# The application of the weak approximation method in inverse problems

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The report is devoted to the application of the splitting method at differential equations (the weak approximation method by N. N. Yanenko [3–5]) for the problems of identification of coefficients in partial differential equations.

Theorem concerning the weak approximation method convergence for integrodifferential equations is formulated and proved. On the basis of this theorem the solvability of inverse problems for partial differential equations is proved. This method is used by investigation of the problem with unknown lowest coefficient in parabolic equation.

## 1. One theorem of the weak approximation method

In the strip $G_{[t_0,t_1]} = \{(t,x,y) \mid t_0 \le t \le t_1, x \in E_n, y \in E_1\}$ we consider the integrodifferential equation

$$\frac{\partial u}{\partial t} = \Psi(t,x,y,\overline{u},J(u)). \tag{1}$$

Here $u = u_1 + \mathrm{i}u_2$, $\Psi = \Psi_1 + \mathrm{i}\Psi_2$ are complex–valued functions, and functions $u_k = u_k(t,x,y)$, $\Psi_k = \Psi_k(t,x,y,\overline{u},J(u))$ are real–valued functions in $G_{[t_0,t_1]}$.

By $\overline{u} = (v^{(0)}, v^{(1)}, \ldots, v^{(p)})$ we denote the vector–function which components are defined in the following way: $v^{(0)} = v$; $v^{(1)}$ is a vector composed by means of all first order derivatives of $v$ with respect to $x_j$, $j = 1,\ldots,n$; $v^{(2)}$ is a vector composed by means of all second order derivatives of $v$ with respect to $x$ and so on; $v^{(p)}$ is a vector composed by means of all order $p$ derivatives of $v$ with respect to $x$.

Thus $\overline{u} = \left(v, \dfrac{\partial v}{\partial x_1}, \ldots, \dfrac{\partial v}{\partial x_n}, \dfrac{\partial^2 v}{\partial x_1^2}, \ldots, \dfrac{\partial^2 v}{\partial x_n^2}, \ldots, \dfrac{\partial^p v}{\partial x_1^p}, \ldots, \dfrac{\partial^p v}{\partial x_n^p}\right)$.

By $J(u)$ we denote the vector–function $J(u) = (J_0(u), J_1(u), \ldots, J_r(u))$, $r \ge 0$ is an integer; $J_k(u) = \int_{-\infty}^{\infty} y^k\, u(t,x,y)\, \mathrm{d}y$, $k = 0, 1, \ldots, r$.

We suppose that $\Psi = \sum\limits_{j=1}^{m} \Psi^j$.

Consider the equation

$$\frac{\partial u^\tau}{\partial t} = \sum_{j=1}^{m} \alpha_{j,\tau}(t)\Psi^j(t,x,y,\overline{u}^\tau, J(u^\tau)), \tag{2}$$

where functions $\alpha_{j,\tau}$ are denoted by relation

$$\alpha_{j,\tau}(\tau,t) = \begin{cases} m, & t_0 + \left(n + \dfrac{j-1}{m}\right)\tau < t \le t_0 + \left(n + \dfrac{j}{m}\right)\tau, \\ 0, & \text{otherwise}, \end{cases}$$

$$n = 0, 1, \ldots, N-1; \quad \tau N = t_1 - t_0.$$

The equation (2) approximates the equation (1) weakly.

Finally we consider the equation

$$\frac{\partial u^\tau}{\partial t} = \sum_{j=1}^{m} \alpha_{j,\tau}(t)\Psi_\tau^j(t,x,y,\overline{u}^\tau, J(u^\tau)), \tag{3}$$

where functious $\Psi_\tau^j(t,x,y,\overline{u}^\tau, J(u^\tau))$ are some functions $\Psi^j(t,x,y,\overline{u}^\tau, J(u^\tau))$ approximations depending on $\tau$.

Below we consider classical solutions to equations (1), ((2), (3)) only. We mean the classical solution to equation (2), ((3)) is a continuous function that has all continuous derivaties with respect to $x$, entering the equation (2), ((3)) and has piecewise continuous derivaties $u_t^\tau$ in the strip $G_{[t_0,t_1]}$ (possibly $u_t^\tau$ has discontinuities

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on hyperplanes $t = (n + j/m)$; $n = 0, 1, \ldots, N-1$; $\tau N = t_1 - t_0$; $j = 0, 1, \ldots, m-1$) and satysfies the equation (2), ((3)) in $G_{[t_0,t_1]}$.

**Assumption 1.** Functions $\Psi^j$ are denoted, and continuous for $(t, x, y) \in G_{[t_0,t_1]}$ and any values of other independent variables.

**Assumption 2.** Let a classical solution $u^{\tau_k}$ to system (3) in $G_{[t_0,t_1]}$ for all $\tau_k \geq 0$ exists. The sequence $\{u^{\tau_k}\}$ with their derivatives with respect to $x$ that are contained in (1) converge to a vector-function $u$ in $G_{[t_0,t_1]}$, and this convergence is uniform in $G^M_{[t_0,t_1]}$ for all fixed M.

**Assumption 3.** Integrals $J_j(u^{\tau_k})$ converge absolutely and uniformly by $\tau_k$ and $(t, x) \in \Pi_{[t_0,t_1]}$. Integrals $J_j(u)$ converge absolutely and uniformly by $(t, x) \in \Pi_{[t_0,t_1]}$, and $J_j(u^\tau_k)$ converge to $J_j(u)$ uniformly in $\Pi^M_{[t_0,t_1]}$ for all fixed $M$ as $\tau_k \to 0$.

**Assumption 4.** For all fixed $M$

$$\lim_{\tau_k \to 0} \max_{G^M_{[t_0,t_1]}} |\Psi^j(t, x, y, \overline{u}^{\tau_k}, J(u^{\tau_k})) - \Psi^j_\tau(t, x, y, \overline{u}^{\tau_k}, J(u^{\tau_k}))| = 0,$$
$$j = 0, 1, \ldots, r.$$

Here $M > 0$ is a constant in Assumptions 1–4.

**Theorem 1.** *Let Assumptions 1–4 be satisfied. Then the function $u(t, x, y)$ is a solution to equation* (1) *in* $G_{[t_0,t_1]}$.

**Proof.** Average functions

$$u^\nu_{av}(t, x, y) = \frac{1}{\nu} \int_t^{t+\nu} u^\nu(t, x, y)\, \mathrm{d}\theta$$

exist in $G^M_{[t_0,t^*]}$ for any $t$ from interval $(t_0, t_1)$ (for sufficienty small $\nu$) and the sequence $u^\nu_{av}$ converges to $u$ as $\nu \to 0$ uniformly in $G^M_{[t_0,t^*]}$.

Let us prove that $\partial u^\nu_{av}/\partial t$ converges to $\partial u/\partial t$ uniformly in $G^M_{[t_0,t^*]}$.

Now we average (3). We deduce the equation

$$\frac{\partial u^\nu_{av}}{\partial t} = \Psi(t, x, y, \overline{u}^\nu, J(u^\nu)) + F_\nu, \tag{4}$$

where

$$\begin{aligned} F_\nu &= F_\nu(t, x, y, \overline{u}^\nu, J(u^\nu)) = \\ &= \tfrac{m}{\nu} \sum_{j=1}^{m} \int_{\sigma_j} \{\Psi^j_\nu(\theta, x, y, \overline{u}^\nu(\theta), J(u^\nu(\theta))) - \Psi^j(t, x, y, \overline{u}^\nu(t), J(u^\nu(t)))\}\, \mathrm{d}\theta. \end{aligned} \tag{5}$$

Let's consider the integrand in (5):

$$|\Psi^j_\nu(\theta, x, y, \overline{u}^\nu(\theta), J(u^\nu(\theta))) - \Psi^j(t, x, y, \overline{u}^\nu(t), J(u^\nu(t)))| \leq$$

$$\leq |\Psi^j_\nu(\theta, x, y, \overline{u}^\nu(\theta), J(u^\nu(\theta))) - \Psi^j(\theta, x, y, \overline{u}^\nu(\theta), J(u^\nu(\theta)))| +$$

$$+ |\Psi^j(\theta, x, y, \overline{u}^\nu(\theta), J(u^\nu(\theta))) - \Psi^j(t, x, y, \overline{u}^\nu(t), J(u^\nu(t)))|.$$

When $\nu \to 0$ the first term in the right-hand part of the latter inequality tends to zero uniformly in $G^M_{[t_0,t^*]}$ due to Assumption 4. The second term also tends to zero uniformly in $G^M_{[t_0,t^*]}$ but due to the uniform continuity of the vector–function $\Psi^j$ on all its arguments for each compact set (see Assumption 1) and equicontinuity with respect to $t$, $x$ in $G^M_{[t_0,t^*]}$ and $\Pi^M_{[t_0,t^*]}$ of the sequences $\{u^\nu(t)\}$ and $\{J(u^\nu(t))\}$ respectively (according to Assumptions 2, 3 and the Arzela's theorem).

Hence, if $\nu \to 0$ then function sequence $F_\nu \to 0$ uniformly in $G^M_{[t_0,t^*]}$. As $\Psi(t, x, y, \overline{u}^\nu(t), J(u^\nu(t)))$ converges uniformly in $G^M_{[t_0,t^*]}$ to $\Psi(t, x, y, \overline{u}(t), J(u(t)))$ (accoding to Assumptions 3–5), then $\partial u^\nu_{av}/\partial t \to \partial u/\partial t = \Psi(t, x, y, \overline{u}(t), J(u(t)))$ uniformly in $G^M_{[t_0,t^*]}$.

By theorem on differentiation of functional sequences $\partial u^\nu_{av}/\partial t \to \partial u/\partial t$ uniformly in $G^M_{[t_0,t^*]}$. Thus $\partial u/\partial t = \Psi(t, x, y, \overline{u}, J(u))$, that is $u$ is a classical solution of the equation (1) in $G^M_{[t_0,t^*]}$.

Considering the average functions

$$\frac{1}{\nu} \int_{t-\nu}^{t} u^\nu(\theta)\, \mathrm{d}\theta,$$

we can prove that $u(t)$ is the solution of the system (1) in $G^M_{[t^*,t_1]}$ for any $t_* \in (t_0, t_1)$ and therefore in $G^M_{[t_0,t_1]}$. Theorem 1 is proved.

## 2. The inverse problem

Below we consider the example connected with the using Theorem 1 to the problem of identification of coefficients for partial differential equations.

Let us consider in $G_{[0,T]} = \{(t,x,z) \mid 0 \le t \le T, x \in E_n,\ z \in E_1\}$, $T =$ const $> 0$, $n \ge 1$, $n$ is an integer, the Cauchy problem

$$u_t(t,x,z) = L_x(u(t,x,z)) + a(t)u_{zz}(t,x,z) + b(t)u_z(t,x,z) + \\ + c(t,x)u(t,x,z) + f(t,x,z), \tag{6}$$

$$u(0,x,z) = u_0(x,z), \quad (x,z) \in E_{n+1}, \tag{7}$$

where

$$L_x(u) = \sum_{j,k=1}^{n} a_{jk} u_{x_j x_k} + \sum_{j=1}^{n} a_j u_{x_j}.$$

We assume that $f(t,x,z)$, $u_0(x,z)$ are the given functions in $G_{[0,T]}$ and $E_{n+1}$, respectively, and the coefficients $a_{jk}(t)$, $a_j(t)$, $j,k = 1, \ldots, n$, and functions $a(t)$, $b(t)$ are the continuous functions of the variable $t$, $a(t) > 0$, $0 \le t \le T$, and the condition

$$\kappa|\xi|^2 \le \sum_{j,k=1}^{n} a_{jk}(t)\xi_j\xi_k, \quad \forall \xi \in E_n, \quad t \in [0,T], \quad \kappa = \text{const} > 0,$$

is satisfied.

We also assume that

$$u(t,x,0) = \varphi(t,x), \quad (t,x) \in \Pi_{[0,T]}, \tag{8}$$

where $\varphi(t,x)$ is a given function satisfying the consistency condition

$$\varphi(0,x) = u_0(x,0), \quad x \in E_n, \tag{9}$$

and in $\Pi_{[0,T]}$

$$0 < \delta \le |\varphi(t,x)|, \quad \delta = \text{const}. \tag{10}$$

In the problem (6) – (8) the coefficient $c(t,x)$ $(\partial c/\partial z \equiv 0)$ and the function $u(t,x,z)$ are unknown.

All the above mentioned functions are real–valued ones.

Assuming the existence of the Fourier transform of the function $u(t,x,z)$ with respect to the variable $z$ and using the condition (8), the problem (6) – (8) is reduced to the problem

$$v_t(t,x,y) = L_x(v(t,x,y)) - y^2 a(t)v(t,x,y) + \mathrm{i}b(t)yv(t,x,y) + \\ + \frac{v(t,x,y)}{\varphi(t,x)}\mathrm{Re}[\psi(t,x) + a(t)\int_{-\infty}^{+\infty} y^2 v(t,x,y)\,\mathrm{d}y - \\ - \mathrm{i}b(t)\int_{-\infty}^{+\infty} yv(t,x,y)\,\mathrm{d}y] + \Phi(t,x,y), \tag{11}$$

$$v(0,x,y) = v_0(x,y), \quad x \in E_n, \quad y \in E_1. \tag{12}$$

Here $\psi = \varphi_t - L_x(\varphi) - f|_{z=0}$ and $v(t,x,y)$, $\Phi(t,x,y)$, and $v_0(x,y)$ are the Fourier transforms with respect to the variable $z$ of the functions $u(t,x,z)$, $f(t,x,z)$, and $u_0(x,z)$, respectively:

$$v(t,x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} u(t,x,z)\mathrm{e}^{-\mathrm{i}zy}\,\mathrm{d}z,$$

$$\Phi(t,x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t,x,z)\mathrm{e}^{-\mathrm{i}zy}\,\mathrm{d}z,$$

$$v_0(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} u_0(x,z)\mathrm{e}^{-\mathrm{i}zy}\,\mathrm{d}z.$$

We reduce the problem (6) – (8) to the problem (11), (12) in the same way as we have reduced the inverse problems to the direct problems in [1, 2].

## 3. The Solvability of the Direct Problem

We weakly approximate the problem (11), (12) by the problem

$$v_t^\tau(t,x,y) = 3L_x(v^\tau(t,x,y)), \quad n\tau < t \le \left(n+\frac{1}{3}\right)\tau, \tag{13}$$

$$v_t^\tau(t,x,y) = 3(\mathrm{i}yb(t) - y^2a(t))v^\tau(t,x,y), \quad \left(n+\frac{1}{3}\right)\tau < t \le \left(n+\frac{2}{3}\right)\tau, \tag{14}$$

$$v_t^\tau(t,x,y) = \frac{3v^\tau(t,x,y)}{\varphi}\mathrm{Re}[\psi(t,x) + a(t)\int_{-\infty}^{+\infty} y^2 v^\tau\left(t-\frac{\tau}{3},x,y\right)\,\mathrm{d}y - \mathrm{i}b(t)\int_{-\infty}^{+\infty} yv^\tau\left(t-\frac{\tau}{3},x,y\right)\,\mathrm{d}y] + 3\Phi(t,x,y), \quad \left(n+\frac{2}{3}\right)\tau < t \le (n+1)\tau, \tag{15}$$

$$v^\tau(0,x,y) = v_0(x,y), \tag{16}$$

where $n = 0, 1, ..., N-1$ and $\tau N = T$; $N$ is an integer.

As concerns the functions $\psi$, $\varphi$, $\Phi$, $v_0$ we suppose that they are sufficiently smooth (they have continuous derivatives with respect to the variables $x, y$ given in (17), (18) in $\Pi_{[0,T]}$, $G_{[0,T]}$ and $E_{n+1}$ respectively. The functions satisfying the relations

$$\begin{gathered} |D_x^\beta v_0| + |D_x^\beta \Phi| + |D_x^\beta \psi| + |D_x^\beta \varphi| \le c_k, \\ |\beta| = k; \quad k = 0,1,...,4; \quad (t,x,y) \in G_{[0,T]}. \end{gathered} \tag{17}$$

$$\begin{gathered} |\frac{\partial}{\partial y} D_x^\gamma v_0| + |\frac{\partial}{\partial y} D_x^\gamma \Phi| \le N_j, \qquad |\gamma| = j;\ j = 0,1,2; \\ |y|^{\nu+\varepsilon}|D_x^\beta v_0| + |y|^{\nu+\varepsilon}|D_x^\beta \Phi| \le M_k, \quad |\beta| = k;\ k = 0,1,...,4; \\ |y|^{\nu+\varepsilon}|\frac{\partial}{\partial y} D_x^\gamma v_0| + |y|^{\nu+\varepsilon}|\frac{\partial}{\partial y} D_x^\gamma \Phi| \le R_j, \quad |\gamma| = j;\ j = 0,1,2. \end{gathered} \tag{18}$$

»From the construction of the solution $v^\tau$ of the problem (13) – (16) and the conditions (18), (17) it follows that for any fixed $\tau$ the solution $v^\tau$ exists, and has continuous in $G_{[0,T]}$ derivatives $D_x^\beta v^\tau$, $|\beta| \le 4$, $D_x^\gamma(\partial v^\tau/\partial y)$, $|\gamma| \le 2$ and

$$|y|^p|D_x^\beta v^\tau(t,x,y)| \le c, \ |\beta| \le 4; (t,x,y) \in G_{[0,T]};\ p = 0, \nu+\varepsilon. \tag{19}$$

$$\left|\frac{\partial}{\partial t} D_x^\alpha v^\tau(t,x,y)\right| \le c, \quad |\alpha| \le 2; \quad (t,x,y) \in G_{[0,T]}. \tag{20}$$

$$\left|\frac{\partial}{\partial y} D_x^\beta v^\tau(t,x,y)\right| \le c, \quad |\beta| \le 2, \quad (t,x,y) \in G_{[0,T]}. \tag{21}$$

»From estimates (19) (under $p = 0$) there follows uniform in $G_{[0,T]}$ boundedness with respect to $\tau$ of the family of the derivatives $\{D_x^\beta v^\tau\}$ for fixed $\beta, |\beta| \le 2$, and from estimates (19), (20) – (21) there follows equicontinuity in $G_{[0,T]}$ with respect to $t, x, y$ of the same family. According to the Arzela's theorem the set $\{D_x^\beta v^\tau\}$ is compact in $C(G_{[0,T]}^M)$, $|\beta| \le 2$, $M > 0$ is a constant.

By means of a diagonal method we choose the subsequence $\{v^\tau\}$ (the notation is not changed) converging together with the derivatives $\{D_x^\beta v^\tau\}$, $|\beta| \le 2$, to some function $v$ in the strip $G_{[0,T]}$ and also uniformly in $G_{[0,T]}^M$ for any $M > 0$:

$$D_x^\alpha v^\tau \to D_x^\alpha v \quad \text{uniformly in} \quad G_{[0,T]}^M \quad \text{as}\ \tau \to 0, \quad |\alpha| \le 2. \tag{22}$$

The function $v$ is continuous in $G_{[0,T]}$ together with the derivatives $D_x^\alpha v$, $|\alpha| \le 2$, and satisfies the inequality (see (19)).

$$|y|^p|D_x^\alpha v(t,x,y)| \le c, \qquad (t,x,y) \in G_{[0,T]}, \quad p = 0, \nu+\varepsilon, \tag{23}$$

and initial data (7).

Thanks to (19), (22), (23) the conditions of the Theorem 1 are fulfilled. By Theorem 1 the function $v(t,x,y)$ is the solution of the equation (6) in $G_{[0,T]}$.

In our case (see Assumptions 1 – 4 of the Theorem 1) $m = 3$, $r = 2$,

$$\Psi(t,x,y,\bar{v}^\tau, J(v^\tau)) = L_x(v^\tau(t,x,y)) + (-a(t)y^2 + \mathrm{i}yb(t))v^\tau(t,x,y) +$$

$$+\frac{v^\tau(t,x,y)}{\varphi(t,x)}\mathrm{Re}[\psi(t,x)+a(t)\int_{-\infty}^{+\infty} y^2 v^\tau(t,x,y)\ \mathrm{d}y-$$
$$-\mathrm{i}b(t)\int_{-\infty}^{+\infty} yv^\tau(t,x,y)\ \mathrm{d}y]+\Phi(t,x,y),$$
$$\Psi^1(t,x,y,\bar v^\tau,J(v^\tau))=\Psi^1_\tau(t,x,y,\bar v^\tau,J(v^\tau))=L_x(v^\tau(t,x,y)),$$
$$\Psi^2(t,x,y,\bar v^\tau,J(v^\tau))=\Psi^2_\tau(t,x,y,\bar v^\tau,J(v^\tau))=$$
$$=(\mathrm{i}yb(t)-y^2a(t))v^\tau(t,x,y),$$
$$\Psi^3(t,x,y,\bar v^\tau,J(v^\tau))=\frac{v^\tau(t,x,y)}{\varphi(t,x)}\mathrm{Re}[\psi(t,x)+$$
$$+a(t)\int_{-\infty}^{+\infty} y^2 v^\tau(t,x,y)\ \mathrm{d}y-\mathrm{i}b(t)\int_{-\infty}^{+\infty} yv^\tau(t,x,y)\ \mathrm{d}y]+\Phi(t,x,y),$$
$$\Psi^3_\tau(t,x,y,\bar v^\tau,J(v^\tau))=\frac{v^\tau(t,x,y)}{\varphi(t,x)}\mathrm{Re}[\psi(t,x)+$$
$$+a(t)\int_{-\infty}^{+\infty} y^2 v^\tau(t-\tfrac{\tau}{3},x,y)\ \mathrm{d}y-$$
$$-\mathrm{i}b(t)\int_{-\infty}^{+\infty} yv^\tau(t-\tfrac{\tau}{3},x,y)\ \mathrm{d}y]+\Phi(t,x,y).$$

Function $v(t,x,y)$ belongs to the class $C^{1,2}_{t,x}(G_{[0,t_*]})$ and satisfies the initial data (12). We have proved the following theorem.

**Theorem 2.** *Let assumptions* (10), (17), (18) *be satisfied. Then there exists the solution* $v(t,x,y)$ *for the problem* (11), (12) *in class* $C^{1,2}_{t,x}(G_{[0,t_*]})$ *so that it satisfies the relations* (23).

There is $C^{1,2}_{t,x}(G_{[0,t_*]})=\{f|f_t,f_x,f_{xx}\in C(G_{[0,t_*]})\}$.

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