GEOMETRICAL AND NUMERICAL ANALYSIS OF COMPREHENSIVE GRID GENERATOR

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В статье представлен единый геометрический подход для построения и анализа разностных сеток в областях и на поверхностях. Конструирование сеток осуществляется с помощью решения уравнений Бельтрами в областях с метрикой, выбор которой позволяет эффективно осуществлять управление качеством сеток.

The paper gives an account of the geometrization of the studies of the comprehensive grid method proposed by the author [1] and described in detail in [2]. It also presents an important extension of the method by developing some procedures for the construction of metric tensors aimed at facilitating the generation of the structured grids with required quality properties. The paper applies some of the relations of the Riemannian geometry [2 – 5] to obtaining new equations for generating grids with prescribed properties. Taking advantage of the relations established, the equations are converted into a compact form convenient for the numerical treatment by available algorithms. Studies of the behavior of the coordinate lines near boundary segments of the monitor surfaces and physical domains are carried out. Some relations of the mean curvatures of the monitor surfaces to the Beltramian equations for grid generation are exhibited.

1. Formulation of Comprehensive Grid Generator

1.1. Mathematical Model

A mathematical model which is claimed to be the foundation of a robust comprehensive grid generator should satisfy the following fundamental properties:
1. well posedness of the mathematical problem formulated by this model for the grid generator,
2. independence of the grid construction of a parametrization of the geometry,
3. malleability (tractability) to a numerical implementation into an automatic code,
4. existence of a straightforward means for efficient control of the grid quality,
5. ability to obtain in a unified manner the domain and surface grids required in practice.

One worthy representative of a mathematical tool to formulate such model is the operator of Beltrami.

The operator is formulated on a set of twice differentiable vector-valued functions $\mathbf{f}(\mathbf{x})$ defined on an arbitrary Riemannian manifold M^n with a covariant metric tensor $g^{\mathbf{x}}_{ij}$ in some local coordinates $x_i,\ i=1,\dots,n$, by the formula

$$\Delta_B[\mathbf{f}] = \frac{1}{\sqrt{g^{\mathbf{x}}}}\frac{\partial}{\partial x_j}\left(\sqrt{g^{\mathbf{x}}}g^{jk}_{\mathbf{x}}\frac{\partial \mathbf{f}(\mathbf{x})}{\partial x_k}\right) , \tag{1.1}$$

where $\mathbf{x}=(x_1,\dots,x_n)$, $g^{\mathbf{x}}=\det\{g^{\mathbf{x}}_{ij}\}$, $g^{jk}_{\mathbf{x}}$ is a contravariant metric tensor of the manifold in the coordinates $x_i,\ i=1,\dots,n$.

Here and further we hold a convention that a summation is carried out over repeated indices undess otherwise noted. It is well-known that the formula (1.1) is the invariant of parametrizations of the manifold M^n.

The Beltramian operator allows one to formulate a mathematical model for generating grids on arbitrary Riemannian manifolds with twice differentiable local metric tensors. Let M^n be such n-dimensional manifold with the metric tensor $g^{\mathbf{s}}_{ij}$ in the local coordinates s_i, $i=1,\dots,n$, whose values lie in some simply connected parametrization domain $S^n\subset R^n$. Thus there is a map $\mathbf{r}(\mathbf{s}): S^n\to M^n$. By the general definition, a local structured grid in M^n is found by mapping a reference grid in a standard logical domain Ξ^n into X^n by the composition of $\mathbf{r}(\mathbf{s})$ and some one-to-one intermediate smooth transformation

$$\mathbf{s}(\boldsymbol{\xi}):\Xi^n\to S^n\ ,\quad \boldsymbol{\xi}=(\xi_1,\dots,\xi_n)\ ,\quad \mathbf{s}=(s_1,\dots,s_n)\ ,$$

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i.e. by $\mathbf{r}(\mathbf{s}(\boldsymbol{\xi})) : \Xi^n \to M^n$. Note the parametrization $\mathbf{r}(\mathbf{s})$ also generates a grid in M^n by mapping some reference grid in S^n. However, this grid may be unsatisfactory and as a rule it is not independent of parametrizations. Besides this, if the geometry of S^n is complex, the reference grid in S^n may require serious efforts to its generation. The role of the intermediate transformation $\mathbf{s}(\boldsymbol{\xi})$ is to make the grid on M^n satisfy the necessary properties, in particular, the property of independence of the choice of a parametrization. While the role of the logical domain Ξ^n is to replace the parametrization domain S^n with a standard parametric domain (n-dimensional cube, simplex, prism, etc) having a simple shape.

The logical domain Ξ^n, its reference grid, and the parametrization $\mathbf{r}(\mathbf{s})$ are chosen by the user. Therefore the local grid in M^n with required properties is defined if a suitable intermediate transformation $\mathbf{s}(\boldsymbol{\xi})$ is found. One of the ways to find this transformation is to use the operator of Beltrami. Namely $\mathbf{s}(\boldsymbol{\xi})$ can be determined as the inverse of the transformation

$$\boldsymbol{\xi}(\mathbf{s}) : S^n \to \Xi^n , \quad \boldsymbol{\xi}(\mathbf{s}) = (\xi_1(\mathbf{s}), \ldots, \xi_n(\mathbf{s}))$$

which is a solution of the boundary value problem

$$\begin{aligned} \Delta_B[\boldsymbol{\xi}] &\equiv \frac{1}{\sqrt{g^{\mathbf{s}}}} \frac{\partial}{\partial s_j}\Big(\sqrt{g^{\mathbf{s}}} g^{jk}_{\mathbf{s}} \frac{\partial \boldsymbol{\xi}}{\partial s_k}\Big) = 0 , \quad j,k = 1, \ldots, n , \\ \Gamma[\boldsymbol{\xi}] &\equiv \boldsymbol{\xi}|_{\partial S^n} = \boldsymbol{\varphi}(\mathbf{s}) : \partial S^n \to \partial \Xi^n , \quad \boldsymbol{\varphi}(\mathbf{s}) = (\varphi_1(\mathbf{s}), \ldots, \varphi_n(\mathbf{s})) , \end{aligned}$$

or in a component form

$$\begin{aligned} \Delta_B[\xi_i] &\equiv \frac{1}{\sqrt{g^{\mathbf{s}}}} \frac{\partial}{\partial s_j}\Big(\sqrt{g^{\mathbf{s}}} g^{jk}_{\mathbf{s}} \frac{\partial \xi_i}{\partial s_k}\Big) = 0 , \quad i,j,k = 1, \ldots, n , \\ \Gamma[\xi_i] &\equiv \xi_i|_{\partial S^n} = \varphi_i(\mathbf{s}) , \quad i = 1, \ldots, n , \end{aligned} \tag{1.2}$$

where ∂S^n and $\partial \Xi^n$ is the boundary of S^n and Ξ^n, respectively, while $\boldsymbol{\varphi}(\mathbf{s})$ is a one-to-one continuous transformation between the boundaries of S^n and Ξ^n. The coordinates $\xi_1, \ldots, \xi_n$, satisfying (1.2), are further referred to as grid coordinates.

Since $\Delta_B[\xi]$ is independent of parametrizations of M^n we obtain that the system (1.2) is equivalent to the following system

$$\begin{aligned} \Delta_B[\xi_i] &\equiv \frac{1}{\sqrt{g^{\boldsymbol{\xi}}}} \frac{\partial}{\partial \xi_j}(\sqrt{g^{\boldsymbol{\xi}}} g^{ji}_{\boldsymbol{\xi}}) = 0 , \quad i,j = 1, \ldots, n , \\ \Gamma[\xi_i] &\equiv \xi_i|_{\partial S^n} = \varphi_i(\mathbf{s}) , \quad i = 1, \ldots, n , \end{aligned} \tag{1.3}$$

where $(g^{ji}_{\boldsymbol{\xi}})$, $i,j = 1, \ldots, n$, is the contravariant metric tensor of M^n in the coordinates $\xi_1, \ldots, \xi_n$,

$$g^{\boldsymbol{\xi}} = J^2 g^{\mathbf{s}} , \quad J = \det\Big(\frac{\partial s_i}{\partial \xi_j}\Big) = 1/\det\Big(\frac{\partial \xi_i}{\partial s_j}\Big) .$$

1.1.1. Realization of Volume and Surface Grids

Grids in Domains. Let M^n be an n-dimensional domain X^n of the Eucledian space R^n which has a local structured grid obtained with the aid of a one-to-one nondegenerate smooth transformation (diffeomorphism)

$$\mathbf{x}(\boldsymbol{\xi}) : \Xi^n \to X^n , \quad \mathbf{x} = (x_1, \ldots, x_n) , \quad \boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) , \tag{1.4}$$

i.e. the local structured grid in X^n is the image of a reference grid in Ξ^n into X^n by $\mathbf{x}(\boldsymbol{\xi})$. Let S^n be the image of the domain Ξ^n in X^n by $\mathbf{x}(\boldsymbol{\xi})$. Then S^n can be formally considered as a local parametric domain of X^n with the coordinates $s_i = x_i$, $i = 1, \ldots, n$. Let

$$\mathbf{r}(\mathbf{s}) : S^n \to R^n , \quad \mathbf{r} = (r_1, \ldots, r_n) , \quad \mathbf{s} = (s_1, \ldots, s_n)$$

be the parametric transformation introducing the coordinates s_i, $i = 1, \ldots, n$, in X^n, i.e.

$$\mathbf{r}(\mathbf{s}) \equiv \boldsymbol{\xi}(\mathbf{s}) \quad \text{for} \quad \mathbf{s} = \mathbf{x}$$

where $\boldsymbol{\xi}(\mathbf{x})$ is the inverse of $\mathbf{x}(\boldsymbol{\xi}) : \Xi^n \to S^n$. Now imposing in X^n a local metric in the coordinates s_i, $i = 1, \ldots, n$, by

$$g^{\mathbf{s}}_{ij} = \mathbf{r}_{s_i} \cdot \mathbf{r}_{s_j} = \frac{\partial \xi_l}{\partial s_i} \frac{\partial \xi_l}{\partial s_j} , \quad i,j,l = 1, \ldots, n , \tag{1.5}$$

we readily find

$$g^{\mathbf{s}} = J^2 , \quad J = \det\Big(\frac{\partial \xi_i}{\partial s_j}\Big) , \quad i,j = 1,\dots,n ,$$

$$g^{ij}_{\mathbf{s}} = \frac{\partial s_i}{\partial \xi_l}\frac{\partial s_j}{\partial \xi_l} , \quad i,j,l = 1,\dots,n .$$

Therefore, applying the equations in (1.2) to the corresponding components of the function $\boldsymbol{\xi}(\mathbf{s})$, which is the inverse of (1.4) with $\mathbf{s} = \mathbf{x}$, we obtain

$$\begin{aligned} \Delta_B[\xi_i] &= \frac{1}{\sqrt{g^{\mathbf{s}}}}\frac{\partial}{\partial s_j}\Big(\sqrt{g^{\mathbf{s}}} g^{jk}_{\mathbf{s}} \frac{\partial \xi_i}{\partial s_k}\Big) = \frac{1}{\sqrt{g^{\mathbf{s}}}}\frac{\partial}{\partial s_j}\Big(\sqrt{g^{\mathbf{s}}}\frac{\partial s_j}{\partial \xi_l}\frac{\partial s_k}{\partial \xi_l}\frac{\partial \xi_i}{\partial s_k}\Big) \\ &= \frac{1}{\sqrt{g^{\mathbf{s}}}}\frac{\partial}{\partial s_j}\Big(J\frac{\partial s_j}{\partial \xi_i}\Big) = 0 , \quad i,j,k,l = 1,\dots,n . \end{aligned}$$

Thus the transformation (1.4) can be obtained as the inverse of the solution of the boundary value problem (1.2) with the metric (1.5) and the boundary condition $\boldsymbol{\varphi}(\mathbf{s}) = \boldsymbol{\xi}(\mathbf{s})$, $\mathbf{s} \in \partial S^n$.

Example of a Metric Deriving Classical Polar Coordinates. In particular, for the polar system of grid lines

$$x = \rho\cos\varphi , \quad y = \rho\sin\varphi ,$$

we find, assuming $x = s_1$, $y = s_2$, $\rho = \xi_1$, $\varphi = \xi_2$,

$$\frac{\partial \mathbf{s}}{\partial \xi_1} = \Big(\frac{s_1}{\rho}, \frac{s_2}{\rho}\Big) , \qquad \frac{\partial \mathbf{s}}{\partial \xi_2} = (-s_2, s_1) ,$$

$$\frac{\partial \boldsymbol{\xi}}{\partial s_1} = \Big(\frac{s_1}{\rho}, -\frac{s_2}{\rho^2}\Big) , \quad \frac{\partial \boldsymbol{\xi}}{\partial s_2} = \Big(\frac{s_2}{\rho}, \frac{s_1}{\rho^2}\Big) ,$$

where $\rho = \sqrt{(s_1)^2 + (s_2)^2}$. So, since (1.5), the elements of the corresponding metric covariant and contravariant tensors in the coordinates s_1, s_2, are as follows:

$$g^{\mathbf{s}}_{ij} = \frac{\partial \boldsymbol{\xi}}{\partial s_i}\cdot\frac{\partial \boldsymbol{\xi}}{\partial s_j} = g^{\mathbf{s}}[\delta^i_j + s_i s_j(1 - g^{\mathbf{s}})] , \quad i,j = 1,2 ,$$

$$g^{ij}_{\mathbf{s}} = \frac{\partial s_i}{\partial \xi_k}\frac{\partial s_j}{\partial \xi_k} = \delta^i_j / g^{\mathbf{s}} - s_i s_j (1 - g^{\mathbf{s}}) , \quad i,j = 1,2 ,$$

where $g^{\mathbf{s}} = \det(g^{\mathbf{s}}_{ij}) = 1/((s_1)^2 + (s_2)^2) = 1/\rho^2$. We readily see that this metric is considerably different from the Eucledian metric.

The formula of the metric realizing the polar coordinate system through the solution of the Beltrami equations prompts one on how to specify more general metrics in a domain or surface M^n. Namely let (g_{ij}), $i,j = 1,\dots,n$, and (g^{ij}), $i,j = 1,\dots,n$, be any covariant and contravariant metric tensor, respectively, of M^n in the coordinates $s_1,\dots,s_n$. Then three smooth functions $f_1(\mathbf{s}), f_2(\mathbf{s}), f(\mathbf{s})$ define the following new covariant and contravariant metric tensors $(\overline{g}_{ij})$ and $(\overline{g}^{ij})$, respectively, of M^n in the coordinates $s_1,\dots,s_n$:

$$\overline{g}_{ij} = f_1 g_{ij} + f_2 \frac{\partial f}{\partial s_i}\frac{\partial f}{\partial s_j} , \quad i,j = 1,\dots,n ,$$

$$\overline{g}^{ij} = \frac{1}{f_1} g^{ij} - l g^{im}\frac{\partial f}{\partial s_m} g^{jp}\frac{\partial f}{\partial s_p} , \quad i,j,m,p = 1,\dots,n ,$$

where

$$l = \frac{f_2}{f_1[f_1 + f_2\nabla(f)]} ,$$

$$\nabla(f) = g^{ij}\frac{\partial f}{\partial s_i}\frac{\partial f}{\partial s_j} , \quad i,j = 1,\dots,n .$$

Note the functions f_1, f_2, and f must be such that $f_1 > 0$ and $\det(\overline{g}_{ij}) \neq 0$.

The metric tensors $g^{\mathbf{s}}_{ij}$ and $g^{ij}_{\mathbf{s}}$ in the coordinates s_1, s_2, considered above for generating polar coordinate system are realized by the metric tensors $\overline{g}_{ij}$ and $\overline{g}^{ij}$, $i,j = 1,2$, respectively, with $g_{ij} = g^{ij} = \delta^i_j$, $i,j = 1,2$,

$$f_1 = \frac{1}{\rho^2} , \quad f_2 = \frac{1}{\rho^2}\Big(1 - \frac{1}{\rho^2}\Big) , \quad f = \frac{1}{2}\rho^2 .$$

1.2. Application to Adaptive Grid Generation

Here we discuss a relation of the operator of Beltrami to adaptive grid generation.

General Concept. In accordance with a concept of Eiseman (1987) adaptive grids in a domain X^n or on a surface $S^{xn} \subset R^{n+k}$ can be generated by projecting quasiuniform grids from a monitor Riemannian manifold (monitor hypersurface) defined as an n-dimensional graph of the values of some generally vector-valued function over X^n or S^{xn}, respectively. One of the techniques realizing this concept is based on the generation of quasiuniform grids on the monitor hypersurface with the use of a smoothness functional, suggested in [1], which generalizes the functional introduced in [6] for generating fixed grids in domains. Since the Euler–Lagrange equations for the smoothness functional are equivalent to the Beltrami equations (1.2) this technique produces the very grids obtained by the comprehensive grid generator proposed above.

Generation of Adaptive Grids in Domains. In the case, important for the generation of adaptive grids in a physical domain $X^n \subset R^n$, the hypersurface is defined as an n-dimensional monitor surface S^{rn} formed by the values of some monitor vector-valued function

$$\mathbf{f}(\mathbf{x}) : X^n \to R^k , \quad \mathbf{x} = (x_1, \dots, x_n) , \quad \mathbf{f} = [f_1(\mathbf{x}), \dots, f_k(\mathbf{x})] , \tag{1.6}$$

over X^n. Thus the monitor surface S^{rn} is the subset of the $(n+k)$-dimensional space R^{n+k} and whose points are $(x_1, \dots, x_n, f_1(\mathbf{x}), \dots, f_k(\mathbf{x}))$, $\mathbf{x} = (x_1, \dots, x_n) \in X^n$. It is apparent that for the parametric domain S^n there can be taken the domain X^n and, consequently, the parametric mapping $\mathbf{r}(\mathbf{s}) : S^n \to R^{n+k}$ is defined as

$$\mathbf{r}(\mathbf{s}) = [\mathbf{s}, \mathbf{f}(\mathbf{s})] = [s_1, \dots, s_n, f_1(\mathbf{s}), \dots, f_k(\mathbf{s})] , \qquad \mathbf{s} = \mathbf{x} . \tag{1.7}$$

If S^{rn} is a monitor surface over the domain S^n formed by the values of a vector-valued function $\mathbf{f}(\mathbf{s})$ then it is obvious that the intermediate transformation $\mathbf{s}(\boldsymbol{\xi})$ found by (1.2) or (1.3) produces, in fact, the very adaptive grid in S^n determined by projecting the quasiuniform grid from S^{rn}. This adaptive grid provides node concentration in the zones of large variation of $\mathbf{f}(\mathbf{s})$.

Note that if the transformation (1.6) is a one-to-one nondegenerate one, which may be possible only when $n = k$, then this very transformation can also be considered as a parametrization of the image of X^n by $\mathbf{f}(\mathbf{x})$, i.e $\mathbf{r}(\mathbf{s}) = \mathbf{f}(\mathbf{s})$, $\mathbf{s} = \mathbf{x}$.

Generation of Adaptive Grids on Surfaces. When the monitor surface is formed by the values of the function $\mathbf{f}(\mathbf{x})$ over a general n-dimensional surface S^{xn} lying in the space R^{n+l} and represented by the parametrization

$$\mathbf{x}(\mathbf{s}) : S^n \to R^{n+l} , \quad \mathbf{x}(\mathbf{s}) = [x_1(\mathbf{s}), \dots, x_{n+l}(\mathbf{s})]$$

from an n-dimensional parametric domain $S^n \in R^n$ then the monitor surface S^{rn} can be described by a parametrization from S^n in the form

$$\mathbf{r}(\mathbf{s}) : S^n \to R^{n+l+k} , \qquad \mathbf{r}(\mathbf{s}) = \{\mathbf{x}(\mathbf{s}), \mathbf{f}[\mathbf{x}(\mathbf{s})]\} . \tag{1.8}$$

In particular, a one-dimensional monitor surface S^{r1} over a curve S^{x1} lying in R^n and represented by $\mathbf{x}(s) : [a, b] \to R^n$, can be defined by the parametrization

$$\mathbf{r}(s) : [a, b] \to R^{n+k} , \qquad \mathbf{r}(s) = \{\mathbf{x}(s), \mathbf{f}[\mathbf{x}(s)]\} .$$

It is evident that the adaptive grid on the surface S^{xn} obtained by projecting the quasiuniform grid from S^{rn} is formed, in fact, by mapping a reference grid in Ξ^n with a composition of $\mathbf{x}(\mathbf{s})$ and the intermediate grid transformation $\mathbf{s}(\boldsymbol{\xi})$, i.e. with $\mathbf{x}(\mathbf{s}(\boldsymbol{\xi})) : \Xi^n \to S^{xn}$.

1.2.1. General Conclusion

Thus the Beltramian operator is a universal tool whose implementation in the grid technology will allow one to generate in a unified manner a required structured grid both in an arbitrary domain and on its boundary surface. So the 5th requirement of the mathematical model (1.2) is held. The metric tensor serves as a means to control the grid quality in this model, hence the 4th requirement is also satisfied. The boundary value problem (1.2) is well posed and malleable to a numerical implementation. The independence of the grid of the choice of a parametrization is the inherent attribute of the operator of Beltrami. So the requirements 1-5 imposed above are held by the mathematical model (1.2).

Note the value of any grid generation method is commonly judged by its ability to rule out the construction of unfolded grids in domains or on surfaces with arbitrary geometry. In the case $n = 2$ the mathematical foundation of this requirement for the technique based on the Beltramian system (1.2) is solid when the computation domain Ξ^2 is convex. It is founded on the following result, derived from a theorem of Rado.

Let M^2 be a simply connected bounded Riemannian manifold. In this case, the Jacobian of the transformation $\boldsymbol{\xi}(\mathbf{s})$ generated by the system (1.2) does not vanish in the interior of S^2, if Ξ^2 is a convex domain and $\boldsymbol{\xi}(\mathbf{s}) : \partial S^2 \to \partial \Xi^2$ is a homeomorphism.

It is likely that the similar result is valid for the three-dimensional geometries, though it has not been proved so far.

Therefore the model formulated is really a promising approach for the development of a comprehensive grid generation code.

2. Grid Equations with Respect to Intermediate Transformations

Here we establish some equivalent forms of the comprehensive grid equations (1.2) with respect to the intermediate transformation $\mathbf{s}(\boldsymbol{\xi})$.

2.1. General Grid Equations

A general form of the grid equations with respect to the intermediate transformation $\mathbf{s}(\boldsymbol{\xi})$ and an arbitrary metric in a Riemannian manifold X^n is obtained after multiplying the system (1.2) by $\partial s_l/\partial \xi_i$. Indeed this multiplication yields

$$\begin{aligned}&\frac{1}{\sqrt{g^{\mathbf{s}}}}\frac{\partial}{\partial s_j}\Big(\sqrt{g^{\mathbf{s}}}g^{jk}_{\mathbf{s}}\frac{\partial \xi_i}{\partial s_k}\Big)\frac{\partial s_l}{\partial \xi_i}\\ &= \frac{1}{\sqrt{g^{\mathbf{s}}}}\Big(\frac{\partial}{\partial s_j}(\sqrt{g^{\mathbf{s}}}g^{jl}_{\mathbf{s}}) - \sqrt{g^{\mathbf{s}}}g^{jk}_{\mathbf{s}}\frac{\partial^2 s_l}{\partial \xi_i \partial \xi_m}\frac{\partial \xi_i}{\partial s_k}\frac{\partial \xi_m}{\partial s_j}\Big)\\ &= \Delta_B[s_l] - g^{im}_{\boldsymbol{\xi}}\frac{\partial^2 s_l}{\partial \xi_i \partial \xi_m} = 0\ , \quad i,j,k,l,m = 1,\dots,n\ .\end{aligned}$$

Therefore we have the following system of the comprehensive grid equations with respect to the functions $s_i(\boldsymbol{\xi})$, $i = 1,\dots,n$,

$$g^{im}_{\boldsymbol{\xi}}\frac{\partial^2 s_l}{\partial \xi_i \partial \xi_m} = \Delta_B[s_l]\ , \quad i,l,m = 1,\dots,n\ . \tag{2.1}$$

Since the value of the Beltramian operator is independent of the choice of parametrizations we also find from (2.1)

$$g^{im}_{\boldsymbol{\xi}}\frac{\partial^2 s_l}{\partial \xi_i \partial \xi_m} = \frac{1}{\sqrt{g^{\boldsymbol{\xi}}}}\frac{\partial}{\partial \xi_k}\Big(\sqrt{g^{\boldsymbol{\xi}}}g^{kp}_{\boldsymbol{\xi}}\frac{\partial s_l}{\partial \xi_p}\Big)\ , \quad i,k,l,m,p = 1,\dots,n\ . \tag{2.2}$$

Analogously we find the following equivalent expression for the grid equations (1.2) in the case of the parametrizations (1.7) or (1.8)

$$g^{mj}_{\boldsymbol{\xi}}(\mathbf{r}_{\xi_m\xi_j}\cdot \mathbf{r}_{\xi_l})\ , \quad j,l,m = 1,\dots,n\ . \tag{2.3}$$

2.1.1. Grid Equations in the Case of the Monitor Surface over a Domain

If the Riemannian manifold is the monitor surface S^{rn} over a domain X^n formed by a vector-valued monitor function $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}),\dots,f_n(\mathbf{x}))$, i.e. the parametrization of S^{rn} is determined by (1.7), then, in the grid coordinates $\xi_1,\dots,\xi_n$, we obtain, assuming $\mathbf{s} = \mathbf{x}$,

$$\begin{aligned}\mathbf{r}_{\xi_m\xi_j} &= (\mathbf{s}_{\xi_m\xi_j},\mathbf{f}_{\xi_m\xi_j})\ , \quad j,m = 1,\dots,n\ ,\\ \mathbf{r}_{\xi_l} &= (\mathbf{s}_{\xi_l},\mathbf{f}_{\xi_l})\ , \quad l = 1,\dots,n\ ,\end{aligned}$$

where

$$\mathbf{f}_{\xi_m\xi_l} = \frac{\partial^2 \mathbf{f}[\mathbf{s}(\boldsymbol{\xi})]}{\partial \xi_m \partial \xi_l}\ , \quad l,m = 1,\dots,n\ .$$

Therefore (2.3), in this case, is as follows:

$$g^{mj}_{\boldsymbol{\xi}}(\mathbf{s}_{\xi_m\xi_j}\cdot \mathbf{s}_{\xi_l} + \mathbf{f}_{\xi_m\xi_j}\cdot \mathbf{f}_{\xi_l}) = 0\ , \quad j,l,m = 1,\dots,n\ ,$$

and the multiplication of this system by $\partial\xi_l/\partial s_i$ yields the following grid system, with respect to $s_i(\boldsymbol{\xi})$, $i = 1,\dots,n$,

$$g^{mj}_{\boldsymbol{\xi}}\Big(\frac{\partial^2 s_i}{\partial\xi_m\partial\xi_j}+\mathbf{f}_{\xi_m\xi_j}\cdot\mathbf{f}_{s_i}\Big)=0\ ,\quad i,j,m=1,\dots,n\ , \tag{2.4}$$

where $\mathbf{f}_{s_i}=\dfrac{\partial f[\mathbf{s}]}{\partial s_i}$, $i = 1,\dots,n$. Note, if $\xi_1,\dots,\xi_n$, are the coordinates satisfying (1.2) and consequently (1.3) then

$$g^{mj}_{\boldsymbol{\xi}}\frac{\partial^2 f_p}{\partial\xi_m\partial\xi_j}=\frac{1}{\sqrt{g^{\boldsymbol{\xi}}}}\frac{\partial}{\partial\xi_j}\Big(\sqrt{g^{\boldsymbol{\xi}}}g^{jm}_{\boldsymbol{\xi}}\frac{\partial f_p}{\partial\xi_m}\Big)=\Delta_B(f_p)\ ,\quad j,m=1,\dots,n\ ,\quad p=1,\dots,k\ ,$$

$$g^{mj}_{\boldsymbol{\xi}}\frac{\partial^2 s_i}{\partial\xi_m\partial\xi_j}=\frac{1}{\sqrt{g^{\boldsymbol{\xi}}}}\frac{\partial}{\partial\xi_j}\Big(\sqrt{g^{\boldsymbol{\xi}}}g^{jm}_{\boldsymbol{\xi}}\frac{\partial s_i}{\partial\xi_m}\Big)=\Delta_B(s_i)\ ,\quad i,j,m=1,\dots,n\ ,$$

so the system (2.4) also has the following forms

$$\Delta_B(s_i)+\Delta_B(f_p)\frac{\partial f_p}{\partial s_i}=0\ ,\quad i=1,\dots,n\ ,\quad p=1,\dots,k\ , \tag{2.5}$$

or

$$g^{mj}_{\boldsymbol{\xi}}\frac{\partial^2 s_i}{\partial\xi_m\partial\xi_j}+\Delta_B(f_l)\frac{\partial f_l}{\partial s_i}=0\ ,\quad i,j,m=1,\dots,n\ ,\quad l=1,\dots,k\ , \tag{2.6}$$

Remind $\Delta_B(f_p)$ is independent of a parametrization of S^{rn} therefore it can be computed through an arbitrary coordinate system, in particular, (1.7).

2.1.2. Grid Equations in the Case of the Monitor Surface over a Surface

If the monitor surface S^{rn} over a surface S^{xn}, represented by

$$\mathbf{x}(\mathbf{s}):S^n\to R^{n+n_1}\ ,\quad \mathbf{x}=(x_1,\dots,x_{n+n_1})\ ,$$

is formed by a monitor function $\mathbf{f}(\mathbf{x})=(f_1(\mathbf{x}),\dots,f_k(\mathbf{x}))$ with the parametrization (1.8) then, in the grid coordinates $\xi_1,\dots,\xi_n$,

$$\mathbf{r}_{\xi_m\xi_j}=(\mathbf{x}_{\xi_m\xi_j},\mathbf{f}_{\xi_m\xi_j})\ ,\quad j,m=1,\dots,n\ ,$$
$$\mathbf{r}_{\xi_l}=(\mathbf{x}_{\xi_l},\mathbf{f}_{\xi_l})\ ,\quad l=1,\dots,n\ ,$$

and consequently the grid equations (2.3) are as follows:

$$g^{mj}_{\boldsymbol{\xi}}(\mathbf{x}_{\xi_m\xi_j}\cdot\mathbf{x}_{\xi_l}+\mathbf{f}_{\xi_m\xi_j}\cdot\mathbf{f}_{\xi_l})=0\ ,\quad j,l,m=1,\dots,n\ . \tag{2.7}$$

Since

$$\begin{aligned}\mathbf{x}_{\xi_m\xi_j}\cdot\mathbf{x}_{\xi_l} &= \frac{\partial}{\partial\xi_m}\Big(\frac{\partial\mathbf{x}}{\partial s_p}\frac{\partial s_p}{\partial\xi_j}\Big)\cdot\frac{\partial\mathbf{x}}{\partial s_a}\frac{\partial s_a}{\partial\xi_l}\\ &= \Big(\frac{\partial^2 s_p}{\partial\xi_m\partial\xi_j}g^{xs}_{ap}+\frac{\partial^2\mathbf{x}}{\partial s_p\partial s_b}\cdot\frac{\partial\mathbf{x}}{\partial s_a}\frac{\partial s_p}{\partial\xi_j}\frac{\partial s_b}{\partial\xi_m}\Big)\frac{\partial s_a}{\partial\xi_l}\ ,\quad a,b,j,l,m,p=1,\dots,n\ ,\end{aligned}$$

we obtain, after multiplying the system (2.7) by $(\partial\xi_l/\partial s_b)g^{bi}_{sx}$,

$$\begin{aligned}&g^{mj}_{\boldsymbol{\xi}}(\mathbf{x}_{\xi_m\xi_j}\cdot\mathbf{x}_{\xi_l}+\mathbf{f}_{\xi_m\xi_j}\cdot\mathbf{f}_{\xi_l})\frac{\partial\xi_l}{\partial s_b}g^{bi}_{sx}\\ &=g^{mj}_{\boldsymbol{\xi}}\Big(\frac{\partial^2 s_i}{\partial\xi_m\partial\xi_j}+\mathbf{f}_{\xi_m\xi_j}\cdot\mathbf{f}_{s_b}g^{bi}_{sx}\Big)+g^{pj}_{\mathbf{s}}(\mathbf{x}_{s_p s_j}\cdot\mathbf{x}_{s_b})g^{bi}_{sx}=0\ ,\quad b,i,j,l,m=1,\dots,n\ .\end{aligned}$$

Thus the grid system with respect to $s_i(\boldsymbol{\xi})$, $i=1,\dots,n$, has the following form

$$g^{mj}_{\boldsymbol{\xi}}\Big(\frac{\partial^2 s_i}{\partial\xi_m\partial\xi_j}+\mathbf{f}_{\xi_m\xi_j}\cdot\mathbf{f}_{s_b}g^{bi}_{sx}\Big)=-g^{pj}_{\mathbf{s}}{}^{\mathbf{x}}\Upsilon^i_{pj}\ ,\quad b,i,j,m,p=1,\dots,n\ , \tag{2.8}$$

where ${}^{\mathbf{x}}\Upsilon^i_{pj}$ is the Christoffel symbol of the second kind of the surface S^{xn} in the coordinates $s_1,\dots,s_n$ [2]. Note

$${}^{\mathbf{x}}\Upsilon^i_{pj}=g^{li}(\mathbf{r}_{s_p s_j}\cdot\mathbf{r}_{s_l})\ ,\quad i,j,l,p=1,\dots,n\ .$$

Analogously to (2.6) we get, in the grid coordinates $\xi_1, \dots, \xi_n$,

$$g_{\boldsymbol{\xi}}^{mj} \frac{\partial^2 s_i}{\partial \xi_m \partial \xi_j} + \Delta_B(f_l) \frac{\partial f_l}{\partial s_b} g_{sx}^{bi} = -g_{\mathbf{s}}^{pj\,\mathbf{x}}\Upsilon_{pj}^i \ , \quad b, i, j, m, p = 1, \dots, n \ , \quad l = 1, \dots, k \ . \tag{2.9}$$

Equations (2.9) in comparison with the equations (2.4), prescribed for generating grids in domains, are more complicated. They also are less malleable for implementation in a numerical code in the case when the process of grid generation on a surface S^{xn} is coupled with the computation of this surface, since the quantity ${}^{\mathbf{x}}\Upsilon_{pj}^i$ includes the second derivatives with respect to s_i, $i = 1, \dots, n$, of the surface parametrization $\mathbf{x}(\mathbf{s})$. However, we can come to the equations of the form (2.4) for generating grids on the surface S^{xn} if we consider as the monitor function over S^{xn} a function $f_1(\mathbf{s}) = (\mathbf{s}, f[\mathbf{x}(\mathbf{s})])$. Then the monitor surface $S^{r_1 n}$ over S^{xn} with this monitor function is represented by the parametrization

$$\mathbf{r}_1(\mathbf{s}) : S^n \to S^{r_1 n} \ , \quad \mathbf{r}_1(\mathbf{s}) = (\mathbf{x}(\mathbf{s}), \mathbf{s}, \mathbf{f}[\mathbf{x}(\mathbf{s})]) \ .$$

Note the monitor surface $S^{r_2 n}$ over S^n with the monitor function $\mathbf{f}_2(\mathbf{s}) = (\mathbf{x}(\mathbf{s}), \mathbf{f}[\mathbf{x}(\mathbf{s})])$, and represented correspondently by

$$\mathbf{r}_2(\mathbf{s}) : S^n \to S^{r_2 n} \ , \quad \mathbf{r}_2(\mathbf{s}) = (\mathbf{s}, \mathbf{x}(\mathbf{s}), \mathbf{f}[\mathbf{x}(\mathbf{s})]) \ ,$$

has the same metric tensor as the surface $S^{r_1 n}$. Hence the equations for generating the intermediate transformation $\mathbf{s}(\boldsymbol{\xi}) : \Xi^n \to S^n$ with these monitor surfaces are identical and have, in accordance with (2.4), the following form

$$g_{\boldsymbol{\xi}}^{mj} \Big(\frac{\partial^2 s_i}{\partial \xi_m \partial \xi_j} + \mathbf{x}_{\xi_m \xi_j} \cdot \mathbf{x}_{s_i} + \mathbf{f}_{\xi_m \xi_j} \cdot \mathbf{f}_{s_i} \Big) = 0 \ , \quad i, j, m = 1, \dots, m \ , \tag{2.10}$$

where $(g_{\boldsymbol{\xi}}^{mj})$ is the contravariant metric tensor of the surface $S^{r_2 n}$ in the coordinates $\xi_1, \dots, \xi_n$. Note, for the covariant metric tensor of $S^{r_2 n}$ we have

$$g_{ij}^{\boldsymbol{\xi}} = \frac{\partial \mathbf{s}(\boldsymbol{\xi})}{\partial \xi_i} \cdot \frac{\partial \mathbf{s}(\boldsymbol{\xi})}{\partial \xi_j} + \frac{\partial \mathbf{x}[\mathbf{s}(\boldsymbol{\xi})]}{\partial \xi_i} \cdot \frac{\partial \mathbf{x}[\mathbf{s}(\boldsymbol{\xi})]}{\partial \xi_j} + \frac{\partial \mathbf{f}\{\mathbf{x}[\mathbf{s}(\boldsymbol{\xi})]\}}{\partial \xi_i} \cdot \frac{\partial \mathbf{f}\{\mathbf{x}[\mathbf{s}(\boldsymbol{\xi})]\}}{\partial \xi_j} \ , \quad i, j = 1, \dots, n \ .$$

Equations (2.10) with respect to the components $s_i(\boldsymbol{\xi})$, $i = 1, \dots, n$, of the intermediate transformation $\mathbf{s}(\boldsymbol{\xi})$ include the first derivatives only of the functions $\mathbf{s}(\boldsymbol{\xi})$ and $\mathbf{f}[\mathbf{x}(\mathbf{s})]$ in s_i, $i = 1, \dots, n$, therefore they are more convenient for implementation in a numerical code in comparison with the equations (2.9). Remind, the grid in S^{xn} is obtained by mapping with $\mathbf{x}(\mathbf{s})$ a grid in S^n generated through $\mathbf{s}(\boldsymbol{\xi})$.

Note that, similarly to (2.5), the equations (2.10) also have the following form

$$\Delta_B[s_i] + \Delta_B[x_j] \frac{\partial x_j}{\partial s_i} + \Delta_B[f_l] \frac{\partial f_l}{\partial s_i} = 0 \ , \quad i = 1, \dots, n \ , \quad j = 1, \dots, n + n_1 \ , \quad l = 1, \dots, k \ .$$

2.1.3. Role of Mean Curvature in the Grid Equations for Domains

In the case of the metric tensor $(g_{ij}^{\mathbf{s}})$ defined by the parametrization (1.7) as $g_{ij}^{\mathbf{s}} = \mathbf{r}_{s_i} \cdot \mathbf{r}_{s_j}$, $i, j = 1, \dots, n$, we have

$$\Delta_B[s_l] = -g_{\mathbf{s}}^{kj} \Upsilon_{kj}^l = -g_{\mathbf{s}}^{kj} g_{\mathbf{s}}^{lm} (\mathbf{r}_{s_k s_j} \cdot \mathbf{r}_{s_m}) \ , \quad j, k, l, m = 1, \dots, n \ .$$

Let $\mathbf{r}(\mathbf{s})$ be specified by (1.7) with a scalar-valued monitor function $f(\mathbf{s})$. Then $S^{rn} \subset R^{n+1}$ and we readily find

$$g_{\mathbf{s}}^{lm} f_{s_m} = \Big(\delta_m^l - \frac{1}{g^{\mathbf{s}}} f_{s_l} f_{s_m} \Big) f_{s_m} = f_{s_l} \Big(1 - \frac{1}{g^{\mathbf{s}}} f_{s_m} f_{s_m} \Big) = \frac{1}{g^{\mathbf{s}}} f_{s_l} \ , \quad l, m = 1, \dots, n \ , \tag{2.11}$$

and, consequently,

$$\Delta_B[s_l] = -g_{\mathbf{s}}^{kj} g_{\mathbf{s}}^{lm} f_{s_k s_j} f_{s_m} = -\frac{1}{g^{\mathbf{s}}} g_{\mathbf{s}}^{kj} f_{s_k s_j} f_{s_l} \ , \quad j, k, l, m = 1, \dots, n \ . \tag{2.12}$$

Now remind that the quantity

$$K_m = \frac{1}{2} g_{\mathbf{s}}^{kj} \mathbf{r}_{s_k s_j} \cdot \mathbf{n} \ , \quad j, k = 1, \dots, n \ ,$$

where $\mathbf{n}$ is the $(n+1)$-dimensional unit normal vector to S^{rn} in R^{n+1}, is the mean curvature of the monitor surface S^{rn}. Note the mean curvature is the invariant of parametrizations of S^{rn}. Since, in our case,

$$\mathbf{r}_{s_i} = (\underbrace{0,\ldots,0}_{i-1}, 1, \underbrace{0,\ldots,0}_{n-i}, f_{s_i}) , \quad i = 1,\ldots,n ,$$

so it is obvious that

$$\mathbf{n} = \frac{1}{\sqrt{g^{\mathbf{s}}}}(-f_{s_1},\ldots,-f_{s_n},1) .$$

Therefore

$$\mathbf{r}_{s_k s_j} \cdot \mathbf{n} = \frac{1}{\sqrt{g^{\mathbf{s}}}} f_{s_k s_j} , \quad j,k = 1,\ldots,n ,$$

and consequently

$$K_m = \frac{1}{2\sqrt{g^{\mathbf{s}}}} g^{kj}_{\mathbf{s}} f_{s_k s_j} , \quad j,k = 1,\ldots,n .$$

So equations (2.1) have the following form

$$\Delta_B[s_l] = -\frac{2}{\sqrt{g^{\mathbf{s}}}} K_m f_{s_l} , \quad l = 1,\ldots,n . \tag{2.13}$$

Thus the grid equations (1.43) for a domain $X^n = S^n$ with a scalar monitor function $f(\mathbf{x})$ are expressed through the mean curvature of S^{rn} as follows:

$$g^{ij}_{\boldsymbol{\xi}} \frac{\partial^2 s_l}{\partial \xi_i \partial \xi_j} = -\frac{2}{\sqrt{g^{\mathbf{s}}}} K_m f_{s_l} , \quad i,j,l = 1,\ldots,n . \tag{2.14}$$

3. Properties of the Coordinate Lines

In this section we establish some relations between the qualitative properties of the coordinate lines generated by the comprehensive grid system (1.2) and the form of monitor functions and geometry characteristics of the physical domain or surface undergoing the gridding process.

3.1. Behavior near Boundary Segments

We consider here an arbitrary surface S^{rn}, $n = 2,3$, locally represented by a parametrization of the form

$$\mathbf{r}(\mathbf{s}) : S^n \to R^{n+k} , \quad \mathbf{r} = (r_1,\ldots,r_{n+k}) , \quad \mathbf{s} = (s_1,\ldots,s_n) , \quad k \geq 0 , \quad n > 0 ,$$

and whose covariant metric tensor $(g^{\mathbf{s}}_{ij})$ in the coordinates s_i, $i = 1,\ldots,n$, is defined through such parametrization as follows:

$$g^{\mathbf{s}}_{ij} = \mathbf{r}_{s_i} \cdot \mathbf{r}_{s_j} , \quad i,j = 1,\ldots,n .$$

We also assume that the logical domain Ξ^n in the boundary value problem (1.2) formulated for generating grids on S^{rn} is the standard unit cube, i.e. $0 \leq \xi_i \leq 1$, $i = 1,\ldots,n$. Besides this, let the (n-1)-dimensional boundary plane $\xi_i = 0$ or $\xi_i = 1$ of Ξ^n for some i, $1 \leq i \leq n$, is mapped into the boundary of S^{rn}. It is well-known [2,4] that when S^{rn} for $n = 2$ and $n = 3$ is a domain with the Eucledian metric then the operator of Beltrami is the Laplace operator and the spacing between the (n-1)-dimensional coordinate surfaces $\xi_i = \text{const}$ in S^{rn} increases near a boundary segment if it is convex and, conversly, the spacing decreases when the boundary segment in concave. It occurs that this is also valid in the case of the Riemannian manifold S^{rn} for $n = 2$ and $n = 3$.

3.1.1. Two-Dimensional Case

Rate of Change of the Coordinate Line Spacing near Boundary Segments of a Monitor Surface. Let ξ_1, ξ_2 be an arbitrary coordinate system of a two-dimensional monitor surface S^{r2} (not necessarily satisfying (1.2)). We first consider in the surface S^{r2} a family of the coordinate lines $\xi_2 = \text{const}$. Note the unit vector $\mathbf{n}_2$ lying in the tangent plane to S^{r2} and which is orthogonal to the line $\xi_2 = c_0$ is expressed in the form

$$\mathbf{n}_2 = \left(g^{2i}_{\boldsymbol{\xi}} / \sqrt{g^{22}_{\boldsymbol{\xi}}}\right) \mathbf{r}_{\xi_i} , \quad i = 1,2 . \tag{3.1}$$

Let us denote by l_h the distance between the two coordinate lines $\xi_2 = c_0$ and $\xi_2 = c_0 + h$ in S^{r2}. We have

$$l_h = (\mathbf{n}_2 \cdot \mathbf{r}_{\xi_2})h + O(h^2) = h\Big(g^{2i}_{\boldsymbol{\xi}}/\sqrt{g^{22}_{\boldsymbol{\xi}}}\Big)\mathbf{r}_{\xi_i} \cdot \mathbf{r}_{\xi_2} + O(h^2) = h/\sqrt{g^{22}_{\boldsymbol{\xi}}} + O(h^2)\ .$$

So the quantity $1/\sqrt{g^{22}_{\boldsymbol{\xi}}}$ reflects the relative spacing between the coordinate grid lines $\xi_2 = c_0 + h$ and $\xi_2 = c_0$ in S^{r2}.

The vector $\mathbf{n}_2$ is orthogonal to the coordinate line $\xi_2 = c_0$, and therefore the rate of change of the relative spacing $1/\sqrt{g^{22}_{\boldsymbol{\xi}}}$ of the coordinate curves $\xi_2 = \text{const}$ near this line is its derivative in the $\mathbf{n}_2$ direction. Using (3.1) we obtain

$$\frac{d}{d\mathbf{n}_2}\Big(\frac{1}{\sqrt{g^{22}_{\boldsymbol{\xi}}}}\Big) = \frac{1}{(g^{22}_{\boldsymbol{\xi}})^2} g^{2i}_{\boldsymbol{\xi}} g^{2l}_{\boldsymbol{\xi}} \Upsilon^2_{li}\ , \quad i,l,k = 1,2\ . \tag{3.2}$$

Note this equation is valid for an arbitrary coordinate system ξ_1, ξ_2. We also have in an arbitrary coordinate system ξ_1, ξ_2 the following relation

$$\begin{aligned} g^{2i}_{\boldsymbol{\xi}} g^{2l}_{\boldsymbol{\xi}} \Upsilon^2_{li} &= g^{22}_{\boldsymbol{\xi}} g^{2l}_{\boldsymbol{\xi}} \Upsilon^2_{l2} + g^{21}_{\boldsymbol{\xi}} g^{2l}_{\boldsymbol{\xi}} \Upsilon^2_{l1} \\ &= g^{22}_{\boldsymbol{\xi}} g^{kl}_{\boldsymbol{\xi}} \Upsilon^2_{lk} + g^{21}_{\boldsymbol{\xi}} g^{2l}_{\boldsymbol{\xi}} \Upsilon^2_{l1} - g^{22}_{\boldsymbol{\xi}} g^{1l}_{\boldsymbol{\xi}} \Upsilon^2_{l1} \\ &= g^{22}_{\boldsymbol{\xi}} g^{kl}_{\boldsymbol{\xi}} \Upsilon^2_{lk} + \Upsilon^2_{l1}[g^{21}_{\boldsymbol{\xi}} g^{2l}_{\boldsymbol{\xi}} - g^{22}_{\boldsymbol{\xi}} g^{1l}_{\boldsymbol{\xi}}] \\ &= g^{22}_{\boldsymbol{\xi}} g^{kl}_{\boldsymbol{\xi}} \Upsilon^2_{lk} + \Upsilon^2_{11}[g^{21}_{\boldsymbol{\xi}} g^{21}_{\boldsymbol{\xi}} - g^{22}_{\boldsymbol{\xi}} g^{11}_{\boldsymbol{\xi}}] = g^{22}_{\boldsymbol{\xi}} g^{kl}_{\boldsymbol{\xi}} \Upsilon^2_{lk} - \frac{\Upsilon^2_{11}}{g^{\boldsymbol{\xi}}}\ . \end{aligned} \tag{3.3}$$

So, taking into account (3.2), and (3.3), we obtain that the rate of change of the relative spacing of the family of the coordinate lines $\xi_2 = \text{const}$ of an arbitrary grid system ξ_1, ξ_2 in S^{r2} is expressed as follows:

$$\frac{d}{d\mathbf{n}_2}\Big(\frac{1}{\sqrt{g^{22}_{\boldsymbol{\xi}}}}\Big) = \frac{1}{g^{22}_{\boldsymbol{\xi}}} g^{kl}_{\boldsymbol{\xi}} \Upsilon^2_{kl} - \frac{1}{g^{\boldsymbol{\xi}}(g^{22}_{\boldsymbol{\xi}})^2} \Upsilon^2_{11}\ , \quad k,l = 1,2\ . \tag{3.4}$$

Similarly, for the family of the coordinate lines $\xi_1 = \text{const}$, its rate of change of the relative spacing has the form

$$\frac{d}{d\mathbf{n}_1}\Big(\frac{1}{\sqrt{g^{11}_{\boldsymbol{\xi}}}}\Big) = \frac{1}{g^{11}_{\boldsymbol{\xi}}} g^{kl}_{\boldsymbol{\xi}} \Upsilon^1_{kl} - \frac{1}{g^{\boldsymbol{\xi}}(g^{11}_{\boldsymbol{\xi}})^2} \Upsilon^1_{22}\ , \quad k,l = 1,2\ , \tag{3.5}$$

where

$$\mathbf{n}_1 = \Big(g^{1i}_{\boldsymbol{\xi}}/\sqrt{g^{11}_{\boldsymbol{\xi}}}\Big)\mathbf{r}_{\xi_i}\ , \quad i = 1,2\ ,$$

is the unit vector orthogonal to the boundary line $\xi_1 = c_0$ in S^{r2}.

Rate of Change of the Grid Line Spacing near Boundary Segments of a Monitor Surface. Let now ξ_1, ξ_2 be the grid coordinate system, i.e. it is obtained by the solution of the system (1.2) with $n = 2$. Then the elements of the contravariant metric tensor and the Christoffel symbols satisfy the grid equations [2,5]

$$g^{kj}_{\boldsymbol{\xi}} \Upsilon^i_{kj} = 0\ , \quad i,k,j = 1,2\ .$$

Taking into account these equations we find from (3.5) at the points (c_0, ξ_2)

$$\frac{d}{d\mathbf{n}_1}\Big(\frac{1}{\sqrt{g^{11}_{\boldsymbol{\xi}}}}\Big) = -\frac{1}{g^{\boldsymbol{\xi}}(g^{11}_{\boldsymbol{\xi}})^2} \Upsilon^1_{22}\ . \tag{3.6}$$

Similarly we find at the points (ξ_1, c_0) from (3.4)

$$\frac{d}{d\mathbf{n}_2}\Big(\frac{1}{\sqrt{g^{22}_{\boldsymbol{\xi}}}}\Big) = -\frac{1}{g^{\boldsymbol{\xi}}(g^{22}_{\boldsymbol{\xi}})^2} \Upsilon^2_{11}\ , \tag{3.7}$$

Availing us of the identities

$$\Upsilon^l_{3-l3-l} = g^{lm}_{\boldsymbol\xi}(\mathbf{r}_{\xi_{3-l}\xi_{3-l}} \cdot \mathbf{r}_{\xi_m}) = \sqrt{g^{ll}_{\boldsymbol\xi}}(\mathbf{r}_{\xi_{3-l}\xi_{3-l}} \cdot \mathbf{n}_l) , \quad l, m = 1, 2 , \tag{3.8}$$

where the repeating index l is fixed (the summation is over m only), we find that the above equations (3.6) and (3.7) give rise to one more system of two-dimensional grid equations

$$\begin{aligned} \frac{d}{d\mathbf{n}_1}\Big(\frac{1}{\sqrt{g^{11}_{\boldsymbol\xi}}}\Big) &= -\frac{1}{g^{\boldsymbol\xi}\sqrt{(g^{11}_{\boldsymbol\xi})^3}}(\mathbf{r}_{\xi_2\xi_2} \cdot \mathbf{n}_1) , \\ \frac{d}{d\mathbf{n}_2}\Big(\frac{1}{\sqrt{g^{22}_{\boldsymbol\xi}}}\Big) &= -\frac{1}{g^{\boldsymbol\xi}\sqrt{(g^{22}_{\boldsymbol\xi})^3}}(\mathbf{r}_{\xi_1\xi_1} \cdot \mathbf{n}_2) . \end{aligned} \tag{3.9}$$

Replacement of the coordinate ξ_{3-l} by the arc length coordinate s, which are related as

$$\frac{ds}{d\xi_{3-l}} = \sqrt{g^{\boldsymbol\xi}_{3-l3-l}} , \quad l = 1, 2 , \quad l \ - \ \text{fixed} ,$$

yields

$$\mathbf{r}_{\xi_{3-l}\xi_{3-l}} \cdot \mathbf{n}_l = g^{\boldsymbol\xi}_{3-l3-l}(\mathbf{r}_{ss} \cdot \mathbf{n}_l) , \quad l = 1, 2 , \quad l \ - \ \text{fixed} .$$

The quantity $\sigma_l = \mathbf{r}_{ss} \cdot \mathbf{n}_l$ is the geodesic curvature of the coordinate line $\xi_l = \text{const}$ in the surface S^{r2} and is invariant of its parametrizations. It is obvious from (3.8) that

$$\sigma_l = (\mathbf{r}_{\xi_{3-l}\xi_{3-l}} \cdot \mathbf{n}_l)/g^{\boldsymbol\xi}_{3-l3-l} = \frac{1}{\sqrt{g^{ll}_{\boldsymbol\xi}}\, g^{\boldsymbol\xi}_{3-l3-l}}\Upsilon^l_{3-l3-l} , \quad l = 1, 2 ,$$

with l fixed. Therefore the equations (3.9) are identical to

$$\frac{d}{d\mathbf{n}_l}\Big(\frac{1}{\sqrt{g^{ll}_{\boldsymbol\xi}}}\Big) = -\frac{1}{\sqrt{g^{ll}_{\boldsymbol\xi}}}\sigma_l , \quad l = 1, 2 , \ \text{fixed} . \tag{3.10}$$

Thus if the geodesic curvature σ_l of the boundary curve $\xi_l = 0$ in S^{r2} is negative (positive) then

$$\frac{d}{d\mathbf{n}_l}\Big(\frac{1}{\sqrt{g^{ll}_{\boldsymbol\xi}}}\Big) > 0 (< 0) , \quad l = 1, 2 , \ \text{fixed} .$$

This means that the coordinate lines $\xi_l = \text{const}$ obtained through (1.2) cluster near the boundary curve $\xi_l = 0$ if $\sigma_l < 0$ and vice versa the coordinate lines become sparser when approaching the curve $\xi_l = 0$ if $\sigma_l > 0$. Note the sign of the geodesic curvature indicates the convexity ($\sigma_l > 0$) or concavity ($\sigma_l < 0$) of the boundary curve $\xi_l = 0$ in S^{r2}, while the condition $\sigma_l = 0$ means that the coordinate line $\xi_l = 0$ is a geodesic line in S^{r2}. Analogous results are held for the lines $\xi_l = 1$, $l = 1, 2$.

So we conclude, finally, that the grid lines obtained from (1.2) are repelled from the convex segments of the boundary lines of S^{r2} and attracted to their concave segments. control the grid spacing near the boundary of S^2.

3.1.2. Three-Dimensional Case

In the case of a three-dimensional monitor surface S^{r3} defined by (1.7) the normal $\mathbf{n}_3$ to the surface $\xi_3 = \text{const}$ in S^{r3} is defined, analogously to (3.1), as follows:

$$\mathbf{n}_3 = \Big(g^{3i}_{\boldsymbol\xi}/\sqrt{g^{33}_{\boldsymbol\xi}}\Big)\mathbf{r}_{\xi_i} , \quad i = 1, 2, 3 . \tag{3.11}$$

Further

$$\begin{aligned} g^{kl}_{\boldsymbol{\xi}}\Upsilon^3_{kl} &= \frac{1}{g^{33}_{\boldsymbol{\xi}}}[g^{33}_{\boldsymbol{\xi}}g^{kl}_{\boldsymbol{\xi}}\Upsilon^3_{kl}] \\ &= \frac{1}{g^{33}_{\boldsymbol{\xi}}}[g^{3k}_{\boldsymbol{\xi}}g^{3l}_{\boldsymbol{\xi}}\Upsilon^3_{kl} + (g^{33}_{\boldsymbol{\xi}}g^{1l}_{\boldsymbol{\xi}} - g^{31}_{\boldsymbol{\xi}}g^{3l}_{\boldsymbol{\xi}})\Upsilon^3_{1l} + (g^{33}_{\boldsymbol{\xi}}g^{2l}_{\boldsymbol{\xi}} - g^{32}_{\boldsymbol{\xi}}g^{3l}_{\boldsymbol{\xi}})\Upsilon^3_{2l}] \\ &= \frac{1}{g^{33}_{\boldsymbol{\xi}}}[g^{3k}_{\boldsymbol{\xi}}g^{3l}_{\boldsymbol{\xi}}\Upsilon^3_{kl} + (g^{33}_{\boldsymbol{\xi}}g^{11}_{\boldsymbol{\xi}} - g^{31}_{\boldsymbol{\xi}}g^{31}_{\boldsymbol{\xi}})\Upsilon^3_{11} + 2(g^{33}_{\boldsymbol{\xi}}g^{12}_{\boldsymbol{\xi}} - g^{31}_{\boldsymbol{\xi}}g^{32}_{\boldsymbol{\xi}})\Upsilon^3_{12} \\ &\quad +(g^{33}_{\boldsymbol{\xi}}g^{22}_{\boldsymbol{\xi}} - g^{32}_{\boldsymbol{\xi}}g^{32}_{\boldsymbol{\xi}})\Upsilon^3_{22}] \\ &= \frac{1}{g^{33}_{\boldsymbol{\xi}}}[g^{3k}_{\boldsymbol{\xi}}g^{3l}_{\boldsymbol{\xi}}\Upsilon^3_{kl} + \frac{1}{g^{\boldsymbol{\xi}}}(g^{\boldsymbol{\xi}}_{22}\Upsilon^3_{11} - 2g^{\boldsymbol{\xi}}_{12}\Upsilon^3_{12} + g^{\boldsymbol{\xi}}_{11}\Upsilon^3_{22})]\ , \quad k,l=1,2,3\ , \end{aligned} \tag{3.12}$$

The two-dimensional surface $\xi_3 = \text{const}$ lying in S^{r3} has a natural covariant metric tensor

$$g_{ij} = g^{\boldsymbol{\xi}}_{ij}\ , \quad i,j=1,2\ ,$$

and, consequently, the elements of its contravariant metric tensor (g^{ij}) are defined as follows:

$$g^{ij} = (-1)^{i+j}g_{3-i3-j}/\det(g_{ij})\ , \quad i,j=1,2\ .$$

It is apparent that

$$g^{\boldsymbol{\xi}} = \det(g_{ij})/g^{33}_{\boldsymbol{\xi}}\ , \quad i,j=1,2\ ,$$

therefore

$$g^{ij} = (-1)^{i+j}g_{3-i3-j}/(g^{\boldsymbol{\xi}}g^{33}_{\boldsymbol{\xi}})\ , \quad i,j=1,2\ . \tag{3.13}$$

Note the equation $g^{ij} = g^{ij}_{\boldsymbol{\xi}}$, $i,j=1,2$, is not valid, in general. Thus, from (3.12) we find

$$g^{kl}_{\boldsymbol{\xi}}\Upsilon^3_{kl} = \frac{1}{g^{33}_{\boldsymbol{\xi}}}[g^{3k}_{\boldsymbol{\xi}}g^{3l}_{\boldsymbol{\xi}}\Upsilon^3_{kl} + g^{33}_{\boldsymbol{\xi}}g^{ij}\Upsilon^3_{ij}]\ , \quad i,j=1,2\ , \quad k,l=1,2,3\ . \tag{3.14}$$

Since

$$\frac{d}{d\mathbf{n}_3}\Big(\frac{1}{\sqrt{g^{33}_{\boldsymbol{\xi}}}}\Big) = \frac{1}{(g^{33}_{\boldsymbol{\xi}})^2}g^{3i}_{\boldsymbol{\xi}}g^{3l}_{\boldsymbol{\xi}}\Upsilon^3_{il}\ , \quad i,l=1,2,3\ ,$$

equations (3.14) are equivalent to

$$g^{kl}_{\boldsymbol{\xi}}\Upsilon^3_{kl} = g^{ij}\Upsilon^3_{ij} + g^{33}_{\boldsymbol{\xi}}\frac{d}{d\mathbf{n}_3}\Big(\frac{1}{\sqrt{g^{33}_{\boldsymbol{\xi}}}}\Big)\ , \quad i,j=1,2\ , \quad k,l=1,2,3\ . \tag{3.15}$$

So if the transformation $\boldsymbol{\xi}(\mathbf{s})$ is defined through (1.2) then we get from (3.15)

$$\frac{d}{d\mathbf{n}_3}\Big(\frac{1}{\sqrt{g^{33}_{\boldsymbol{\xi}}}}\Big) = -\frac{1}{g^{33}_{\boldsymbol{\xi}}}g^{ij}\Upsilon^3_{ij}\ , \quad i,j=1,2\ . \tag{3.16}$$

Note that the quantity

$$K^3_m = \frac{1}{2}\frac{1}{\sqrt{g^{33}_{\boldsymbol{\xi}}}}g^{ij}\Upsilon^3_{ij}\ , \quad i,j=1,2\ ,$$

is the mean curvature of the surface $\xi_3 = \text{const}$ in S^{r3}. So (3.16) yields

$$\frac{d}{d\mathbf{n}_3}\Big(\frac{1}{\sqrt{g^{33}_{\boldsymbol{\xi}}}}\Big) = -\frac{2}{\sqrt{g^{33}_{\boldsymbol{\xi}}}}K^3_m\ .$$

Similar equations are valid for the coordinate surfaces $\xi_i = \text{const}$, $i=1,2$.

Thus we conclude, analogously to the two-dimensional case considered above, that the grid surfaces obtained from (1.2) are repelled from the boundary segment whose mean curvature in S^{r3} is positive and attracted to it if its mean curvature in S^{r3} is negative.

3.2. Numerical Experiment

As a confirmation of the robustness of the grid technology described above we demonstrate an example of the generation of a two-dimensional grid by equations (2.6). The numerical algorithm is based on the iterative scheme of fractional steps [7]. As an initial grid there was taken a grid whose all interior points coincide with a single point. The monitor function has large gradients near the corner points of the domain. Figure 1 demonstrates the grid obtained after 5, 100, and 1000 iterations.

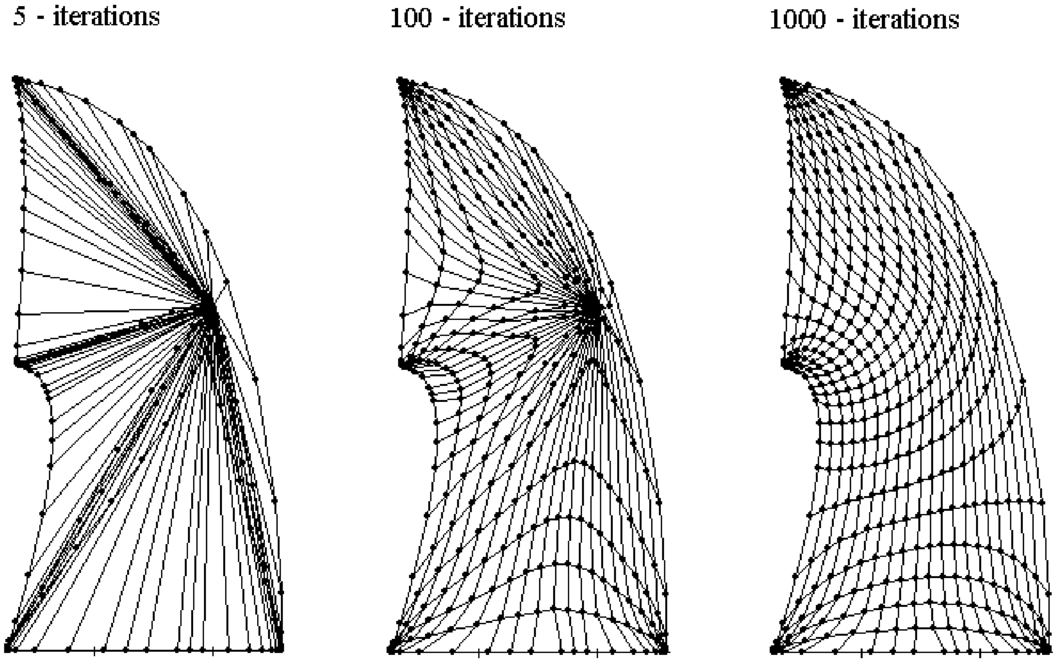

Fig. 1. Steps of grid evolution.

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