of the program of proof-theoretic ordinal analysis of theories, which so far only dealt with the characteriziations of Π-classes of provable sentences (usually Π^1_1 or Π^0_2).

References

[1] P. Hájek and F. Montagna. The logic of Π_1-conservativity. *Archive for Mathematical Logic*, 30(2):113–123, 1990.

[2] P. Hájek and P. Pudlák. *Metamathematics of First Order Arithmetic.* Springer-Verlag, Berlin, Heidelberg, New York, 1993.