INVERSE EXTREMUM PROBLEMS FOR STATIONARY EQUATIONS OF HEAT AND MASS TRANSFER IN VISCOUS FLUID

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Рассматриваются обратные экстремальные задачи для стационарной системы уравнений тепломассопереноса в ограниченной области с липшицевой границей. Указанные задачи заключаются в нахождении неизвестных параметров среды либо плотностей источников масс и тепла по определенной информации о решении. Исследуется разрешимость обратных экстремальных задач, обосновывается применение принципа Лагранжа, выводятся и анализируются системы оптимальности. Библиогр. 13.

Introduction

In this paper inverse extremum problems are studied for the following Boussinesq heat and mass transfer model:

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\mathrm{grad})\mathbf{u} + \mathrm{grad}p = (\beta_C C - \beta_T T)\mathbf{G} \text{ in } \Omega,\ \mathrm{div}\mathbf{u} = 0 \text{ in } \Omega, \mathbf{u} = \mathbf{g} \text{ on } \Gamma, \tag{1}$$

$$-\lambda\Delta T + \mathbf{u}\cdot\mathrm{grad}T = 0 \text{ in } \Omega,\ T = 0 \text{ in } \Gamma_D,\ \lambda\left(\frac{\partial T}{\partial n} + \alpha T\right) = \chi \text{ on } \Gamma_N, \tag{2}$$

$$-\lambda_c\Delta C + \mathbf{u}\cdot\mathrm{grad}C - w_0\frac{\partial C}{\partial z} + kC = f \text{ in } \Omega,\ C = 0 \text{ on } \Gamma_D,\ \lambda_c\frac{\partial C}{\partial n} = \chi_c \text{ on } \Gamma_N. \tag{3}$$

Here Ω is a bounded domain of the space $\mathbf{R}^3$, with a Lipschitz continuous boundary Γ which consists of two parts: Γ_D and Γ_N or Γ_D^c and Γ_N^c. The other notations are usual (see for example [4]). In particular, $\mathbf{u}$, p, T and C are velocity, pressure, temperature and concentration of some substantion in the fluid, $\nu > 0$, $\lambda > 0$, $\lambda_c > 0$ are the kinematic viscosity, thermal conductivity and diffusion coefficients (the constants), f is the volume density of mass sources, $w_0 = \mathrm{const}$ is the speed of vertical sedimentation of substantion, $\mathbf{G} = (0, 0, G)$ is the gravitational acceleration, β_T, β_C, k, α, χ, χ_C and $\mathbf{g}$ are some functions.

Direct problem (1) – (3) contains some physical parameters and functions. All of these must be given for finding its solution. Sometimes some of these parameters are not known. In these cases the necessity of solving of inverse problems for the model (1) – (3) arises. These inverse problems consist of finding unknown parameters of the model given an additional information about the solution. This information may take many different forms. One can take for example the set of values $C_d(x)$ of the concentration C at the points of some set $Q \subset \Omega$. In this case unknown parameters are found from the condition $C(x) = C_d(x)$, $x \in Q$. One can consider and more general situation when unknown parameters are found from the condition $(\Lambda C)(x) = C_d(x)$ $x \in Q$, where Λ is a certain operator.

Information of another type is used in extremum inverse problems. A cost functional is used in these problems which depends both the parameters and the solution and it is required to minimize it for example by choosing unknown sources of pollutant. A number of papers is devoted to investigation of these problems. Among them we mention [2–5, 8, 11, 12]. It is important that the solution of inverse problems can be also reduced to solving extremum problems under the respective choice of a cost functional. So we shall study below in more details extremum problems for the model (1) – (3) where an element $u \equiv (f, k, \chi_c, \chi, \mathbf{g})$ shall play the role of the control.

We begin with discussion some properties of solutions of the direct problem (1) – (3). For brevity we shall refer to this problem as Problem 1.

1. Some properties of the solution of the direct problem

As usual we shall use the weak formulation of the direct problem (1) – (3). It is based on using the Sobolev spaces $H^s(D)$, $s \in \mathbf{R}$ and $L^r(D)$, $1 \le r \le \infty$ where D is either the domain Ω or the boundary Γ, or a certain part Γ_0 of Γ. The corresponding spaces of vector-functions we shall denote by $\mathbf{H}^s(D)$ or $\mathbf{L}^r(D)$. The functional norms in the spaces $H^s(D)$, $H^s(\Gamma)$ or $H^s(\Gamma_0)$ and $L^r(\Omega)$ and its vector analogues we shall denote by $\|\cdot\|_{s,\Omega} \equiv \|\cdot\|_s$, $\|\cdot\|_{s,\Gamma}$,

*This work was partially supported by Russian Foundation of Basic Research under grant No. 99–01–00214.

$\|\cdot\|_{s,\Gamma_0}$, $\|\cdot\|_{L^r}(\Omega)$. The scalar products in $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ will be denoted by $(\cdot,\cdot)$, the scalar products in $L^2(\Gamma)$ and $\mathbf{L}^2(\Gamma)$ or in $L^2(\Gamma_0)$ and $\mathbf{L}^2(\Gamma_0)$ will be denoted by $(\cdot,\cdot)_\Gamma$ or by $(\cdot,\cdot)_{\Gamma_0}$. By $(\cdot,\cdot)_1$, $\|\cdot\|_1$ and $|\cdot|_1$ we shall denote scalar product, norm and seminorm in $H^1(\Omega)$ and $\mathbf{H}^1(\Omega)$ respectively. The duality pairing between an arbitrary Hilbert or Banach space X and its dual X^* will be denoted by $<\cdot,\cdot>_{X^*\times X}$ or by $<\cdot,\cdot>$ simply. Set $\mathbf{V} = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) : \text{div}\mathbf{v} = 0\}$, $L_0^2(\Omega) = \{p \in L^2(\Omega) : (p,1) = 0\}$, $Z_1 = H^1(\Omega, \Gamma_D) \equiv \{S \in H^1(\Omega) : S|_{\Gamma_D} = 0\}$, $Z_2 = H^1(\Omega, \Gamma_D^c)$, $L_+^2(\Omega) = \{k \in L^2(\Omega) : k \geq 0 \text{ in } \Omega\}$.

Introduce the bilinear and trilinear forms $a : H^1(\Omega) \times H_0^1(\Omega) \to \mathbf{R}$, $b : \mathbf{V} \times L_0^2(\Omega)$, $c : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}_0^1(\Omega) \to \mathbf{R}$, $\tilde{a} : H^1(\Omega) \times H^1(\Omega) \to \mathbf{R}$, $b_1 : Z_1 \times \mathbf{H}_0^1(\Omega) \to \mathbf{R}$, $b_2 : Z_2 \times \mathbf{H}_0^1(\Omega) \to \mathbf{R}$, $c_i : \mathbf{H}^1(\Omega) \times Z_i \times Z_i \to \mathbf{R}$ defined by

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} d\Omega,\ b(\mathbf{v},q) = -\int_\Omega \text{div}\mathbf{v} d\Omega,\ c(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega [(\mathbf{u}\cdot\text{grad})\mathbf{v}]\cdot\mathbf{w} d\Omega,$$

$$\tilde{a}(T,S) = \int_\Omega \nabla T \cdot \nabla S d\Omega,\ b_i(S,\mathbf{v}) = \int_\Omega \mathbf{b}_i S \cdot \mathbf{v} d\Omega,\ c_i(\mathbf{u},\varphi,\psi) = \int_\Omega (\mathbf{u}\cdot\text{grad}\varphi)\psi d\Omega, \tag{4}$$

where $\mathbf{b}_1 = \beta_T \mathbf{G}$, $\mathbf{b}_2 = \beta_C \mathbf{G}$.

We note that the forms a and $\tilde{a}$ are $\mathbf{V}$–elliptic and Z_i–elliptic respectively so there hold inequalities

$$a(\mathbf{v},\mathbf{v}) \geq \alpha_0 \|\mathbf{v}\|_1^2\ \forall \mathbf{v} \in \mathbf{V},\ \tilde{a}(S,S) \geq \alpha_1 \|S\|_1^2\ \forall S \in Z_i,\ i = 1,2. \tag{5}$$

Here α_0 and α_1 are positive constants. Set $\mathcal{N} = \|c\|$, $\mathcal{N}_i = \|c_i\|$, $\beta_i = \|b_i\|$, $i = 1,2$.

We shall assume in what follows that the following conditions take place:

(i) Ω is a bounded connected domain of $\mathbf{R}^d$ with the boundary $\Gamma \in C^{0,1}$;

(ii) $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ where $\Gamma_D \in C^{0,1}$, $\text{meas}\Gamma_D > 0$, $\Gamma_D \cap \Gamma_N = \emptyset$;
$\Gamma = \overline{\Gamma^c}_D \cup \overline{\Gamma^c}_N$ where $\Gamma_D^c \in C^{0,1}$, $\text{meas}\Gamma_D^c > 0$, $\Gamma_D^c \cap \Gamma_N^c = \emptyset$;

(iii) $\mathbf{b}_i \in \mathbf{L}^2(\Omega)$, $0 \leq \beta_i < \infty$, $i = 1,2$, $\alpha \in L^\infty(\Gamma_N)$ and $\alpha \geq 0$ on Γ_N, $\lambda_* \equiv \lambda_c\alpha_1 - |w_0| > 0$.

Let us divide the data of the problem (1)–(3) into two sets: fixed data set and control set. The first set contains fixed data: Ω, Γ_D, Γ_N, λ, λ_c, w_0, $\mathbf{b}_1$, $\mathbf{b}_2$ and α. The second one contains functions f, k, χ, χ_c and $\mathbf{g}$. They will play the role of controls below. We suppose that f, k, χ, χ_c and $\mathbf{g}$ are elements of some subsets K_1, K_2, K_3, K_4 and K_5 respectively where

(iv) $K_1 \subset L^2(\Omega)$, $K_2 \subset L_+^2(\Omega)$, $K_3 \subset L^2(\Gamma_N^c)$, $K_4 \subset L^2(\Gamma_N)$, $K_5 \subset \tilde{\mathbf{H}}^{1/2}(\Gamma)$, are nonempty closed convex sets. Here $\tilde{\mathbf{H}}^{1/2}(\Gamma) = \{\mathbf{w}|_\gamma : \mathbf{w} \in \tilde{\mathbf{H}}^1(\Omega)\}$ where $\tilde{\mathbf{H}}^1(\Omega) = \{\mathbf{w} \in \tilde{\mathbf{H}}^1(\Omega) : \mathbf{u}\cdot\mathbf{n} = 0$ on $\Gamma_N \cup \Gamma_N^c$, $\int_{\Gamma(i)} \mathbf{u}\cdot\mathbf{n} d\Gamma = 0, i = 1,2,...,N\}$.

Let us introduce a notion of a weak solution of Problem 1. Let $\mathbf{x} = \mathbf{u}, p, T, C)$, $u = (f, k, \chi, \chi_c, \mathbf{g})$, $K = K_1 \times K_2 \times K_3 \times K_4 \times K_5$, $Y = \mathbf{H}^{-1}(\Omega) \times L_0^2 \times (\tilde{\mathbf{H}}^{1/2}(\Gamma))^* \times Z_1^* \times Z_2^*$, $X = \tilde{\mathbf{H}}^1(\Omega) \times L_0^2(\Omega) \times Z_1 \times Z_2$,

$$a_1(T,S) = \lambda\tilde{a}(T,S) + \lambda(\alpha T, S)_{\Gamma_N},\ a_2(C,h) = \lambda_c\tilde{a}(C,h) - w_0(\frac{\partial C}{\partial z}, h) + (kC, h). \tag{6}$$

Multiply the first equation in (1) by a function $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, the second one in (1) by $q \in L_0^2(\Omega)$, the equation in (2) by $S \in Z_1$ and the equation in (3) by $h \in Z_2$. After integrating and using boundary conditions we shall come to the weak formulation. It consists of finding a quadruple $\mathbf{x} = (\mathbf{u}, p, T, C) \in X$ such that

$$F(\mathbf{x}, u) \equiv F(\mathbf{u}, p, T, C, f, k, \chi_c, \chi, \mathbf{g}) = 0. \tag{7}$$

Here an operator $F \equiv (F_1, F_2, F_3, F_4, F_5) : X \to Y$ is defined by

$$< F_1(\mathbf{x},u), \mathbf{v} >= \nu a(\mathbf{u},\mathbf{v}) + c(\mathbf{u},\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) + b_1(T,\mathbf{v}) - b_2(C,\mathbf{v}), < F_2(\mathbf{x},u), q >= b(\mathbf{u},q),$$

$$< F_3(\mathbf{x},u) >= \mathbf{u}|_\Gamma - \mathbf{g},\ < F_4(\mathbf{x},u), S >= a_1(T,S) + c_1(\mathbf{u},T,S) - < l, S >, \tag{8}$$

$$< F_5(\mathbf{x},u), h >= a_2(C,h) + c_2(\mathbf{u},C,h) - < l_c, h >,$$

where the functionals $l \in Z_1^*$ and $l_c \in Z_2^*$ are defined by

$$< l, S >= (\chi, S)_{\Gamma_N} \equiv \int_{\Gamma_N} \chi S d\Gamma,\ < l_c, h >= \int_\Omega fh d\Omega + \int_{\Gamma_N^c} \chi_c h d\Gamma. \tag{9}$$

A quadruple $(\mathbf{u}, p, T, C) \in X$ satisfying (7), (8) will be called a weak solution to Problem 1.

While studying inverse problems we shall need some facts about solvability of (7). We shall use the following results which are proved in [1,5].

Lemma 1. *Under conditions (i) for any function* $\mathbf{g} \subset \tilde{\mathbf{H}}^{1/2}(\Gamma)$ *and* $\varepsilon > 0$ *there exist a vector-function* $\mathbf{u}_\varepsilon \in \mathbf{H}^1(\Omega)$ *such that*

$$\mathrm{div}\mathbf{u}_\varepsilon = 0 \text{ в } \Omega, \ \mathbf{u}_\varepsilon|_\Gamma = \mathbf{g}, \ \|\mathbf{u}_\varepsilon\|_1 \leq c_\varepsilon \|\mathbf{g}\|_{1/2,\Gamma}, |c(\mathbf{v}, \mathbf{u}_\varepsilon, \mathbf{v})| \leq \varepsilon \|\mathbf{g}\|_{1/2,\Gamma} \|\mathbf{v}\|_1^2 \ \forall \mathbf{v} \in \mathbf{V}.$$

Here a constant c_ε *depends on* ε *and* Ω.

Lemma 2. *Assume that conditions (i)-(iii) are satisfied. Then: 1) the bilinear form* $a_1 : Z_1 \times Z_1 \to \mathbf{R}$ *in (6) is continuous and elliptic with a constant* α_* *dependent on* Ω, α *and* λ; *2) the bilinear form* $a_2 : Z_2 \times Z_2 \to \mathbf{R}$ *in (6) is continuous and elliptic with a constant* $\lambda_* = \lambda_c \alpha_1 - |w_0|$.

It follows from Lemma 2 that

$$a_1(T,T) \geq \alpha_* \|T\|_1^2 \ \ \forall T \in Z_1, \ \ a_2(C,C) \geq \lambda_* \|C\|_1^2 \ \ \forall C \in Z_2.$$

Using Lemmas 1, 2 and the Schauder's theorem one can prove the following theorem (see [5]).

Theorem 1. *Under conditions (i)–(iii) for any element* $u \in K$ *there exists a weak solution* $(\mathbf{u}, p, T, C)$ *to Problem 1 and the following estimates hold:*

$$\|\mathbf{u}\|_1 \leq M_{\mathbf{u}}(u), \ \ \|T\|_1 \leq M_T(u), \ \ \|C\|_1 \leq M_C(u). \tag{10}$$

Here $M_{\mathbf{u}}$, M_T and M_C are nondecreasing continuous functions of $\|f\|$, $\|k\|$, $\|\chi_c\|_{\Gamma_N^c}$, $\|\chi\|_{\Gamma_N}$ and $\|\mathbf{g}\|_{1/2,\Gamma}$.

Let us introduce generalized Reinolds number Re, Rayleigh number Ra and diffusive Rayleigh number Ra_c by

$$\mathrm{Re}(u) = \frac{1}{\alpha_0 \nu} \mathcal{N} M_{\mathbf{u}}, \ \ \mathrm{Ra}(u) = \frac{1}{\alpha_0 \nu} \frac{\beta_1 \mathcal{N}_1}{\alpha_*} M_T, \ \ \mathrm{Ra}_c(u) = \frac{1}{\alpha_0 \nu} \frac{\beta_2 \mathcal{N}_2}{\lambda_*} M_C.$$

Theorem 2. *Let in addition to conditions (i)–(iii)*

$$\mathrm{Re}(u) + \mathrm{Ra}(u) + \mathrm{Ra}_c(u) < 1 \ \ \forall u \in K. \tag{11}$$

Then the weak solution of Problem 1 is unique for any $u \in K$.

We shall also consider a weak formulation of the linear analogue of Problem 1. It consists of finding a quadruple $(\mathbf{u}, p, T, C) \in X$ from identities

$$\nu a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}_0, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + b_1(T, \mathbf{v}) - b_2(C, \mathbf{v}) = 0 \ \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \ b(\mathbf{u}, q) = 0 \ \forall q \in L^2(\Omega),$$

$$a_1(T,S) + c_1(\mathbf{u}_0, T, S) = < l, S > \ \ \forall S \in Z_1, \ a_2(C,h) + c_2(\mathbf{u}_0, C, h) = < l_c, h > \ \ \forall h \in Z_2 \tag{12}$$

which are linear analogues of (7), (8). Here $\mathbf{u}_0 \in \tilde{\mathbf{H}}^1(\Omega)$ is a given function. Let us introduce linear operators $A : \mathbf{H}^1(\Omega) \to \mathbf{H}^{-1}(\Omega)$, $B : \mathbf{H}_0^1(\Omega) \to (L_0^2(\Omega))^* \equiv L_0^2(\Omega)$, $B^* : L_0^2(\Omega) \to \mathbf{H}^{-1}(\Omega)$, $A_1 : Z_1 \to Z_1^*$, $A_2 : Z_2 \to Z_2^*$, $B_1 : Z_1 \to \mathbf{H}^{-1}(\Omega)$, $B_2 : Z_2 \to \mathbf{H}^{-1}(\Omega)$ defined by

$$< A\mathbf{u}, \mathbf{v} >= \nu a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}_0, \mathbf{u}, \mathbf{v}), \ < B\mathbf{v}, q >= b(\mathbf{v}, q) =< B^* q, \mathbf{v} >,$$

$$< A_1 T, S >= a_1(T,S) + c_1(\mathbf{u}_0, T, S), \ < A_2 C, h >= a_2(C,h) + c_2(\mathbf{u}_0, C, h),$$

$$< B_1 T, \mathbf{v} >= b_1(T, \mathbf{v}), \ \ < B_2 C, \mathbf{v} >= b_2(C, \mathbf{v}). \tag{13}$$

Taking into account (13) we may rewrite (12) as

$$A\mathbf{u} + B^* p + B_1 T - B_2 C = 0, \ \ B\mathbf{u} = 0, \ \ A_1 T = l, \ \ A_2 C = l_c. \tag{14}$$

Let us introduce an operator $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5) : X \to Y$ by

$$\Phi_1(\mathbf{x}) = \tilde{\Phi}_1(\mathbf{x}) + B_1 T - B_2 C, \ \ \tilde{\Phi}_1(\mathbf{x}) = A\mathbf{u} + B^* p, \ \ \Phi_2(\mathbf{x}) = \mathrm{div}\mathbf{u},$$

$$\Phi_3(\mathbf{x}) = \mathrm{div}\mathbf{u}, \ \ \Phi_4(\mathbf{x}) = A_1 T, \ \ \Phi_5(\mathbf{x}) = A_2 C. \tag{15}$$

Simple analysis shows that the linear operators $(\Phi_1, \Phi_2, \Phi_3) : \tilde{\mathbf{H}}^1(\Omega) \times L_0^2(\Omega) \to \mathbf{H}^{-1}(\Omega) \times L_0^2(\Omega)$, $A_1 : Z_1 \to Z_1^*$ and $A_2 : Z_2 \to Z_2^*$ for $k \in L_+^2(\Omega)$ are isomorphisms. So the next theorem is valid.

Theorem 3. *Let under conditions (i)–(iii)* $\alpha \in L^\infty(\Gamma_N^c)$, $\alpha \geq 0$, $k \in L_+^2(\Omega)$, $\mathbf{u}_0 \in \tilde{\mathbf{H}}^1(\Omega)$. *Then the operator* $\Phi : X \to Y$, *defined by (14), (15) is an isomorphism.*

2. Stament and study of inverse problems

Let $\tilde{J}$ be a cost functional, $\mu_j > 0$ some constants. Let us introduce an auxiliary functional

$$J(\mathbf{x}, u) = \tilde{J}(\mathbf{x}) + \frac{\mu_1}{2}||f|| + \frac{\mu_2}{2}||\kappa|| + \frac{\mu_3}{2}||\chi_c||_{\Gamma_N^c} + \frac{\mu_4}{2}||\chi||_{\Gamma_N} + \frac{\mu_5}{2}||\mathbf{g}||_{1/2,\Gamma_D}. \quad (16)$$

Let the following conditions hold in addition to (iv):

(v) $\mu_j > 0$ and K_j is a bounded set, $j = 1, 2, 3, 4, 5$.

We shall study the following optimization problem

$$J(\mathbf{x}, u) \to \inf, \quad F(\mathbf{x}, u) = 0, \quad (\mathbf{x}, u) \in X \times K. \quad (17)$$

The following types of the cost functional will be considered below:

$$J_1(\mathbf{x}) = \frac{1}{2}\int_\Omega |\nabla \mathbf{u}|^2 d\Omega, \; J_2(\mathbf{x}) = \frac{1}{2}\int_\Omega |C - C_d|^2 d\Omega, \; J_3(\mathbf{x}) = \frac{1}{2}\int_\Omega rCd\Omega, \; r \in L^2_+(\Omega). \quad (18)$$

One can read about physical sense of functionals J_1, J_2 and J_3 in [1,2,10]. Let

$$Z_{ad} = \{(\mathbf{x}, u) \in X \times K : F(\mathbf{x}, u) = 0, \;\; J(\mathbf{x}, u) < \infty\}.$$

Theorem 4. *Let the conditions (i) – (v) hold; $\tilde{J} : X \to \mathbf{R}$ is a weakly lower semicontinuous functional which is independent of p and the set Z_{ad} is nonempty. Then the problem (17) has at least one solution.*

Proof. Denote by $(x_m, u_m) \equiv (u_m, p_m, T_m, C_m, f_m, \kappa_m, \chi^c_m, \chi_m, \mathbf{g}_m) \in Z_{ad}$ a minimizing sequence for the functional J, where

$$\lim_{m\to\infty} J(x_m, u_m) = \inf_{(x,u)\in Z_{ad}} J(x, u) = J^*. \quad (19)$$

Due to (v) we have the estimates

$$||f_m|| \le M_1, \qquad ||k_m|| \le M_2, \qquad ||\chi^c_m||_{\Gamma^c_N} \le M_3, \qquad ||\chi_m||_{\Gamma_N} \le M_4. \quad (20)$$

Here and below $M_1, M_2, ...$ will be certain constants independent of m. It follows from (20) and Theorem 1 that the following estimates hold for u_m, T_m and C_m:

$$||u_m||_1 \le M_5, \qquad ||T_m||_1 \le M_6, \qquad ||C_m||_1 \le M_7, \;\; m = 1, 2, ... \quad (21)$$

From (20) and (21) it follows that there exist weak limits $f^* \in Z^*, k^* \in L^2_+(\Omega), \chi^*_c \in L^2(\Gamma^c_N), \chi^* \in L^2(\Gamma_N), \mathbf{g}^* \in \tilde{\mathbf{H}}^{1/2}(\Gamma), u^* \in H^1(\Omega), T^* \in H^1(\Omega)$ and $C^* \in Z$ of some subsequences of sequences $\{f_m\}$, $\{k_m\}$, $\{\chi^c_m\}$, $\{\chi_m\}$, $\{\mathbf{g}_m\}$, $\{\mathbf{u}_m\}$, $\{T_m\}$ and $\{C_m\}$.

As K_1, K_2, K_3, K_4 and K_5 are convex closed sets they are closed in the weak topology as well. So $f^* \in K_1, k^* \in K_2, \chi^*_c \in K_3, \chi^* \in K_4, \mathbf{g}^* \in K_5$. Let us prove that $F_j(x^*, u^*) = 0, j = 1, 2, 3, 4$. To this end we note that $\mathbf{u}_m$, T_m and C_m satisfy for each m the relations

$$\nu a(\mathbf{u}_m, \mathbf{v}) + c(\mathbf{u}_m, \mathbf{u}_m, \mathbf{v}) + b_1(T_m, \mathbf{v}) - b_2(C_m, \mathbf{v}) = 0 \;\; \forall \mathbf{v} \in \mathbf{V}. \quad (22)$$

Let us pass to the limit in (22) as $m\to \infty$. As $\mathbf{u}_m \to \mathbf{u}^*$ weakly in $\mathbf{H}^1(\Omega)$ and strongly in $\mathbf{L}^4(\Omega)$ and $T_m \to T^*, C_m \to C^*$ weakly in $H^1(\Omega)$ we have $a(\mathbf{u}_m, \mathbf{v}) \to a(\mathbf{u}, \mathbf{v})$, $c(\mathbf{u}_m, \mathbf{u}_m, \mathbf{v}) \to c(\mathbf{u}^*, \mathbf{u}^*, \mathbf{v})$, $b_2(C_m, \mathbf{v}) \to b_2(C^*, \mathbf{v})$, $b_1(T_m, \mathbf{v}) \to b_1(T^*, \mathbf{v})$ $\forall \mathbf{v} \in \mathbf{V}$. So passing to the limit in (22) as $m\to \infty$ we easily obtain that

$$\nu a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b_1(T, \mathbf{v}) - b_2(C, \mathbf{v}) = 0 \;\; \forall \mathbf{v} \in \mathbf{V}.$$

It follows from here that there exists a function $p^* \in L^2_0(\Omega)$ [9] such that $F_1(\mathbf{x}^*, u^*) = 0$. Here $\mathbf{x}^* = (\mathbf{u}^*, p^*, T^*, C^*)$, $u^* = (f^*, k^*, \chi^*_c, \chi^*, \mathbf{g}^*)$. Similar considerations lead us to relations $F_j(\mathbf{x}^*, u^*) = 0$, j=2,3,4,5. Besides it follows from the weak lower semicontinuity of J that $J(\mathbf{x}^*, u^*) = J^*$. The theorem is proved.

We remark that every of functional J_1, J_2, J_3 in (18) is weakly lower semicontinuous and in independent of p. So we have

Corollary 1. Under conditions of Theorem 4 there exists at least one solution of the extremum problem (17) for $\tilde{J} = J_i$, $i = 1, 2, 3$.

In the case where K_j are nonbounded sets the statements of Theorem 4 hold if in addition to conditions (i) – (v) the functional $\tilde{J}$ is bounded from below and the constants μ_j are positive. This condition is valid for functionals J_1 and J_2 as they are nonnegative. As for functional J_3 the latter takes place in the case when the component C of the solution $(\mathbf{u}, p, T, C)$ is nonnegative for any $u \in K$. Let us formulate these results.

Theorem 5. *Let under conditions (i) – (v) $J : X \to \mathbf{R}$ be a weakly lower semicontinuous and bounded from below functional which is independent of p and let $\mu_j > 0$ and the set Z_{ad} is nonempty. Then the problem (17) has at least one solution.*

Corollary 2. Under conditions of Theorem 5 there exists at least one solution of the extremum problem (17) for $\tilde{J} = J_i$, $i = 1, 2$. If besides $r \in L^2_+(\Omega)$ in Ω and the concentration C is nonnegative on the set K then the problem (17) for $\tilde{J} = J_3$ has at least one solution.

Now we deduce the optimality system for the problem (17). Similarly [2-5] we shall use the extremal principle in smooth-convex problems of conditional minimization [10]. For this purpose let us find the Frechet derivative of F with respect to $\mathbf{x}$. One can easily see that this derivative in any point $(\hat{\mathbf{x}}, \hat{u}) = (\hat{\mathbf{u}}, \hat{p}, \hat{T}, \hat{C}, \hat{f}, \hat{k}, \hat{\chi}_c, \hat{\chi}, \hat{\mathbf{g}}) \in X \times K$ is a linear continuous operator $F'_x(\hat{\mathbf{x}}, \hat{u}) : X \to Y$ which maps each element $(\mathbf{w}, r, \tau, \mu) \in X$ into the element $F'_x(\hat{\mathbf{x}}, \hat{u})(\mathbf{w}, r, \tau, \mu) = (y_1, y_2, y_3, y_4, y_5) \in Y$ where

$$< y_1, \mathbf{v} >= \nu a(\mathbf{w}, \mathbf{v}) + c(\hat{\mathbf{u}}, \mathbf{w}, \mathbf{v}) + c(\mathbf{w}, \hat{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, r) + b_1(\tau, \mathbf{v}) - b_2(\mu, \mathbf{v}),$$

$$< y_2, q >= b(\mathbf{w}, q), \ y_3 = \mathbf{w}|_\Gamma, \ < y_4, S >= a_1(\tau, S) + c_1(\hat{\mathbf{u}}, \tau, S) + c_1(\mathbf{w}, \hat{T}, S),$$

$$< y_5, h >= a_2(\mu, h) + c_2(\hat{\mathbf{u}}, \mu, h) + c_2(\mathbf{w}, \hat{C}, h). \tag{23}$$

Let $\mathbf{y}^* = (\xi, \sigma, \zeta, \theta, \eta) \in Y^* = \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega) \times (\tilde{\mathbf{H}}^{1/2}(\Gamma))^* \times Z_1 \times Z_2$. Introduce the Lagrangian $\mathcal{L} : X \times K \times \mathbf{R} \times Y^* \to \mathbf{R}$ as follows

$$\mathcal{L}(\mathbf{x}, u, \lambda_0, \mathbf{y}^*) = \lambda_0 \ J(\mathbf{x}, u) + < \mathbf{y}^*, F(\mathbf{x}, u) >\equiv \lambda_0 J(\mathbf{x}, u) + < F_1(\mathbf{x}, u), \xi > +$$

$$+ (F_2(\mathbf{x}, u), \sigma) + < \zeta, F_3, (\mathbf{x}, u) >_\Gamma + < F_4(\mathbf{x}, u), \theta > + < F_5(\mathbf{x}, u), \eta > . \tag{24}$$

Let $\mathbf{R}^+ = \{\lambda \in \mathbf{R} : \lambda \geq 0\}$. The following theorem is a main result of this Section.

Theorem 6. *Let under conditions (i) – (iv) $(\hat{\mathbf{x}}, \hat{u}) \in X \times K$ be the point of local minimum in problem (17) and let $J(\mathbf{x}, \cdot) : K \to \mathbf{R}$ be a convex functional for any point $\mathbf{x} \in X$. Suppose the function $\mathbf{x} \to J'_{\mathbf{x}}(\mathbf{x}, u)$ with values in X^* belongs to the space C^0 in $\hat{\mathbf{x}}$ for any $u \in K$. Then there exists a non-zero Lagrange multiplier $(\lambda_0, \mathbf{y}^*) \in \mathbf{R}^+ \times Y^*$ such that there hold the Euler-Lagrange equation*

$$\lambda_0 < J'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{u}), (\mathbf{w}, r, \tau, \mu) >_{X^* \times X} + < y^*, F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{u})(\mathbf{w}, r, \tau, \mu) >_{Y^* \times Y} = 0 \ \forall (\mathbf{w}, r, \tau, \mu) \in X \tag{25}$$

and the variational inequality (minimum principle)

$$< f - \hat{f}, \eta > -((k - \hat{k})\hat{C}, \eta) + (\chi_c - \hat{\chi}_c, \eta)_{\Gamma_N} + (\chi - \hat{\chi}, \theta)_\Gamma + < \zeta, \mathbf{g} - \hat{\mathbf{g}} >_\Gamma \leq$$

$$\leq \lambda_0 [J(\hat{\mathbf{x}}, u) - J(\hat{\mathbf{x}}, \hat{u})] \quad \forall u \in K. \tag{26}$$

Proof. We shall use the theorem from [10, p.79]. By this theorem it suffice to prove that the operator $F'_x(\hat{\mathbf{x}}, \hat{u}) : X \to Y$ is a Fredholm operator. By (23) we have $F'_x(\hat{\mathbf{x}}, \hat{u}) = \Phi + \Phi' = \Phi + (\Phi'_1, 0, 0, \Phi'_4, \Phi'_5)$ where the operator Φ is introduced in (15) and the operators Φ'_1, Φ'_4 and Φ'_5 are defined by $< \Phi'_1(\mathbf{w}, r, \tau, \mu), \mathbf{v} >= c(\mathbf{w}, \hat{\mathbf{u}}, \mathbf{v})$, $< \Phi'_4(\mathbf{w}, r, \tau, \mu), S >= c_1(\mathbf{w}, \hat{T}, S)$, $< \Phi'_5(\mathbf{w}, r, \tau, \mu), h >= c_2(\mathbf{w}, \hat{C}, h)$. By Theorem 3 the operator $\Phi : X \to Y$ is an isomorphism and from properties of forms c, c_1 and c_2 it follows [3] that the operator $(\Phi'_1, \Phi'_4, \Phi'_5)$ which depends only on $\mathbf{w}$ is continuous from $\mathbf{L}^4(\Omega)$ in $\mathbf{H}^{-1}(\Omega) \times Z_1^* \times Z_2^*$. The latter implies the compactness of the operator $(\Phi'_1, \Phi'_4, \Phi'_5) : \mathbf{H}^1_0(\Omega) \to \mathbf{H}^{-1}(\Omega) \times Z_1^* \times Z_2^*$. Therefore the operator $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{u}) = \Phi + \Phi'$ is a Fredholm operator.

Let us consider equation (25). Let J is independent of p, so that

$$< J'_{\mathbf{x}}(\hat{\mathbf{x}}, u), (\mathbf{w}, r, \tau, \mu) >=< J'_{\mathbf{u}}(\hat{\mathbf{x}}, \hat{u}), \mathbf{w} > + < J'_T(\hat{\mathbf{x}}, \hat{u}), \tau > + < J'_C(\hat{\mathbf{x}}, \hat{u}), \mu > .$$

Setting in (25) at first $r = 0, \tau = 0, \mu = 0$ then $\mathbf{w} = 0, \tau = 0, \mu = 0$; $\mathbf{w} = 0, r = 0, \mu = 0$ and $\mathbf{w} = 0, r = 0, \tau = 0$ we obtain the following identities for the Lagrange multipliers $\xi \in \mathbf{H}^1_0(\Omega)$, $\theta \in Z_1$ and $\eta \in Z_2$:

$$\nu a(\mathbf{w}, \xi) + c(\hat{\mathbf{u}}, \mathbf{w}, \xi) + c(\mathbf{w}, \hat{\mathbf{u}}, \xi) + c_1(\mathbf{w}, \hat{T}, \eta) + c_2(\mathbf{w}, \hat{C}, \eta) + b(\mathbf{w}, \sigma) +$$

$$\lambda_0 < J'_{\mathbf{u}}(\hat{\mathbf{x}}, \hat{u}), \mathbf{w} >= 0 \ \ \forall \mathbf{w} \in \mathbf{H}^1_0(\Omega), \ \ b(\xi, r) = 0 \ \ \forall r \in L^2_0(\Omega), \tag{27}$$

$$a_1(\tau, \theta) + c_1(\hat{\mathbf{u}}, \tau, \theta) + b_1(\tau, \xi) + \lambda_0 < J'_T(\hat{\mathbf{x}}, \hat{u}), \tau >= 0 \ \ \forall \tau \in Z_1, \tag{28}$$

$$\hat{a}_2(\mu, \eta) + c_2(\hat{\mathbf{u}}, \mu, \eta) - b_2(\mu, \xi) + \lambda_0 < J'_C(\hat{\mathbf{x}}, \hat{u}), \mu >= 0 \ \ \forall \mu \in Z_2. \tag{29}$$

Relations (27) – (29) together with the minimum principle (25) and the operator restriction (7) which is a weak formulation of the problem (1) – (3) form the optimality system for the problem (17). One must solve it in order to find the desired controls $\hat{f}, \hat{k}, \hat{\chi}_c, \hat{\chi}, \hat{\mathbf{g}}$ and the optimal state, i.e. velocity, pressure, temperature and concentration fields in the domain Ω.

Note that the optimality system consists of three parts. The first part has a form of a weak formulation (7) of Problem 1; the second part is a variational inequality (26) for controls $\hat{f}, \hat{k}, \hat{\chi}_c$, $\hat{\chi}$ and $\hat{\mathbf{g}}$. Finally the last part consists of identities (27) – (29) for the Lagrange multipliers ξ, σ, ζ, θ and η.

We can show arguing as in [3] that identities (27) – (29) are a weak formulation of some boundary value problem for ξ, σ, ζ, θ and η. We limit ourself by formulation of corresponding theorem (see details in [5]). Introduce the linear continuous operator $S_{\mathbf{H}} : (\tilde{\mathbf{H}}^1(\Omega))^* \to \mathbf{H}^{-1}(\Omega)$, $S_H : Z_i^* \to H^{-1}(\Omega)$ acting by formula

$$< S_{\mathbf{H}}\mathbf{l}, \mathbf{v} >_{\mathbf{H}^{-1}(\Omega) \times \mathbf{H}^1_0(\Omega)} = < \mathbf{l}, \mathbf{v} >_{(\tilde{\mathbf{H}}^1(\Omega))^* \times \tilde{\mathbf{H}}^1(\Omega)} \ \ \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega) \subset \tilde{\mathbf{H}}^1(\Omega), \ \mathbf{l} \in (\tilde{\mathbf{H}}^1(\Omega))^*,$$

$$< S_H l, S >_{H^{-1}(\Omega) \times H^1_0(\Omega)} = < l, S >_{(Z_1^* \times Z_2} \ \ \forall S \in H^1_0(\Omega) \subset Z_i, \ l \in Z_i^*, i = 1, 2.$$

Theorem 7. Let the conditions of Theorem 6 hold and a functional J is independent of p. Then there exist functions (Lagrange multipliers) $\xi \in \mathbf{H}^1_0(\Omega)$, $\sigma \in L^2_0(\Omega)$, $\zeta \in \tilde{\mathbf{H}}^{1/2}(\Gamma)$, $\theta \in Z_1$, $\eta \in Z_1$ and the constant $\lambda_0 \geq 0$ which together with the solution $(\hat{\mathbf{x}}, u)$ of the extremum problem (17) satisfy the equations

$$-\nu\Delta\xi - (\hat{\mathbf{u}} \cdot \nabla)\xi + \nabla\hat{\mathbf{u}} \cdot \xi + \nabla\sigma + \theta\nabla\hat{T} + \eta\nabla\hat{C} = -\lambda_0 J'_{\mathbf{u}}(\hat{\mathbf{x}}, \hat{u}), \ \ \mathrm{div}\xi = 0, \tag{30}$$

$$-\lambda\Delta\theta - \hat{\mathbf{u}} \cdot \nabla\theta + \mathbf{b}_1 \cdot \xi = -\lambda_0 S_H J'_T(\hat{\mathbf{x}}, \hat{u}), \tag{31}$$

$$-\lambda_c\Delta\eta - \hat{\mathbf{u}} \cdot \nabla\eta + w_0\frac{\partial\eta}{\partial z} + \hat{k}\eta - \mathbf{b}_2 \cdot \xi = -\lambda_0 S_H J'_C(\hat{\mathbf{x}}, \hat{u}), \tag{32}$$

integral identities (27) – (29) and the variational inequality (26).

For linear model of heat and mass transfer the optimality system can be deduced in a similar way. Simple analysis shows that in the linear case the optimality system consists of (12), the variational inequality (24) and identities

$$\nu a(\mathbf{w}, \xi) + c(\mathbf{u}_0, \mathbf{w}, \xi) + b(\mathbf{w}, \sigma) + \lambda_0 < J'_{\mathbf{u}}(\hat{\mathbf{x}}, \hat{u}), \mathbf{w} >= 0 \ \forall \mathbf{w} \in \mathbf{H}^1_0(\Omega), b(\xi, r) = 0 \ \forall r \in L^2_0(\Omega), \tag{33}$$

$$a_1(\tau, \theta) + c_1(\mathbf{u}_0, \tau, \theta) + b_1(\tau, \xi) + \lambda_0 \ \ < J'_T(\hat{\mathbf{x}}, \hat{u}), \tau) >= 0 \ \forall \tau \in Z_1, \tag{34}$$

$$a_2(\mu, \eta) + c_2(\mathbf{u}_0, \mu, \eta) - b_2(\mu, \xi) + \lambda_0 \ \ < J'_C(\hat{\mathbf{x}}, \hat{u}), \mu) >= 0 \ \forall \mu \in Z_2. \tag{35}$$

In the particular case when J is independent of $\mathbf{u}$ it follows from (33) that $\xi = 0$, $\sigma = 0$. If furthermore J is independent as well of T then besides $\theta = 0$. In this case the optimality system consists of the last equation in (12) having due to (9) the form

$$a_2(C, h) + c_2(\mathbf{u}_0, C, h) = < f, h > \ +(\chi_c, h)_{\Gamma_N}, \tag{36}$$

variational inequality (24) and the identity (35). The latter in the particular case of functional J_2 is a weak formulation of the next boundary value problem with respect to η:

$$-\lambda_c\Delta\eta - \mathbf{u}_0 \cdot \nabla\eta + w_0\frac{\partial\eta}{\partial z} + \hat{k}\eta = -\lambda_0 r(\hat{C} - C_d) \text{ in } H^{-1}(\Omega), \tag{37}$$

$$\eta|_{\Gamma^c_D} = 0, \ \ (\lambda_c\frac{\partial\eta}{\partial n} - w_0 n_3\eta)|_{\Gamma^c_N} = 0. \tag{38}$$

In the case where $J = J_3$ one should replace the equation (37) by

$$-\lambda_c\Delta\eta - \mathbf{u}_0 \cdot \nabla\eta + w_0\frac{\partial\eta}{\partial z} + \hat{k}\eta = -\lambda_0 r \text{ in } \Omega. \tag{39}$$

Let us compare the optimality systems which correspond to the functionals J_1, J_2 and J_3 in the linear case. For brevity we denote them by (S_1), (S_2) and (S_3) respectively. The first optimality system (S_1) consists of 12 scalar integral identities (12), (33) – (35) with respect to initial $(\hat{\mathbf{u}}, \hat{p}, \hat{T}, \hat{C})$ or adjoint $(\xi, \sigma, \zeta, \theta, \eta)$ states and the variational inequality (24) with respect to controls $\hat{f}, \hat{k}, \hat{\chi}_c, \hat{\chi}, \hat{\mathbf{g}}$. At the same time (S_2) and (S_3) are separate in a certain sense systems with respect to the pair of adjoint variables (C, η) and remaining variables. The latter is connected with the fact that corresponding problem for finding of the pair (C, η) does not contain hydrodynamic variables and T. So one can solve the optimality system with respect to C and η independently of the hydrodynamic problem. In other words the solution of the corresponding minimization problem in this case is reduced to finding variables C, η and controls $\hat{f}, \hat{k}, \hat{\chi}_c$ from equations (35), (36) and the corresponding variational inequality (24). Furthermore in the case of the system (S_3) the next formula holds for the minimal value $J_3^{\min}$ of the functional J_3:

$$J_3^{\min} \equiv \int\limits_\Omega r\hat{C}d\Omega = -\frac{1}{\lambda_0}\left(\int\limits_\Omega \hat{f}\eta d\Omega + \int\limits_{\Gamma_N} \chi_c \eta d\Gamma\right). \qquad (40)$$

In order to obtain (40) it suffice to put $h = \eta$ in (35), $J = J_3$, $\xi = 0$, $\mu = C$ in (35) and to subtract obtained relations.

For finding $J_3^{\min}$ by (40) one must find an adjoint state η and controls $\hat{f}, \hat{k}, \hat{\chi}_c$ from (35) and the variational inequality

$$< f - \hat{f}, \eta > \; -((k - \hat{k})\hat{C}, \eta) + (\chi_c - \hat{\chi}_c, \eta)_{\Gamma_N} \le 0 \; \forall (f, k, \chi_c) \in K_1 \times K_2 \times K_3. \qquad (41)$$

If furthermore the coefficient $\hat{k}$ is given so that $K_2 = \{\hat{k}\}$ then the solution of the latter problem is simplified and is reduced to subsequent finding a function η by solving an elliptic boundary value problem (37), (38) for η, determination of controls $\hat{f}$ and $\hat{\chi}_c$ from (41) at $k = \hat{k}$ and finding a minimal value $J_3^{\min}$ by (40). Formula (40) gives an explicit representation of the minimal value $J_3^{\min}$ of J_3 by virtue of Lagrange multipliers η and controls $\hat{f}, \hat{\chi}_c$. This formula is a generalization of the corresponding formula obtained in [11] under solving the problem of the optimal arrangement of enterprises near ecologocally important regions. Just an idea of minimization of the functional J_3 was lay down as the base of the economic method of solving this problem.

It should be emphasized that this approach is applicable only for linear with respect to C functional J_3 and only in the case of the linear model of mass transfer. If the cost functional is not linear one must solve the general optimality system. The latter is enough complicated problem even for the linear model of mass transfer in a viscous fluid and requires construction of effective numerical algorithms for solving such type problem. This problem will be considered by the author in further papers.

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