One approach to Vekua matrix equation and its applications

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Abstract. *In this paper, the procedure for solving the Vekua type system of partial differential equations, introduced in [1,2], is appiled to determining particle motion equations.*

Introduction and results

In [1,2] the following matrix procedure was introduced for solving a Vekua type system of partial differential equations.

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Let $A_{ij}(z,\overline{z}), B_{ij}(z,\overline{z}), F_{ij}(z,\overline{z})$ be analytical functions of z and $\overline{z}$ defined in the bounded region of a complex plane.

The system of partial differential equations

$$\frac{\partial w_i}{\partial \overline{z}} = \sum_{j=1}^{n} A_{ij}(z,\overline{z})w_j + \sum_{j=1}^{n} B_{ij}(z,\overline{z})\overline{w_j} + F_i(z,\overline{z}) \qquad j = 1,\ldots,n \tag{1}$$

can be given in the matrix form

$$\frac{\partial W}{\partial \overline{z}} = AW + B\overline{W} + F$$

where

$$W = \begin{bmatrix} w_1 \\ w_2 \\ \cdots \\ w_n \end{bmatrix} \qquad \overline{W} = \begin{bmatrix} \overline{w_1} \\ \overline{w_2} \\ \cdots \\ \overline{w_n} \end{bmatrix} \qquad \frac{\partial W}{\partial \overline{z}} = \begin{bmatrix} \frac{\partial w_1}{\partial \overline{z}} \\ \frac{\partial w_2}{\partial \overline{z}} \\ \cdots \\ \frac{\partial w_n}{\partial \overline{z}} \end{bmatrix} \qquad F = \begin{bmatrix} F_1 \\ F_2 \\ \cdots \\ F_n \end{bmatrix}$$

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \qquad B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{bmatrix}.$$

Replacement with $W = U\overline{V}$ where U and $\overline{V}$ are matrix colons gives

$$\frac{\partial W}{\partial \overline{z}} = \frac{\partial U}{\partial \overline{z}}\overline{V} + U\frac{\partial V}{\partial \overline{z}},$$

which implies

$$\frac{\partial U}{\partial \overline{z}}\overline{V} + U\frac{\partial V}{\partial \overline{z}} = AU\overline{V} + B\overline{U}V + F.$$

So we have:

$$\left[\frac{\partial U}{\partial \overline{z}} - AU\right]\overline{V} + \left[U\frac{\partial V}{\partial \overline{z}} - B\overline{U}V - F\right].$$

If

$$\frac{\partial U}{\partial \overline{z}} - A(z,\overline{z})U = 0$$

then solution of the system is reduced to solving a Teodoresku type non-homogeneous equation:

$$U\frac{\partial V}{\partial \overline{z}} = B\overline{U}V + F,$$

which can be written as follows

$$U^{-1}U\frac{\partial V}{\partial \overline{z}} = U^{-1}B\overline{U}V + U^{-1}F,$$

if U is a regular matrix. Replacement with $T = \overline{V}$ gives

$$\frac{\partial T}{\partial \overline{z}} = U^{-1}BU\overline{T} + U^{-1}F. \tag{2}$$

Thus solution of the system (1) is reduced to solving matrix differential equations (2).

An application in Mechanics

Descartes's coordinates x_1, x_2, x_3 of the particle position in spatial motion can be expressed with two variables: length of covered distance - s and time elapsed from start of motion t. The problem of determining position coordinates leads us to the following system of six partial differential equations of the first order:

$$\frac{\partial x_1}{\partial s} = p_1(s,t)x_1 + q_1(s,t)x_2 + g_1(s,t)x_3 + h_1(s,t)$$
$$\frac{\partial x_1}{\partial t} = p_2(s,t)x_1 + q_2(s,t)x_2 + g_2(s,t)x_3 + h_2(s,t)$$
$$\frac{\partial x_2}{\partial s} = p_3(s,t)x_1 + q_3(s,t)x_2 + g_3(s,t)x_3 + h_3(s,t)$$
$$\frac{\partial x_2}{\partial t} = p_4(s,t)x_1 + q_4(s,t)x_2 + g_4(s,t)x_3 + h_4(s,t)$$
$$\frac{\partial x_3}{\partial s} = p_5(s,t)x_1 + q_5(s,t)x_2 + g_5(s,t)x_3 + h_5(s,t)$$
$$\frac{\partial x_3}{\partial t} = p_6(s,t)x_1 + q_6(s,t)x_2 + g_6(s,t)x_3 + h_6(s,t).$$

Replacement with $z = s + ti$ and $\overline{z} = s - ti$ from which follows $s = \frac{z + \overline{z}}{2}$ and $t = \frac{z - \overline{z}}{2i}$ and replacement of

$$w_1 = x_1 + ix_2$$
$$w_2 = x_2 + ix_3$$
$$\frac{\partial w_1}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial x_1}{\partial s} - \frac{\partial x_2}{\partial t} + i(\frac{\partial x_1}{\partial t} + \frac{\partial x_2}{\partial s}))$$
$$\frac{\partial w_2}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial x_2}{\partial s} - \frac{\partial x_3}{\partial t} + i(\frac{\partial x_2}{\partial t} + \frac{\partial x_3}{\partial s}))$$

reduces the starting system to the following system:

$$\frac{\partial w_1}{\partial \overline{z}} = A_{11}(z,\overline{z})w_1 + A_{12}(z,\overline{z})w_1 + A_{13}(z,\overline{z})\overline{w_1} + A_{14}(z,\overline{z})\overline{w_1} + A_{15}(z,\overline{z})$$
$$\frac{\partial w_2}{\partial \overline{z}} = A_{21}(z,\overline{z})w_1 + A_{22}(z,\overline{z})w_1 + A_{23}(z,\overline{z})\overline{w_1} + A_{24}(z,\overline{z})\overline{w_1} + A_{25}(z,\overline{z})$$

that is of the Vekua type and can be solved using the procedure described in the paper by Dimitrovski and Rajović [1,2] if A_{ij} are analytical functions of A_{ij} and $\overline{z}$ defined in the bounded region of the complex plane.

References

[1] D. Dimitrovski, M. Rajović, On Matrix equation of Vekua, Proc. of 4^{th} Symp. Diff. eq. and Analysis, Ohrid 1994, 105-110.

[2] D. Dimitrovski, M. Rajović, L'idée d'une équation matricelle de Vécua, Int. Symp. on Diff. eq. Plovdiv 1995.

[3] I. N. Vekua, Generalized analytic functions, (in Russian) Nauka, Moscow 1989.

[4] N. P. Vekua, Systems of singular integral equations, (in Russian) Fizmatgiz, Moscow.