On Mass Problems of Presentability [1]

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Let $\mathfrak{M}$ be a countable structure of computable signature. A presentation of $\mathfrak{M}$ on natural numbers, or simply a *presentation* of $\mathfrak{M}$, is any structure $\mathfrak{C}$ such that $\mathfrak{C} \cong \mathfrak{M}$ and the domain of $\mathfrak{C}$ is a subset of ω. We can also treat the atomic diagram $D(\mathfrak{C})$ of a presentation $\mathfrak{C}$ as a subset of ω, using some Gödel numbering of the atomic formulas of the signature of $\mathfrak{M}$. Recall that a *mass problem* [1] is any set of total functions from ω to ω. A mass problem can be considered as a set of "solutions"(in form of functions from ω to ω) of some "informal problem". Important examples [2,4] of mass problems are the following: for a set $A \subseteq \omega$ the *problem of solvability of* A is the mass problem $\mathcal{S}_A = \{\chi_A\}$, where χ_A is the characteristic function of A, and the *problem of enumerability of* A is the mass problem $\mathcal{E}_A = \{f : \omega \to \omega \mid \mathrm{rng}(f) = A\}$.

Definition 1 *The problem of presentability of* $\mathfrak{M}$ *is the mass problem* $\underline{\mathfrak{M}}$ *consisting of characteristic functions of the atomic diagrams of all possible presentations of* $\mathfrak{M}$*:*

$$\underline{\mathfrak{M}} = \{ \chi_{D(\mathfrak{C})} \mid \mathfrak{C} \textit{ is a presentation of } \mathfrak{M} \}$$

We denote by $\leqslant$ the relation of Medvedev reducibility [1] on mass problems, while $\leqslant_w$ denotes the relation of Muchnik reducibility [2]. It follows from the definitions that for any mass problems $\mathcal{A}, \mathcal{B}$, we have $\mathcal{A} \leqslant \mathcal{B} \Rightarrow \mathcal{A} \leqslant_w \mathcal{B}$. A.A.Muchnik [2] established a sufficient condition under which these reducibilities are equivalent. Unfortunately, it is useless in case of problems of presentability. The theorem below describes the situation for such problems and shows that there is a natural connection between problems of presentability and admissible superstructures [3].

Theorem 1 *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be countable structures. The following are equivalent:*

1) $\underline{\mathfrak{M}} \leqslant_w \underline{\mathfrak{N}}$*;*

2) $\underline{\mathfrak{M}} \leqslant \underline{(\mathfrak{N}, \bar{n})}$ *for some* $\bar{n} \in N^{<\omega}$*;*

3) $\mathfrak{M}$ *is* Δ*-definable in* $\mathbb{HF}(\mathfrak{N})$*.*

We call a structure $\mathfrak{M}$ $*$*-uniform* if $\underline{(\mathfrak{M}, \bar{m})} \leqslant \underline{\mathfrak{M}}$ for any $\bar{m} \in M^{<\omega}$. If $\mathfrak{M}$ is homogeneous structure of finite relational signature, then $\mathfrak{M}$ is $*$-uniform. Also, $\mathfrak{M}$ is $*$-uniform in case when $\mathfrak{M}$ is constructivizable (i.e. has a computable presentation). From Theorem 1 we get

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Corollary 1 *If $\mathfrak{N}$ is $*$-uniform then, for any structure $\mathfrak{M}$, $\underline{\mathfrak{M}} \leqslant \underline{\mathfrak{N}}$ if and only if $\underline{\mathfrak{M}} \leqslant_w \underline{\mathfrak{N}}$.*

Theorem 2 *$\langle \omega_1^{CK}; \leqslant \rangle$ is $*$-uniform, while $\langle \omega_1^{CK}+1; \leqslant \rangle$ is not $*$-uniform.*

As a corollary we obtain an example of structures $\mathfrak{M}$, $\mathfrak{N}$ such that $\underline{\mathfrak{M}} \leqslant_w \underline{\mathfrak{N}}$ but $\underline{\mathfrak{M}} \nleqslant \underline{\mathfrak{N}}$. Also, we establish the following characterization of mass problems of enumerability (decidability) which are reducible to mass problems of presentability.

Theorem 3 *Let $\mathfrak{M}$ be a countable structure, and $A \subseteq \omega$, $A \neq \varnothing$. The following are equivalent:*
1) $\mathcal{E}_A \leqslant_w \underline{\mathfrak{M}}$;
2) $\mathcal{E}_A \leqslant \underline{(\mathfrak{M}, \bar{m})}$ for some $\bar{m} \in M^{<\omega}$;
3) A is Σ-definable in $\mathbb{HF}(\mathfrak{M})$.

Theorem 4 *Let $\mathfrak{M}$ be a countable structure, and $A \subseteq \omega$. The following are equivalent:*
1) $\mathcal{S}_A \leqslant_w \underline{\mathfrak{M}}$;
2) $\mathcal{S}_A \leqslant \underline{(\mathfrak{M}, \bar{m})}$ for some $\bar{m} \in M^{<\omega}$;
3) A is Δ-definable in $\mathbb{HF}(\mathfrak{M})$.

For mass problems $\mathcal{A}$ and $\mathcal{B}$ we denote by $\mathcal{A} \equiv \mathcal{B}$ the fact that $\mathcal{A} \leqslant \mathcal{B}$ and $\mathcal{B} \leqslant \mathcal{A}$. We say that a countable structure $\mathfrak{M}$ has *strong presentability dimension* α (denote $\text{Pr-dim}_s(\mathfrak{M}) = \alpha$), where α is a cardinal, if $\underline{\mathfrak{M}} \equiv \mathcal{B}$ for some $\mathcal{B} \subseteq \underline{\mathfrak{M}}$, $\text{card}(\mathcal{B}) = \alpha$, and α is the least cardinal satisfying these conditions. In the same way we can define the notion of weak presentability dimension. It is clear that for any countable $\mathfrak{M}$ we have $1 \leqslant \text{Pr-dim}_w(\mathfrak{M}) \leqslant \text{Pr-dim}_s(\mathfrak{M}) \leqslant 2^\omega$. We introduce some examples of structures with different presentability dimensions.

References

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